# Chapter 8

## Strong interactions: Quantum Chromodynamics (QCD)

1) Evidence of 3 coloured quark
2) QCD: Lagrangian and Feynman rules
3) Some tests: asymptotic freedom and confinement
   - Running of $\alpha_s$: asymptotic freedom
   - $e^+e^- \to$ jets
4) $q\bar{q}$ interaction in QCD: bound states
5) SU(3) FLAVORS and Spectrum of light hadrons

# 1 Evidence of 3 colours of quarks

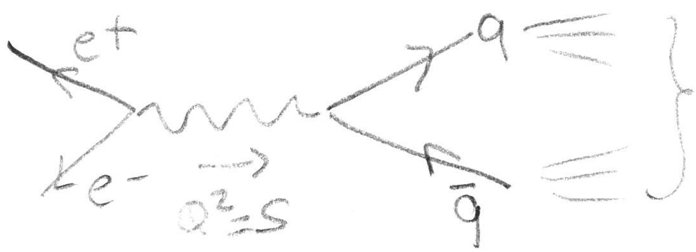
We have seen in chapter 7 that at large $Q^2$

$$e^- p \to e^- \underset{\text{hadrons}}{X}$$

could be understood as $e^- q \to e^- q$

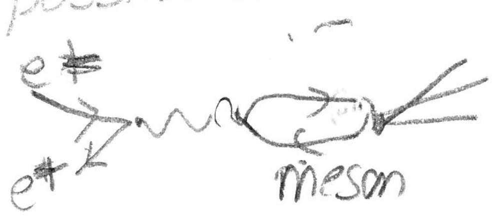
Equivalently $e^+e^- \to$ hadrons at large $Q^2$

can be understood as : $e^+e^- \to q\bar{q}$




or if $s^2 = m^2_{meson} \equiv (q\bar{q})$ bound state : $\rho, \omega, \phi, \psi$

it is possible also




in this case there is a resonant peak

In QED

$$\sigma(e^+e^- \to \mu^+\mu^-)(s) = \frac{4\pi\alpha^2}{3s} \qquad (s \gg 4m_\mu^2)$$

so in this picture

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{3s} \underset{\substack{\uparrow \\ \#\text{colours of quark}}}{N_c} \sum_{\text{quark}} \underset{\substack{\uparrow \\ \text{charge of quark}}}{e_q^2} \qquad \text{for } (4m_q^2 \ll s)$$

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})(s)}{\sigma(e^+e^- \to \mu^+\mu^-)(s)} = N_c \sum_{\substack{\text{quark} \\ \text{with } 4m_q^2 \ll s}} e_q^2.$$

So using $m_u, m_d, m_s \ll 100$ MeV , $m_c \sim$ GeV , $m_b \sim 5$ GeV
$m_t \sim 175$ GeV

- For $\sqrt{s} \ll 3$ GeV $q = u, d, s$ $R = N_c\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = \frac{2}{3} N_c$
- For $3\text{GeV} \ll \sqrt{s} \ll 10$ GeV $q = u, d, s, c$ $R = \frac{2}{3} N_c + \left(\frac{2}{3}\right)^2 N_c = \frac{10}{9} N_c$
- For $10\text{GeV} \ll \sqrt{s} \leq 300$ GeV $q = u, d, s, c, b$: $R = \frac{10}{9} N_c + \frac{1}{9} N_c = \frac{11}{9} N_c$

Figure : From data $N_c = 3$.

the constituent quark model leaves unanswered the question of why quarks are never seen as free states

It also does not explain why quarks behave as free at high-$Q^2$ collisions (so we can make such full predictions for $e^- p \to e^- X$ or $e^+e^- \to q\bar{q}$ treating them as we do a $\mu^-$.

