Chapter 5

QED I : Feynman amplitudes and Feynman diagrams.

1) Non-relativistic perturbation theory
2) Interaction of e^- with electromagnetic field
3) $\mu^- e^- \rightarrow \mu^- e^-$
4) Propagators
5) Feynman rules for QED
6) Lagrangian for QED: as a gauge theory

1) Non-relativistic perturbation theory

Start point are the solution for free particle Schroedinger Eq

$$H_0 \phi_n = E_n \phi_n \qquad H_0 \equiv \text{time indep} \Rightarrow \phi_n(\vec{x})$$

normalized as $\int_V d^3x \, \phi_m^*(x) \phi_n(x) = \delta_{nm} \Rightarrow$ 1 particle in a box of unit volume V

If we switch a potential $V(\vec{x}, t)$ the Eq is

$$[H_0 + V(\vec{x}, t)] \phi = i \frac{\partial \phi}{\partial t}$$

and its solution can be expressed in the basis of ϕ_n states

$$\psi = \sum_n a_n(t) \phi_n(\vec{x}) e^{-iE_n t}$$

So to solve eq $\equiv$ find the coefficient $a_n(t)$ ocurs

$$i \sum_m \frac{da_m}{dt} \phi_m(\vec{x}) e^{-iE_m t} = \sum_n V(\vec{x}, t) a_n \phi_n(\vec{x}) e^{-iE_n t}$$

(using that $H_0 \phi_n = E_n \phi_n$
multiplying by ϕ_f^* and integrating over d^3x
and using orthogonality one gets

$$i \frac{da_f}{dt} = \sum_n a_n(t) \int \phi_f^*(\vec{x}) e^{iE_f t} V(\vec{x}, t) \phi_n(\vec{x}) e^{-iE_n t} d^3x$$

③

Formally we can write the ev eqn integral form with so initial conditions at some time $t \equiv -T/2$ $a_i(-T/2) = 1$ $a_{j \neq i}(-T/2) = 0$

$$\Rightarrow a_f(t) = \delta_{fi} - i \int_{-T/2}^{t} dt' \sum_n a_n(t') \int \phi_f^* e^{iE_f t'} V \phi_n e^{-iE_n t'} d^3x \ldots$$

order zero
state does not change

which we can solve iteratively. At 1st order we introduce δ_{fi} in $a_n(t')$

$$a_f^{(1)}(t) = \delta_{fi} - i \int_{-T/2}^{t} dt' \underbrace{\phi_f^*(\vec{x}) e^{iE_f t'}}_{\phi_f^*(x)} \underbrace{V(\vec{x}, t)}_{V(x)} \underbrace{\phi_i(\vec{x}) e^{-iE_i t'}}_{\phi_i(x)} d^3x$$

$a_f^{(0)}$ ← which

In particular at time $t = T/2$ when the interaction has ceased

$$T_{fi}^{(1)} \equiv a_f^{(1)}(T/2) - \delta_{fi} = -i \underbrace{\underbrace{\int_{-T/2}^{T/2} dt \int d^3x}_{d^4x} \phi_f^*(x) V(x) \phi_i(x)}_{\text{written in covariant form}}$$

We define the transition rate per unit time and volume

$$w_{fi} = \lim_{T \to \infty} \frac{|T_{fi}|^2}{VT}$$

↑ volume

For a time independent potential

$$T_{fi}^{(1)} = -i \underbrace{\int \phi_f^*(\vec{x}) V(\vec{x}) \phi_i(\vec{x})}_{= V_{fi}} \underbrace{\int_{-\infty}^{\infty} dt' \, e^{i(E_f - E_i)t}}_{2\pi \delta(E_i - E_f)}$$

the evolution eq for time indep potential

$$a_f(t) = \underbrace{\delta_{fi}}_{= a_f^{(0)}} - i \int_{-T/2}^{t} dt' \sum_n a_n(t') V_{fn} e^{i(E_f - E_n)t'}$$

and we found at 1st order $a_f^{(1)}(t) = a_f^{(0)} - i \int_{-T/2}^{t} V_{fi} e^{i(E_f - E_i)t'} dt'$

at second order

$$a_f(t) = a_f^{(1)}(t) \underbrace{- i \int_{-T/2}^{t} dt' \sum_n (-i) \int_{-T/2}^{t'} V_{ni} e^{i(E_n - E_i)t''} dt'' \, V_{fn} e^{i(E_f - E_n)t'}}_{a_f^{(2)}(t)}$$

$$T_{fi}^{(2)} = (-i)^2 \sum_n V_{fn} V_{ni} \underbrace{\int_{-\infty}^{\infty} dt \, e^{i(E_f - E_n)t} \int_{-\infty}^{t} e^{i(E_n - E_i)t'} dt'}$$

