1) We are going to be usig that

(a) $[a, bc] = b[a, c] + [a, b]c$

(b) $[ab, cd] = a[b, c]d + ac[b, d] + [a, c]db$
$+ c[a, d]b$

1.1) $H = \vec{\alpha}\vec{P} + m\beta = \sum_i \alpha_i P^i + m\beta$

$[P^j, H] = [P^j, \sum_i \alpha_i P^i] + [P^j, m\beta]$

using (a) $= \sum_i \alpha_i \underbrace{[P^j, P^i]}_{0} + \sum_i [P^j, \alpha_i] P^i$

$+ [P^j, m]\beta + m[P^j, \beta] = 0 + 0 + 0 + 0$

each of the terms is zero because α^i, m, and β do not depend on x and $[P^j, P^i] = 0$

$\vec{L} = \vec{X} \wedge \vec{P} \Rightarrow L_a = \sum_{b,c} \epsilon_{abc} X^b P^c$

$[L_a, H] = [\sum_{bc} \epsilon_{abc} X^b P^c, \sum_i \alpha_i P^i]$

$+ [\sum_{bc} \epsilon_{abc} X^b P^c, m\beta] = (*)$

using (b)

$$(*) = \sum_{bc} \epsilon_{abc} \sum_i x^b \underbrace{[P^c, \alpha_i]}_{=0} P^i + \sum_{bc} \sum_i \epsilon_{abc} x^b \alpha_i \underbrace{[P^c, P^i]}_{=0}$$

$$+ \sum_{abc} \sum_i \epsilon_{abc} \underbrace{[x^b, \alpha_i]}_{=0} P^i P^c + \sum_{bc} \sum_i \epsilon_{abc} \alpha_i [x^b, P^i] P^c$$

$$= \sum_{bc} \sum_i \epsilon_{abc} \alpha_i (i) \delta^{bi} P^c = i \sum_{bc} \epsilon_{abc} \alpha^b P^c$$

1.2) $\vec{J} = \vec{L} + \frac{1}{2} \vec{\Sigma}$ $\quad \Sigma_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}$ this is for either Dirac or Chiral rep

$$[J_a, H] = [L_a, H] + [\Sigma_a, H]$$

we already have $[L_a, H] = i \sum_{bc} \epsilon_{abc} \alpha^b P^c$

now $[\Sigma_a, H] = [\Sigma_a, \sum_i \alpha_i P^i] + [\Sigma_a, m\beta]$

$$= \sum_i \alpha_i \underbrace{[\Sigma_a, P^i]}_{=0} + \sum_i [\Sigma_a, \alpha_i] P^i$$

$$+ m [\Sigma_a, \beta] + \underbrace{[\Sigma_a, m]}_{=0} \beta$$

we can use the Dirac rep in which

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

so $[\Sigma_a, \beta] = \frac{1}{2}\begin{pmatrix}\sigma_a & \\ & \sigma_a\end{pmatrix}\begin{pmatrix}I & \\ & -I\end{pmatrix} - \frac{1}{2}\begin{pmatrix}I & \\ & -I\end{pmatrix}\begin{pmatrix}\sigma_a & \\ & \sigma_a\end{pmatrix}$

$= 0$

$$[\Sigma_a, \alpha^i] = \frac{1}{2}\left\{\begin{pmatrix}\sigma_a & 0\\ 0 & \sigma_a\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\begin{pmatrix}\sigma_a & 0\\ 0 & \sigma_a\end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix}0 & \sigma_a\sigma_i - \sigma_i\sigma_a\\ \sigma_a\sigma_i - \sigma_i\sigma_a & 0\end{pmatrix} = i\sum_c \epsilon_{aic}\underbrace{\begin{pmatrix}0 & \sigma^c\\ \sigma^c & 0\end{pmatrix}}_{\alpha^c}$$

[illegible] $\sigma_a\sigma_i - \sigma_i\sigma_a = 2i\sum_c \epsilon_{aic}\sigma^c$

so $[\Sigma_a, H] = i\sum_{ci}\epsilon_{aic}\alpha^c p^i = i\sum_{cb}\epsilon_{abc}\alpha^c p^b$

$= -i\sum_{bc}\epsilon_{acb}\alpha^c p^b = -i\sum_{bc}\epsilon_{abc}\alpha^b p^c$