The theory of strong interactions must explain these

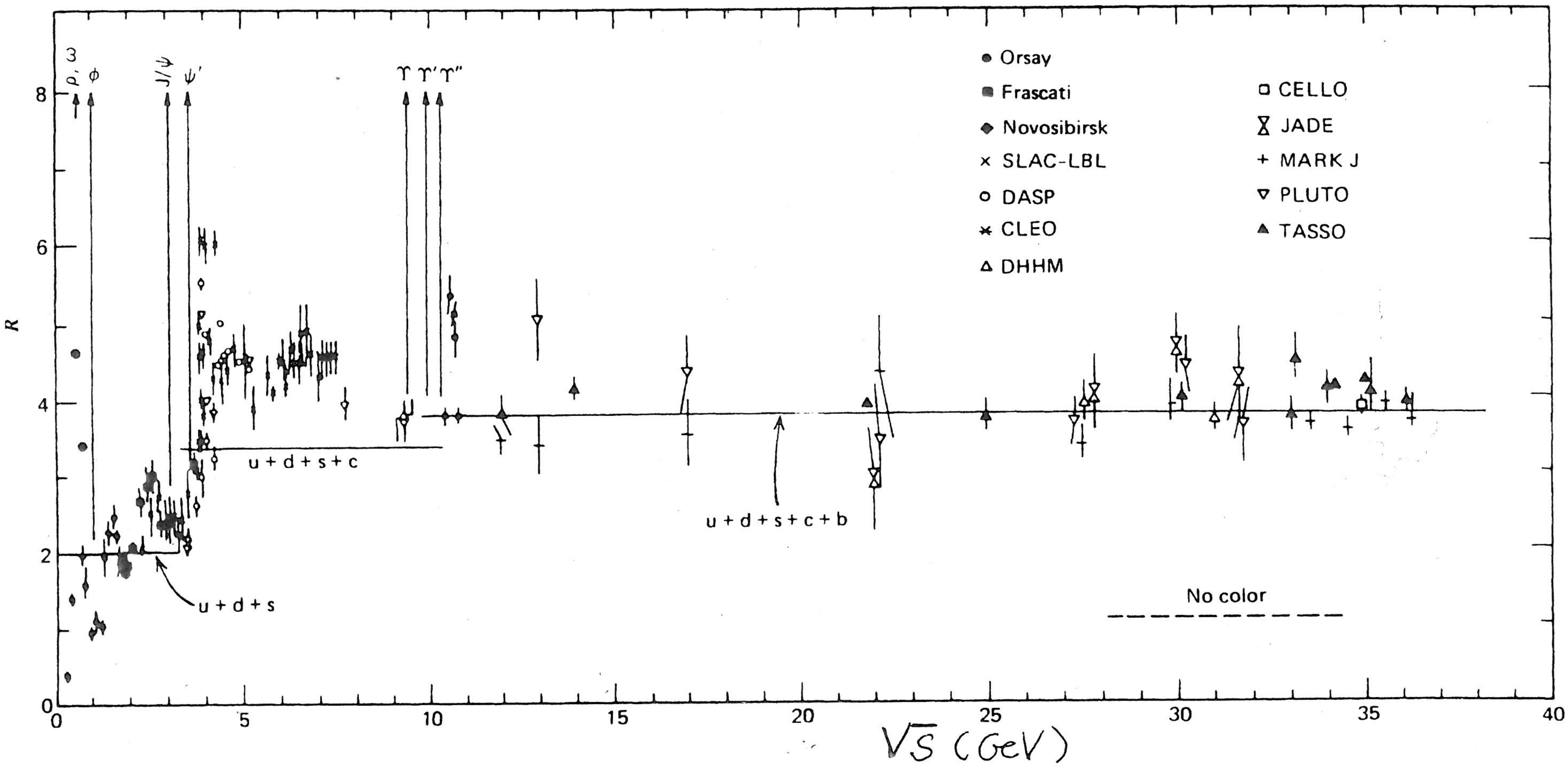



Fig. 11.3 Ratio $R$ of (11.6) as a function of the total $e^-e^+$ center-of-mass energy. (The sharp peaks correspond to the production of narrow $1^-$ resonances just below or near the flavor thresholds.)

# 2) Quantum chromodynamics

In principle there is no reason why the answer to these two questions have anything to do with the colour quantum # of the quarks. But as we are going to see they can be explained if we assume that these facts are due to colour because colour is "the charge" responsible of strong interactions.
We call Quantum Chromodynamics (QCD) the gauge theory implementing this idea. Its basic assumptions:

- each quark has a colour quantum # in 3 possible states which we call r, g, b ≡ red, green, blue.