To perform in dt' we introduce a small ϵt and then will take the limit $\epsilon \to 0$

$$\int_{-\infty}^{t} dt' e^{i(E_n - E_i - i\epsilon)t} = i \frac{e^{i(E_n - E_i - i\epsilon)t}}{E_i - E_n + i\epsilon}$$

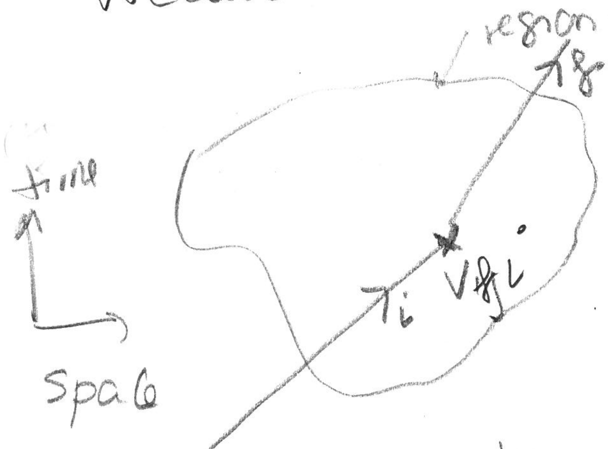
So and integrating $\int dt\, e^{i(E_f-E_n)t}\, e^{i(E_n-E_i)t}$

$$=(2\pi)\,\delta(E_f-E_i)$$

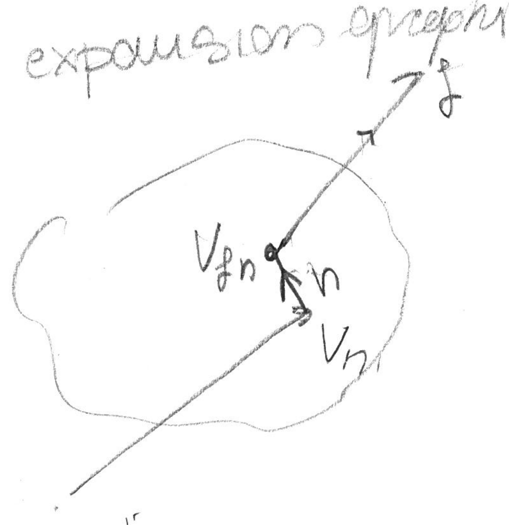
So at second order

$$T_{fi}^{(2)}=(-i)^2(2\pi i)\sum_n \frac{V_{fn}V_{ni}}{E_i-E_n+i\epsilon}\,\delta(E_f-E_i)$$

We can think of this perturbative expansion graphically



1st order



2nd order

For each interaction vertex we get a factor "V_{ni}"

the intermediate state "n" is represented by

a "propagator" $\dfrac{i}{E_i-E_n+i\epsilon}$

the intermediate state is "virtual" (does not conserve energy since $E_i \neq E_n$). But Energy conservation in the full process is guaranteed by $\delta(E_i-E_f)$