$= -[L_a, H]$

$\Rightarrow [J_a, H] = 0$

1.3)

$$[J_a, P_i] = [L_a, P_i] + [\Sigma_a, P_i]$$

$$= [\sum_{bc} \epsilon_{abc} x^b p^c, p^i]$$

$$= \sum_{bc} \epsilon_{abc} x^b \underbrace{[p^c, p^i]}_{=0} + \sum_{bc} \underbrace{[x^b, p^i]}_{(i)\delta_{bi}} p^c$$

$$= i \sum_c \epsilon_{aic} P^c \neq 0$$

(1.4) $\vec{J}\vec{P} = \vec{L}\cdot\vec{P} + \vec{\Sigma}\cdot\vec{P}$

$$\vec{L}\cdot\vec{P} = \sum_a L_a P_a = \sum_a \sum_{bc} \underbrace{\epsilon_{abc}}_{\text{antisymmetric under } c\leftrightarrow a} x^b \underbrace{p^c p^a}_{\text{symmetric under } c\leftrightarrow a} = 0$$

$$\vec{J}\vec{P} = \vec{\Sigma}\vec{P} = \sum_a \Sigma_a P^a$$

$$[\Sigma_a P^a, H] = [\Sigma_a P^a, \alpha_j p^j] + [\Sigma_a, m\beta]$$

$$= \Sigma_a \underbrace{[P^a, \alpha_j]}_{=0} p^j + [\Sigma_a, \alpha_j] p^a p^j +$$

$$\sum_a \alpha_j \cancel{[P^a, P^j]} + \alpha_j [\Sigma_a, P^j] P^a$$

$$+ [\Sigma_a, m]\beta + m[\Sigma_a, \beta]$$

$$= [\Sigma_a, \alpha_j] p^a p^j = i \underbrace{\epsilon_{ajc}}_{\text{antisymmetric } a\leftrightarrow j} \alpha^c \underbrace{p^a p^j}_{\text{symmetric } a\leftrightarrow j} = 0$$

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(2) $u^{\pm}(\vec{p}) = \begin{pmatrix} \sqrt{E \mp |\vec{p}|}\ \xi_p^{\pm} \\ \sqrt{E \pm |\vec{p}|}\ \xi_p^{\pm} \end{pmatrix}$ with $\xi_p^{+} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$ $\xi_p^{-} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$ (1)

$$v^{\pm}(\vec{p}) = \pm \begin{pmatrix} \sqrt{E \pm |\vec{p}|}\ \xi_{-p}^{\mp} \\ -\sqrt{E \mp |\vec{p}|}\ \xi_{-p}^{\mp} \end{pmatrix}$$

we need $v^{\pm}(-\vec{p})$

$\vec{p} = |\vec{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in spherical coordinates with $0 \leq \theta \leq \pi$ $\quad 0 \leq \varphi \leq 2\pi$

the $-\vec{p} = |\vec{p}|(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$

with $\theta' = \pi - \theta$ $\quad$ so $\quad \cos\theta' = -\cos\theta \quad \cos\phi' = -\cos\phi$

$\phi' = \pi + \phi \qquad \sin\theta' = \sin\theta \quad \sin\phi' = -\sin\phi$

$\Rightarrow \quad \cos\frac{\theta'}{2} = \cos\frac{\pi}{2}\cos\frac{\theta}{2} + \sin\frac{\pi}{2}\sin\frac{\theta}{2} = \sin\frac{\theta}{2}$

$\sin\frac{\theta'}{2} = -\cos\frac{\pi}{2}\sin\frac{\theta}{2} + \sin\frac{\pi}{2}\cos\frac{\theta}{2} = +\cos\frac{\theta}{2}$

$e^{\pm i\phi'} = \underbrace{-}_{e^{i\pi}} e^{\pm i\phi}$

$$\xi^{+}_{-p} = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} \qquad \xi^{-}_{-p} = \begin{pmatrix} +e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$\bar{u}^{\pm}(p) = [u^{\pm}(p)]^{\dagger} \underset{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}{\underset{\parallel}{\gamma^{0}}}$$

$$\Rightarrow \bar{u}^{\pm}(p) = \left(\sqrt{E \pm |\vec{p}|}\,(\xi^{\pm}_{p})^{\dagger},\ \sqrt{E \mp |\vec{p}|}\,(\xi^{\pm}_{p})^{\dagger}\right)$$