- to specify a quark's state besides the spinor we need also a vector $C^{(i)}$ to specify its colour state which will be a linear combination of the basics $C_1 \equiv r \equiv \begin{pmatrix}1\\0\\0\end{pmatrix}$, $C_2 \equiv b \equiv \begin{pmatrix}0\\1\\0\end{pmatrix}$ $C_3 \equiv g = \begin{pmatrix}0\\0\\1\end{pmatrix}$

So we represent a fermion wave function

$$q_i(x) \equiv \psi_{q_i}(x) = \underset{\uparrow \text{ spinor part}}{\psi_q(x)} \; C^{(i)} \leftarrow \text{colour}$$

(corresponding 3 anticolours $\bar{r}, \bar{g}, \bar{b}$ for antiquarks)

- colour is the "charge" responsible of strong interaction and QCD is its gauge theory with gauge group

$SU(3)_c$

[ $SU(3) \equiv$ dimension 8 non-abelian group

Its algebra $[T^a, T^b] = i f_{abc} T^c$ ]

$T^a, T^b$: generators of group $a, b = 1 \ldots 8$

$f_{abc}$: structure constants

$f_{abc}$ are totally antisymmetric $\Rightarrow f_{abc} = -f_{bac} = -f_{cba}$ (*)

$f_{123} = 1$, $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$

$f_{458} = f_{678} = \frac{\sqrt{3}}{2}$ ] all other not related to these by (*) are zero

So $(r, g, b)$ are a basis for the triplet representation of $SU(3)_c$. In this representation

$$T_a = \frac{\lambda_a}{2} \leftarrow \text{Gellmann matrices}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Impossing gauge invariance $\Rightarrow$

$$\sum_j (D^\mu_{ij}\, q_j)' = \sum_{k,j} \left(e^{i\sum_a \alpha_a \frac{\lambda^a}{2}}\right)_{ij} (D^\mu)_{kj}\, q_j$$

after some algebra

$$\Rightarrow G^{a\prime}_\mu = \underbrace{G^a_\mu - \frac{1}{g_s}\partial_\alpha \alpha_\mu^a}_{\text{same as QED}} \underbrace{- \sum_{b,c} f^{abc} \alpha^b G^c_\mu}_{\text{new due to } f^{abc} \neq 0}$$

We factorize also the colour index of the gluon field $G^a_\mu \equiv G_\mu(x)\, a^{(a)}$ ← vector in 8 dimensional space

So in QCD there is an additional pice in the gauge transf of the gluon wrt photons of QED because SU(3) in non-abelian (new piece $\propto f_{abc} \neq 0$)

Next we need to add the Lag for the gluons

In QED $\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

In QCD

$$\mathcal{L}_G = -\frac{1}{4}\sum_a G^{(a)}_{\mu\nu} G^{\mu\nu(a)} \quad \text{with } G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu \underbrace{- i g_s \sum_{bc} f_{abc} G^b_\mu G^c_\nu}_{\text{new wrt}}$$

under an $SU(3)_c$ transf.

$$q_i(x) \to q_i'(x) = \underbrace{\Big[ e^{i \overset{\downarrow\ \text{8 parameters of transf.}}{\alpha_a(x)} \overset{\swarrow\ 3\times3\ \text{matrix}}{\frac{\lambda^a}{2}}} \Big]_{ij}}_{3\times3\ \text{matrix}} q_j(x)$$

- the coupling constant is $g_s$

We can build a Lagrangian for $q$ which is invariant under $SU(3)_c$ in analogy of what we did for $e^-$ in QED with group: $U(1)$

$$\mathcal{L} = \sum_{ij} \bar{q}_i \left[ i\gamma^\mu D_\mu \right]_{ij} q_j - \sum_q m \bar{q}_i q_i$$

$ij \leftarrow$ colour indexes.

where to define $D_\mu$ we need to introduce gauge bosons $G_\mu^{(a)}$ $a = 1 \ldots 8$ which we call gluons. there are 8 because dim of $SU(3)$ is 8.