2 Interaction of e^- with em field

So far we have done non-relativistic perturbation expansion. However we have been able to write the first order in a covariant form

$$T_{fi} = -i \int d^4x \, \phi_f^*(x) \, V(x) \, \phi_i(x)$$

which is also valid for relativistic systems

So Lets start with electron with 4-momentum p^μ

((described by a 4-component spin

$$\psi(x) = u(p) \, e^{-ipx} \quad p_\mu x^\mu$$

which satisfies $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

In classical em the motion of a particle of charge $-e$ in an electromagnetic field A^μ is obtained with the substitution

$$p^\mu \to p^\mu + eA^\mu$$

the QM version

$$P^\mu \to P^\mu + eA^\mu$$

$$i\partial^\mu \to i\partial^\mu + eA^\mu$$

)

so in presence of em field the Dirac Eq is

$$i\gamma^\mu \partial_\mu \Psi - m\Psi = -e\gamma_\mu A_\mu \Psi \equiv \gamma^0 V \Psi$$

with $V \equiv -e\gamma^0\gamma^\mu A_\mu$

the γ^0 is introduced so the $E \Rightarrow E + V$

Using this potential we get at 1st order in perturbation theory

$$T_{fi} = ie\int d^4x \underbrace{\Psi_f^+(x)\,\gamma^0}_{\bar\Psi_f}\gamma_\mu A(x)\,\Psi_i(x) \equiv -i\int j_\mu^{fi}(x) A^\mu(x)\, d^4x$$

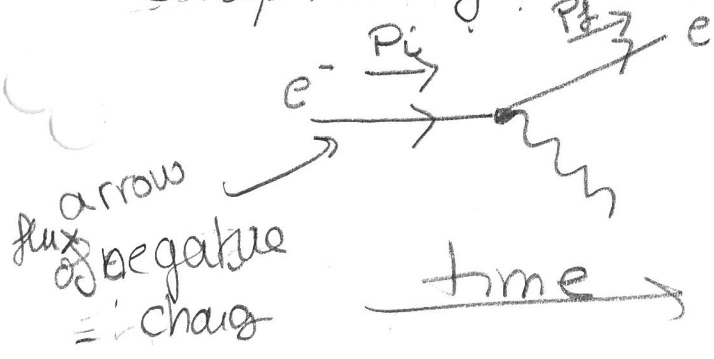
$$\text{with } j_\mu^{fi}(x) \equiv -e\,\bar\Psi_f\gamma_\mu\Psi_i = -e\,\underset{\bar u_f}{\bar u(p_f)}\,\gamma_\mu\,\underset{u_i}{u(p_i)}\,e^{i(p_f-p_i)x}$$

is the em current for the electron $i \to f$ transition

(notation $\gamma^\mu v_\mu \equiv \cancel{v}$)

$$\Rightarrow T_{fi} = \int d^4x\,[\underbrace{ie\,\bar u_f \cancel{A} u_i}]\,e^{i(p_f-p_i)x}$$

Graphically we can represent this process



and we can write T_{fi} in momentum space

by writing the factor

		factor
- incoming e^- with momentum	p_i	$u_i(p_i)$
- outgoing e^- " "	p_f	$\bar{u}_f(p_f)$
- interaction vertex		$ie\gamma^\mu$

and writing these factor from right to left following the arrow representing the flux of negative charge

For a position we can write the corresponding eq for spinor ψ^c with charge $+e$

$$i\gamma_\mu \partial^\mu \psi^c - m\psi^c = e\gamma_\mu A^\mu \psi^c = -\gamma^0 V \psi$$

$$= [\gamma_\mu (i\partial^\mu - eA^\mu) - m]\psi^c = 0$$

there must be a one to one correspondance between ψ and ψ^c because they both represent the e^- and e^+

If we complex conjugate the eq for ψ

$$[\gamma^\mu (i\partial_\mu + eA_\mu) - m]^* \psi = 0$$

$$[-\gamma^{\mu *}(i\partial_\mu - eA_\mu) - m]\psi^* = 0 \qquad \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu +}$$

$$\gamma^0\gamma^{\mu T}\gamma^0 = \gamma^{\mu *}$$

$$[-\gamma^0\gamma^{\mu T}(i\partial_\mu - eA_\mu) - m]\underbrace{\gamma^0\psi^*}_{\bar{\psi}^T} = 0$$

notice that $C = i\gamma^2\gamma^0$ verify (check)

$$-C\gamma^0\gamma^{\mu T} = \gamma^\mu C \quad \text{for } \mu = 0, 1, 2, 3$$

So we get the positron eq with

$$\psi^c = C\,\bar{\psi}^T \qquad \left[\, C = -C^{-1} = -C^{*T} = -C^T \,\right]$$

$$\Rightarrow \bar{\psi}^c = -\psi^T C^{-1} = \psi^T C$$

So to get the amplitude for a positron we can follow the same procedure

current will be

$$j^\mu_c = -e\,\bar{\psi}^c_f \gamma^\mu \psi^c_i = e\,\psi_f^T \overbrace{C^{-1}\gamma^\mu C}^{-\gamma^{\mu T}} \bar{\psi}_i^T$$