$$(\xi^{+}_{p})^{\dagger}\xi^{+}_{-p} = \left(\cos\frac{\theta}{2},\ e^{-i\phi}\sin\frac{\theta}{2}\right)\begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} =$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$$

$$(\xi^{+}_{p})^{\dagger}\xi^{-}_{-p} = \left(\cos\frac{\theta}{2},\ e^{-i\phi}\sin\frac{\theta}{2}\right)\begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= e^{-i\phi}\left(\cos^{2}\frac{\theta}{2} + \sin^{2}\frac{\theta}{2}\right) = e^{-i\phi}$$

$$(\xi^{-}_{p})^{\dagger}\xi^{+}_{-p} = \left(-e^{i\phi}\sin\frac{\theta}{2},\ \cos\frac{\theta}{2}\right)\begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$= -e^{i\phi}$$

$$(\xi^{-}_{p})^{\dagger}\xi^{-}_{-p} = \left(-e^{i\phi}\sin\frac{\theta}{2},\ \cos\frac{\theta}{2}\right)\begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\theta/2 \end{pmatrix} = -\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0$$

So $\bar{u}^{+}(p)\, v^{-}(-p) = \bar{u}^{-}(p)\, v^{+}(-p) = 0$

$$\bar{u}^{+}(p)\, v^{+}(-p) = \left(\sqrt{E+|\vec{p}|}\,,\ \sqrt{E-|\vec{p}|}\right)\begin{pmatrix}\sqrt{E+|\vec{p}|}\\ -\sqrt{E-|\vec{p}|}\end{pmatrix}(\xi^{+}_{p})^{\dagger}\,\xi^{-}_{-p}$$

$$= \left[(E+|\vec{p}|) - (E-|\vec{p}|)\right] e^{-i\phi} = 2\,|\vec{p}|\, e^{-i\phi}$$

$$\bar{u}^{-}(p)\, v^{-}(-p) = \left(\sqrt{E-|\vec{p}|}\,,\ \sqrt{E+|\vec{p}|}\right)(-)\begin{pmatrix}\sqrt{E-|\vec{p}|}\\ -\sqrt{E+|\vec{p}|}\end{pmatrix}(\xi^{-}_{p})^{\dagger}\,\xi^{+}_{-p}$$

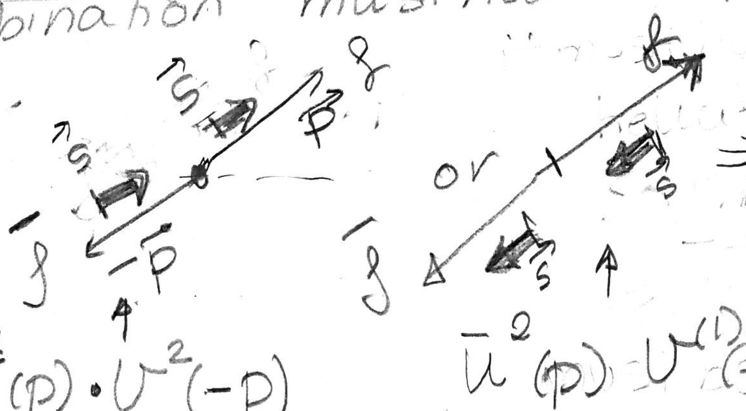
$$= (-)\left(E-|\vec{p}| - (E+|\vec{p}|)\right)\ (-)\, e^{i\phi}$$

$$= -2\,|\vec{p}|\, e^{i\phi}$$

Let us think what this means. You are evaluating the product of two spinors $\bar{u}^{s_1}(p) \cdot v^{s_2}(-p)$ this is from $\bar{\Psi}, \Psi \equiv$ Lorentz scalar

f so this combination must have 0 spin

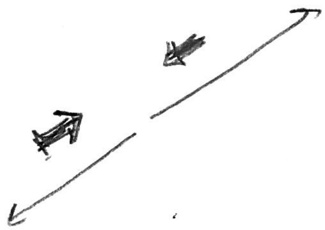
$\Rightarrow$ since



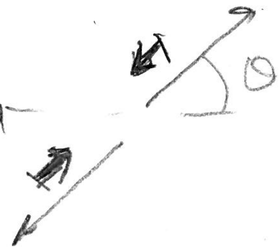
$\Rightarrow \vec{S}^{TOT} \neq 0$ must be zero