So

$$(D^\mu)_{ij} q_j = \left[ \partial^\mu \delta_{ij} + i g_s \sum_{a=1}^{8} \frac{(\lambda^a)_{ij}}{2} G_a^\mu \right] q_j$$

New part ⇒ $\mathcal{L}_G$ there are terms like

$$\sum_{abc} f^{abc}\, \partial_\nu G^a_\mu\, G^{\mu,b}\, G^{\nu,c} \longrightarrow \text{3 gluon vertices}$$

$$\sum_{abcde} f^{abc} f^{ade}\, G^b_\mu G^c_\nu\, G^{\mu,d} G^{\nu e} \longrightarrow \text{4 gluon vertices}$$

So unlike in QED gluons 'interact' with themselves.

So with this the FR of QCD are

- External lines $\overrightarrow{p,s,i}$ ← color

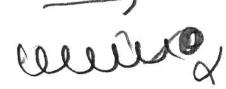
quark incoming  $c^{(i)}\, u^s(p)$

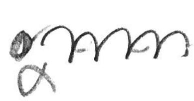
quark outgoing  $\bar{u}^s(p)\, c^{(i)+}$

anti q incoming  $\bar{v}^s(p)\, c^{(i)+}$

anti q outgoing  $c^{(i)}\, v^s(p)$

$\overrightarrow{p,\lambda,a}$

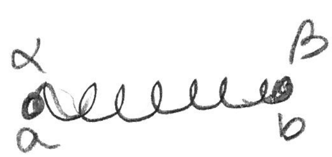
gluon incoming  $\epsilon^\lambda_\alpha(p)\, a^{(a)}$

gluon out  $[\epsilon^\lambda_\alpha(p)]^*\, a^{(a)+}$

gluon represent with a "spring" line

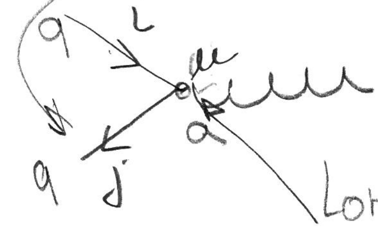
- Propagator $\overrightarrow{p}$

quark  $\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$

gluon  $\dfrac{-i g^{\alpha\beta}\,\delta_{ab}}{p^2+i\epsilon}$

∘ Vertices

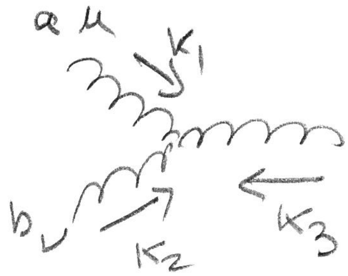
same flavour different colour




Lorentz index $\mu$ and gluon colour index $a$

$$-ig\gamma^\mu \frac{\lambda^a_{ij}}{2}$$

Equiv to QED only with "charge" $\frac{\lambda^a_{ij}}{2}$



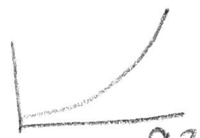

$$g_s f^{abc}\left[(k_1-k_2)_\rho g_{\mu\nu} + (k_2-k_3)_\mu g_{\nu\rho} + (k_3-k_1)_\nu g_{\mu\rho}\right]$$


$$= -ig_s^2 \sum_e \Big[ f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\rho\lambda}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\nu\lambda}) \Big]$$

or equivalently if we measure at some $\mu^2 \gg m_e^2$

$$\alpha(q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \ln \frac{q^2}{\mu^2}}$$

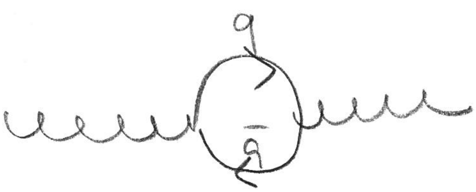
negative $\Longrightarrow$ $\alpha(q^2)$



$q^2$

QED strength grows with $q^2$

For QCD we have to follow a similar procedure but now at one-loop there are additional diagrams due to gluon self-interactions



like QED

but also

3 gluon vertex

4 gluon vertex

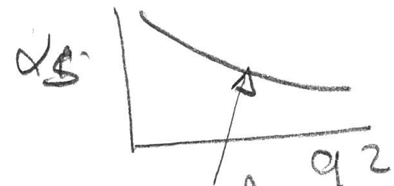
these additional pieces can be computed using the FR of QCD and they lead to a totally different running of $\alpha_s$