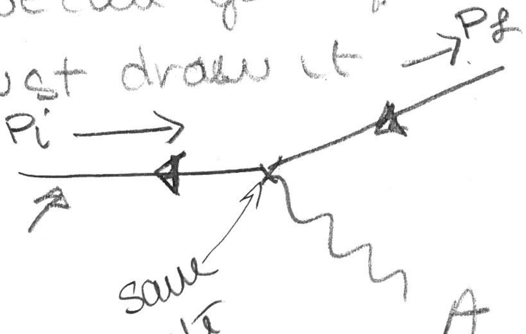
from anticommuting fermion

$$= -e\,\psi_f^T \gamma^{\mu T} \bar{\psi}_i^T = (-)(-)\ \bar{\psi}_i \gamma^\mu \psi_f$$

So for the antifermion spinors $\psi = v(p)\, e^{ipx}$

$$T_{fi} = -i\int j^c_\mu A^\mu = \int d^4x\,(-)\left[ie\,\bar{v}_i \not{A}\, v_f\right] e^{i(p_f - p_i)x}$$

and we can get it from a diagram but now we must draw it



arrow of flux of negative cha

same vertex because

γ is defined the same sign wrt the e^- case

factor

incoming e^+ $\quad \bar{v}(p_i)$

outgoing e^+ $\quad v(p_f)$

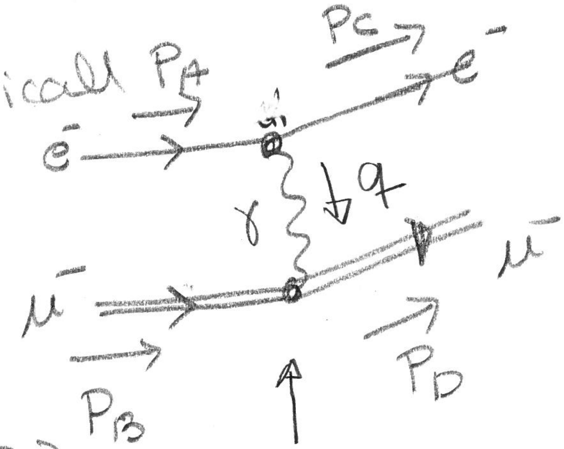
same vertex $\quad ie\gamma^\mu$

but a relative $(-)$

3) $e^- \mu^- \to e^- \mu^-$ scattering ; propagators

Using what we derive for e^- int with and em field let us now assume that the em encountered by the e^- is due to an μ^-

Using Maxwells eq . Graphically



$$\partial^\nu \partial_\nu A^\alpha = j^\alpha_{muon}$$

We call this graphic rep a Feynman Diagram (more later)

$$j^\alpha_{muon} = -e\, \bar{u}_{D,m} \gamma^\alpha u_{B,m}\, e^{i\underbrace{(P_D - P_B)}_{q} x}$$

Since $\partial^\mu \partial_\mu e^{iqx} = -q^2 e^{iqx}$

the solution is $A^\alpha = -\frac{1}{q^2} j^\alpha_{muon}$

So $T_{fi} = -i \int j^\alpha_{ele} \left(-\frac{1}{q^2}\right) j_{\alpha,muon}\, d^4x$

$$= \int (ie)\, \bar{u}_{Ce} \gamma^\alpha u_{Ae} \frac{-i g_{\alpha\beta}}{q^2} (ie)\, \bar{u}_{D,m} \gamma^\beta u_{B,m}$$

$$\underbrace{\int d^4x\, e^{-i(P_A + P_B - P_C - P_D)}}_{= (2\pi)^4 \delta^4 (P_A + P_B - P_C - P_D)}$$