$\vec{J}$

Further more



and



have $\vec{S}=0$

irrespective of θ, φ

so $|\bar{u}^{(1)}_{(p)} v^{(1)}_{(-p)}| = |\bar{u}^{(2)}_{(p)} v^{(2)}_{(-p)}|$ and must be independent of θ, φ

3 Under gauge transformation of function $\chi(x)$

1) $A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu \chi$

$$A'^\mu = \epsilon'^\mu e^{-ip_\alpha x^\alpha}$$

$$A^\mu = \epsilon^\mu e^{-ip_\alpha x^\alpha}$$

$$\chi = i\kappa\, e^{-ip_\alpha x^\alpha} \rightarrow \partial^\mu \chi = i\kappa(-ip^\mu)\, e^{-ip_\alpha x^\alpha}$$

so al together

$$\epsilon'^\mu e^{-ip_\alpha x^\alpha} = \epsilon^\mu e^{-ip_\alpha x^\alpha} + \kappa p^\mu e^{-ip_\alpha x^\alpha}$$

$$= (\epsilon^\mu + \kappa p^\mu)\, e^{-ip_\alpha x^\alpha}$$

$$\Rightarrow \epsilon'^\mu = \epsilon^\mu + \kappa p^\mu$$

2) ~~If we~~ In any gauge the Maxwell's eq are

$$0 = \partial_\mu F^{\mu\nu} \equiv \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

For A'^{μ} the equations are

$$\underbrace{p_\mu p^\mu}_{\substack{\parallel \\ m_\gamma^2 = 0}} \epsilon'^{\nu} e^{-ip_\alpha x^\alpha} - p_\mu p^\nu \epsilon'^{\mu} e^{-ip_\alpha x^\alpha} = 0$$

So the equations $\Rightarrow (p_\mu \epsilon'^{\mu}) p^\nu = 0$

$$\Rightarrow p_\mu \epsilon'^{\mu} = 0 \Rightarrow \text{Lorentz gauge}$$

For $\kappa = -\frac{\epsilon^0}{p^0} \Rightarrow \epsilon'^0 = \epsilon^0 - \frac{p^0 \epsilon^0}{p^0} = 0$

So with this κ Maxwell' eq $\Rightarrow \vec{p} \cdot \vec{\epsilon} = 0$

Alternatively one could use that gauge transf. must conserve the norm of the A^μ

$$\Rightarrow \cancel{\epsilon'^{\mu} \epsilon'_{\mu}} = \epsilon^\mu \epsilon_\mu = (\epsilon'^{\mu} - \kappa p^\mu)(\epsilon'_{\mu} - \kappa p_\mu)$$

$$= \cancel{\epsilon'^{\mu} \epsilon'_{\mu}} - 2\kappa \epsilon'^{\mu} p_\mu + \kappa^2 \underbrace{p^\mu p_\mu}_{\substack{\parallel \\ m_\gamma^2 = 0}}$$

$$\Rightarrow 0 = \epsilon'^{\mu} p_\mu$$

Could I find this without using $p^2 = m_\gamma^2 = 0$?
explicitly
(or what is the same
the free Maxwell eq.)

We have

$$\epsilon'^\mu = \epsilon^\mu + \kappa p^\mu$$

to be in the Lorentz gauge $\epsilon'^\mu p_\mu = 0$

we must chose $\kappa = -\frac{(\epsilon^\mu p_\mu)}{p^2} = \frac{-\epsilon^0 p^0 + \vec{\epsilon}\vec{p}}{p^2}$

now if we imposse the specific value $\kappa = -\frac{\epsilon^0}{p^0}$

$$-\frac{\epsilon^0 p^0 + \vec{\epsilon}\vec{p}}{p^2} = -\frac{\epsilon^0}{p^0}$$

both things are required

$$\cancel{-\epsilon_p^0 p^0} + \vec{\epsilon}\vec{p} = -\frac{\epsilon^0}{p^0}(\cancel{p^{0\,2}} - |\vec{p}|)^2$$

also

$$\Rightarrow \vec{\epsilon}\vec{p} = \frac{\epsilon_0}{p^0}|\vec{p}| = -\kappa|\vec{p}| \Rightarrow \kappa = -\frac{\vec{\epsilon}\cdot\vec{p}}{|\vec{p}|}$$

So $\vec{\epsilon}'\vec{p} = \vec{\epsilon}\vec{p} + \kappa|\vec{p}|^2 = \vec{\epsilon}\vec{p} - \vec{\epsilon}\vec{p} = 0$