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \underbrace{\frac{\alpha_s(\mu^2)}{12\pi}}_{>0} (33 - 2N_f) \ln \frac{q^2}{\mu}}$$

# flavoured quarks with $4m_q^2 \ll q^2$

For any $q^2$ $\underset{u,d,c,s,b,t}{N_f \leq 6}$ $\Rightarrow 3 - 2N_f > 11 > 0$

So unlike QED in QCD




much faster than QED also

- So at $q^2 \to 0$ $\alpha_s \to$ very large
  $\Rightarrow$ confinement of non-relativistic quarks inside the hadrons

  $\Rightarrow$ non-perturbative regime $\equiv$ Lattice QCD or phenomenological hadron models

- At $q^2 \to$ large $\alpha_s \to$ small
  $\Rightarrow$ strong interactions become weak enough $\equiv$ asymptotic freedom

$\Rightarrow$ even if quarks are confined inside the hadrons when we probe them with high $Q^2$ (like in DIS) we can treat them as "free". this is the basics of ~~why~~ the parton model works.

If we define $\Lambda$ as the scale at which $\alpha$ becomes very large

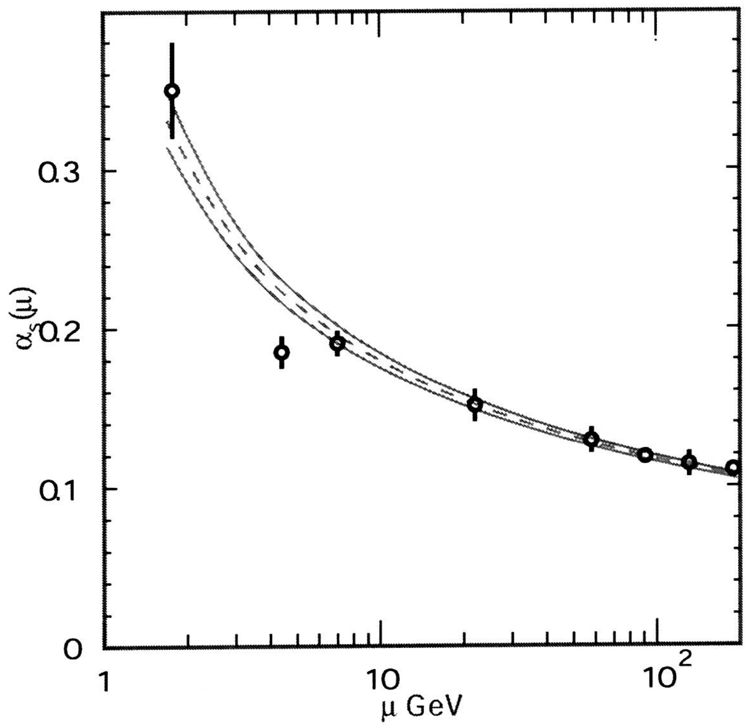
$$0 \simeq \frac{1}{\alpha(\Lambda^2)} = \frac{1}{\alpha(\mu^2)} + \frac{33-2N_f}{12\pi} \ln\frac{\Lambda^2}{\mu^2}$$

$$\Lambda^2 = \mu^2 e^{\left[-\frac{12\pi}{(33-2N_f)\alpha_s(\mu^2)}\right]} \Rightarrow \alpha_s(q^2) \simeq \frac{12\pi}{(33-2N_f)\ln\frac{Q^2}{\Lambda^2}}$$

QCD prediction

$$\alpha_s(q^2) = \frac{\alpha_s(q_0^2)}{1 + \frac{\alpha_s(q_0^2)}{12\pi}(33 - 2N_f)\ln\left(\frac{q^2}{q_0^2}\right)}$$

Comparing to data



Consequences:

- At short distances the strong potential between two quarks separated by $r$ is Coulomb-like

$$\frac{\alpha_S(q^2 \sim r^{-2})}{r}$$

  If we separate them apart $r$ grows, and so does $\alpha$

  $\Rightarrow$ effective strong potential grows (predicted to grow linearly) with the separation