We define the Feynman amplitud iM_{fi}

so that for any process init $\longrightarrow$ final

$$T_{fi} \equiv (2\pi)^4 \delta^4(P_{init} - P_{fin}) \; (iM_{fi})$$

So for $\underset{P_A}{e^-}\underset{P_B}{\mu^-} \rightarrow \underset{P_C}{e^-}\underset{P_D}{\mu^-}$

$$ \; s_a \; s_b \qquad s_c \; s_d$$

$$iM_{fi}^{s_a s_b} = (ie)\left(\bar u^{s_c}_{C,e}\gamma^\mu u^{s_a}_{A,e}\right)\left[\frac{-ig_{\mu\nu}}{q^2}\right](ie)\left(\bar u^{s_d}_{D,m}\gamma^\nu u^{s_b}_{B,m}\right)$$

$$q^2 \equiv (P_A - P_C)^2 = (P_B - P_D)^2$$

$\dfrac{-ig_{\mu\nu}}{q^2}$ $\equiv$ propagator for a "virtual" photon

It is clearly virtual because $q^2 \neq 0$

So we could have infered T_{fi} from the diagram we wrote with the aditional rule that for

$\longleftarrow$ different index in the

we must include a factor $\dfrac{-ig_{\alpha\beta}}{q^2}$

4 Propagators

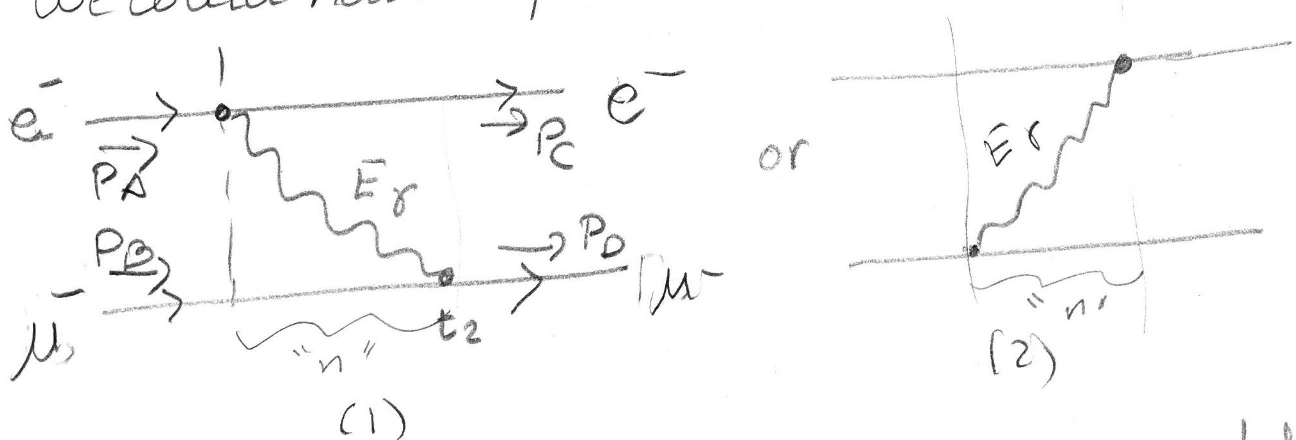
In non-relativistic perturbation theory we write

$$T_{fi} = +2\pi\,\delta(E_f-E_i)\left[(-i)V_{fi} + (-i)^2\sum_n V_{fn}\,\frac{1}{E_i-E_n}\,V_{ni}\right]$$

and $\dfrac{1}{E_i-E_n}$ is called the propagator in QM.

For QED we wrote for the photon $\dfrac{1}{q^2}$, why?