  $\Rightarrow$ quarks cannot escape the potential

  $\Rightarrow$ quarks are *confined* inside the hadrons

- If we insist in breaking the hadron, once the quarks are at sufficient distance from each other the potential energy is huge

  $\Rightarrow$ Energetically favorable to create a $q\bar{q}$ and bind them to the original quarks to form hadrons

  $\Rightarrow$ quarks cannot be observed in isolation

- Conversely going to smaller and smaller distances, or equivalently, to larger and larger energies, the strong coupling constant becomes weak.
  For example at $q^2 = (91\,\text{GeV})^2$, $\alpha_s \simeq 0.12$.

  $\Rightarrow$ perturbation theory can be trusted.

  $\Rightarrow$ quarks behave as free asymptotic states when probed at very high energies.

  We say that QCD is an *asymptotically-free* theory.

  Asymptotic freedom:

  $\Rightarrow$ We can treat quarks as free outgoing fermions when produced in high energy collisions

  $\Rightarrow$ We can treat quarks as free incoming fermions inside the hadrons when we collide the hadrons at very high energies

  $\Rightarrow$ We can use Feynman calculus to compute perturbatively the expectation from QCD

  Ultimately it was the reason why QCD was accepted as the theory of strong interactions which unified the hadronic low energy physics with the very high-energy strong interaction effects.

The strong coupling constant becomes small at high energies

$\Rightarrow$ one can calculate perturbatively the expectations in QCD for quarks and gluons at high energies.

But in the real world the quarks and gluons have to *hadronize* ($\equiv$ become real white hadrons)

$\Rightarrow$ comparison is not straight-forward

The cleanest data to do this comparison is to use data on

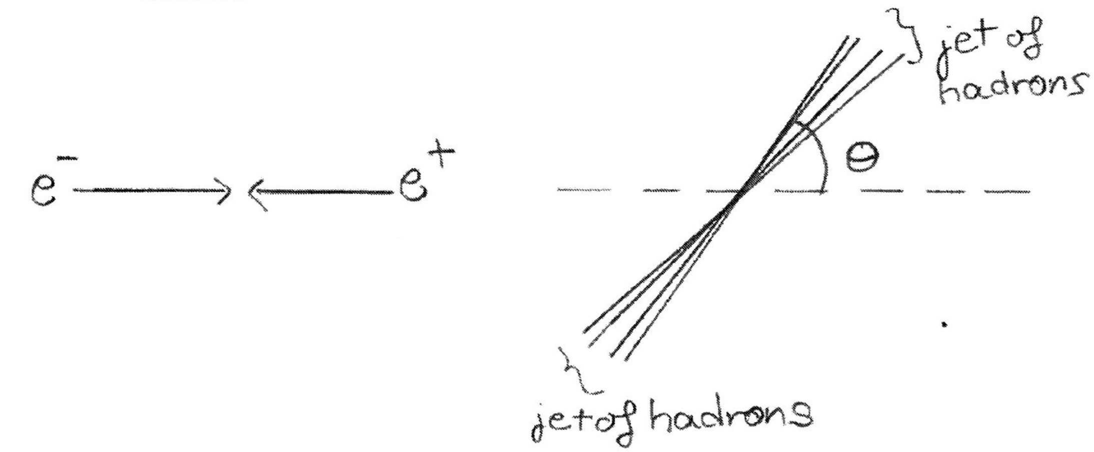
$$e^+e^- \to \text{hadrons} \qquad \text{at large} \qquad s = (p_e^+ + p_e^-)^2$$