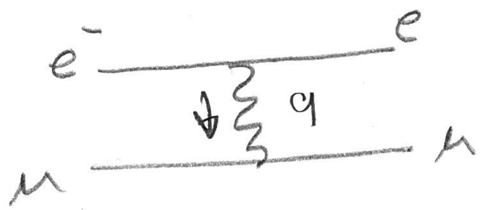
When we do QFT a Feynman diagram represents all "time ordered" diagrams with same particle contents. This means that for example in $\mu^- e^- \to \mu^- e^-$ we could have 2 possible time sequences



$\xrightarrow[t]{\qquad}$ in this class time left to right.

and both contributions are possible: one with positive energy propagates forward in time and the other with negative energy (≡ antiparticle which for the photon is same) backward in time

So when we write



we are adding both

Following with non-relativistic perturbation theory

In (1) $E_{n_1} = E_\gamma + E_C + E_B$ $\quad E_i = E_A + E_B = E_C + E_D$

(2) $E_{n_2} = E_\gamma + E_A + E_D$

$$\text{So } \frac{1}{E_i - E_{n_1}} + \frac{1}{E_i - E_{n_2}} = \frac{1}{E_A - E_C - E_\gamma} + \frac{1}{-(E_A - E_C) - E_\gamma}$$

$$= -\frac{2E_\gamma}{(E_A - E_C)^2 - E_\gamma^2}$$

$\xrightarrow{}$ related with state normalization

Now in non-relativistic pert theory 3-momentum is conserved at vertices but not energy so

$$E_\gamma^2 = |\vec{p}_\gamma|^2 + m_\gamma^2 = |\vec{p}_A - \vec{p}_C|^2 + m_\gamma^2$$

$$(E_A - E_C)^2 = (\underset{\text{4-moment}}{p_A} - p_C)^2 + |\vec{p}_A - \vec{p}_C|^2$$

$$= q^2 + E_\gamma^2 - (m_\gamma^2)$$

$$\Rightarrow \frac{1}{(E_A - E_C)^2 - E_\gamma^2} = \frac{1}{q^2 - (m_\gamma^2)}$$

So the propagators of any particles ...

is of the form $\sim \dfrac{1}{q^2-m^2}$ $q \equiv$ 4-momentum in propag obtained from E-mass conservation in vertex $q^2 \neq m^2$

the numerator is different for different particles.

We notice that in QM we saw

$$T_{fi}^{(2)} = -2\pi i\,\delta(E_I - E_i) \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni}$$

$V_{fn} \equiv \langle f|V|n\rangle$, $V_{ni} \equiv \langle n|V|i\rangle$

with $H_0|n\rangle = E_n|n\rangle$ and since $\sum_n |n\rangle\langle n| \equiv I$

we can write this formally as

$$= 2\pi\,\delta(E_i - E_n)\,\langle f|(-iV)\frac{i}{E_i - H_0}(-iV)|i\rangle$$

So now the popagator $\dfrac{i}{E_i - H_0}$ is the inverse ~~inversed~~ op

of the Schrodinger Eq $-i(E-H_0)\psi = -iV\psi$

So all we need to do is to find the equ. operators for our particles

Let us find them. We will work for states with well defined 4-momentum.

- For fermions the Dirac Eq is

$$\gamma E \quad (\not p - m)\, u(p) = \underbrace{-e\gamma^\mu A_\mu}_{V}\, u(p) \qquad \times(-i) \text{ to have } -iV\gamma^0$$

$\gamma^0 V$

$$\underbrace{iS_F(p)}_{\text{electron propag}} = \frac{1}{-i(\not p - m)} = \frac{i(\not p + m)}{(\not p - m)(\not p + m)} = \frac{i(\not p + m)}{p^2 - m^2}$$

Graphically 

P in same direction

arrow of e^- charge

which has in the numerator $\not p + m = \sum_{\text{spins}} u(p)\bar u(p)$

so all virtual spin states propagate

- For scalar particle the K-G eq for plane wa

$$(-q^2 + m^2)\,\phi(q) = -V\,\phi(q) \qquad \times\ i$$

$$\underbrace{i\Delta(q)}_{\text{Scalar prop}} = \frac{1}{i(-q^2 + m^2)} = \frac{i}{q^2 - m^2}$$

 not in QED so far

• For photon: we have to work in some gauge.

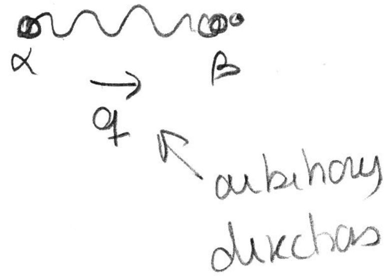
In the Lorenz gauge the Eq

$$g^{\mu\lambda}\partial^\alpha\partial_\alpha A_\lambda = j^\mu \to g^{\mu\lambda}(-q^2) A_\lambda(q) = j^\mu(q)$$

since $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$ we can invert $\Rightarrow (g_{\mu\nu})^{-1} =$ " "

$= g^{\nu\mu} = g^{\mu\nu}$

photon propag $\equiv i D^{\mu\nu}_{(q)} = \dfrac{-i\, g^{\mu\nu}}{q^2}$



Notice that we found that

$$-g^{\mu\nu} = \sum_{r=0}^{3} \varepsilon_r\, \epsilon_r^{*\mu}(p)\, \epsilon_r^{\nu}(p)$$

↑ summed over both physical and unphysical polarizations. All "propagate" in these "virtual" photons

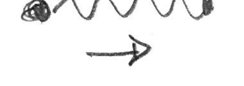
• Massive vector:

Procca Eq $[g^{\mu\lambda}(\partial^\alpha\partial_\alpha + M^2) - \partial^\mu\partial^\lambda] B_\lambda = j^\mu$

(free)

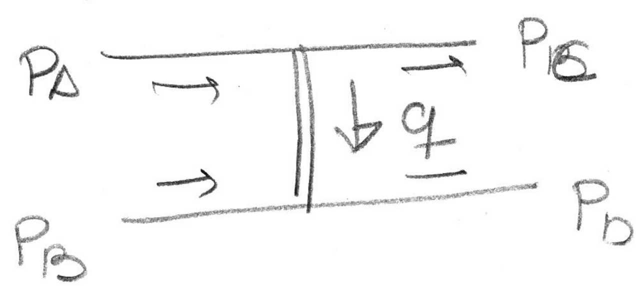
In momentum space $(g^{\mu\lambda}[-q^2+M^2] + q^\mu q^\lambda)$

So $\underbrace{i D_V^{\mu\nu}(q)}_{\text{Propag}} = \dfrac{i(-g^{\mu\nu} + q^\mu q^\nu/M^2)}{q^2 - M^2}$



So far we have parked the problem of poles at $E_n - E_i$. To solve them we add $+i\epsilon$ in the denominator of propagators

Notice that for any propagator in diagram like



energy momentum conservation

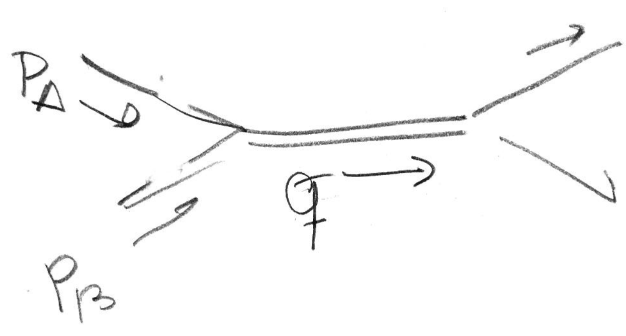
$$q^2 = (P_C - P_A)^2 < 0$$

so it is never possible

$$q^2 = m^2$$

so the $+i\epsilon$ prescription is irrelevant.

But for diagram with propagator



$$q^2 = (P_A + P_B)^2 > 0$$

$$\Rightarrow q^2 = m^2 \text{ is possible}$$

In QED since $m_\gamma = 0$ this is not possible but we will go back to this latter.

5 Feynman rules for QED

For any process $|I\rangle \rightarrow |F\rangle$ we can obtain the amplitude

convention depending book

$$T_{fi} \equiv (2\pi)^4 \delta^4(P_I^{TOT} - P_F^{TOT}) \, (-) i \, \mathcal{M}_{FI}$$

where $\mathcal{M}_{FI} = \sum_n \mathcal{M}_{FI}^{(n)}$

$\mathcal{M}_{FI}^{(n)}$ can be obtained from all diagrams connecting $|I\rangle$ to $|F\rangle$ containing "n" vertices. The $\mathcal{M}_{FI}^{(n)}$ of each diagram is obtained following these rules

same flavor

(0) Assign arrows to all fermion lines following the flux of negative charge
- Label all external 4-momentum and helicities.
- For fermion prop the momenta must follow the arrow of neg charge
- For photon prop the momentum direction is arbitrary
- Obtain the value of all momentum in propag. by energy-momentum conservation in vertex
- Assign a Lorentz index to each vertex

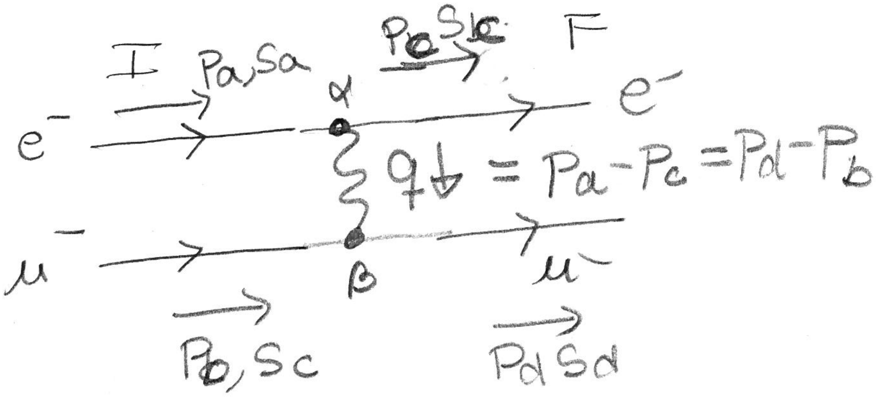
take $e^- \; \mu^- \longrightarrow e^- \; \mu^-$

$p_a \quad p_b \qquad p_c \quad p_d$ 4-momen)

$s_a \quad s_b \qquad s_c \quad s_d$ helich

time $\longrightarrow$

Rule (0)



$$\Rightarrow q^2 = (p_a - p_c)^2 = (p_d - p_b)^2 \equiv t$$

(more latter)

Factor

1) For each vertex ~~~α (f, f) $\qquad -ieQ_f \gamma^\alpha$

$Q_f = (-1)$ for $f = e^\pm$

2) For each external leg

incoming f $\qquad u^s(p)$

outgoing f $\qquad \bar{u}^s(p)$

incomin $\bar{f}$ $\qquad \bar{v}^s(p)$

outgoing $\bar{f}$ $\qquad v^s(p)$

incoming γ $\qquad [\epsilon_\lambda(p)]_\alpha$ ← Lorentz index; λ pol. index

outgoing γ (phot) $\qquad [\epsilon^*_\lambda(p)]_\alpha$

3) Internal fermion lin

q in same direction

$$\frac{i(\not{q}+m)}{q^2 - m^2 + i\epsilon}$$

4) Internal photon $\alpha \sim\sim \beta$

$$\frac{-ig^{\alpha\beta}}{q^2 + i\epsilon}$$

5) For each fermion line in the diagram write these factors from right to left following the arrow of charge to get iM_{FT}

$$iM_{e\bar{\mu}\to e\bar{\mu}^-} = \left[\bar{u}^{s_c}_{elec}(p_c)\,(ie\gamma^\alpha)\,u^{s_a}_{elec}(p_a)\right]\frac{-ig_{\alpha\beta}}{(p_A-p_c)^2}\left[\bar{u}^{s_d}_{muon}(p_d)(ie\gamma^\beta)u^{s_b}_{(p_b)}\right]$$

$$\Rightarrow M_{e\bar{\mu}\to e\bar{\mu}^-} = \frac{-e^2}{(p_a-p_c)^2+i\epsilon}\left[\bar{u}^e_C\gamma^\alpha u^e_D\right]\left[\bar{u}^{muon}_D\gamma_\alpha u^{mun}_B\right]$$

(6) Diagrams differing in exchange of

- identical initial fermions
- " final "
- " initial anti-ferms
- " final "
- initial fermion ⟷ final antifermion
- initial anti " ⟷ final fermion

relative (−) sign.

(7) For each loop in the diagram $\int \frac{d^4k}{(2\pi)^4}$ factor

undetermined 4-momentum in loop (more later)

(8) For each fermion loop (−) Tr [propag in loop)]