and see if it can be understood in terms of

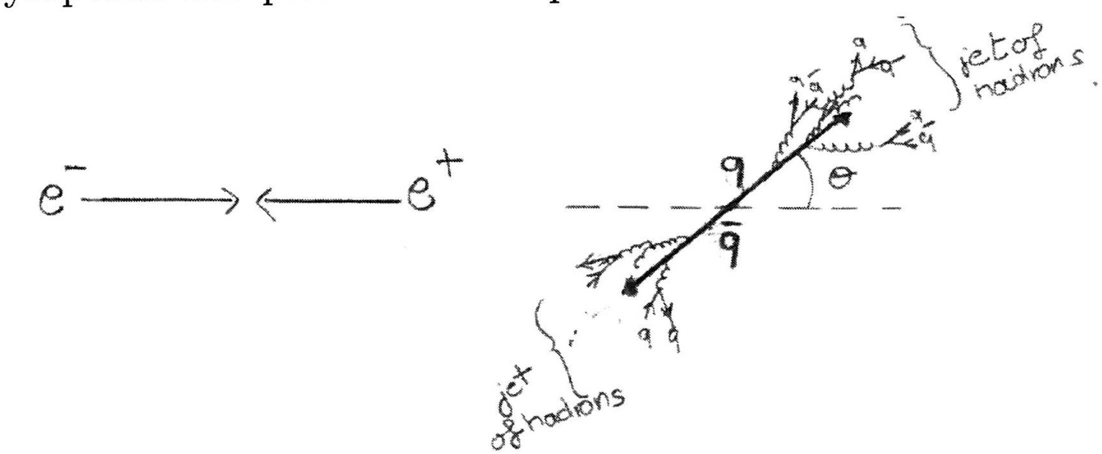
$$e^+e^- \to \text{quarks and gluons}$$

For example we can look at the process with two hadron jets in the final state. In COM

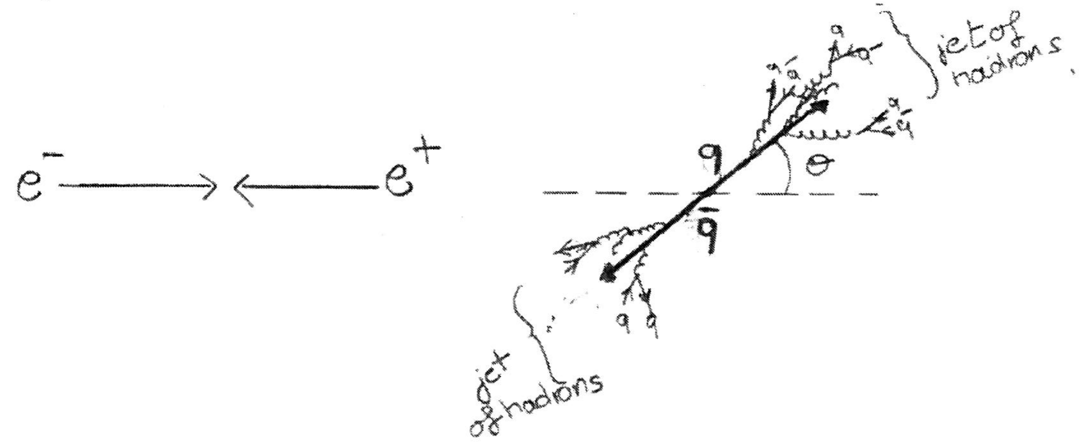
Initial | Final




In QED+QCD the asymptotic free picture for this process is



In QED+QCD the asymptotic free picture for $e^+e^- \to 2$ hadron; jets is




So we predict

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2 \text{ jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}} \times \text{Prob}(\text{final } q \to \text{jet}) \times \text{Prob}(\text{final } \bar{q} \to \text{jet})$$

where the sum extend over all possible quark flavors light enough to be produced

But quarks and gluons always hadronize $\Rightarrow$ Prob(final $q \to$ jet) = Prob(final $\bar{q} \to$ jet) = 1

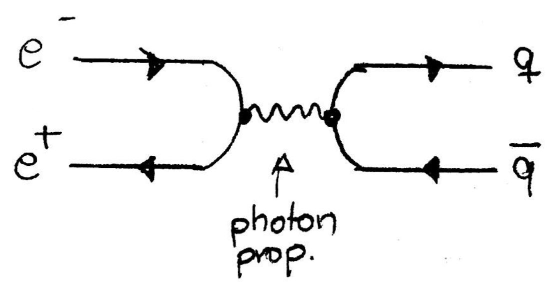
So we predict

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2 \text{ jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}}$$

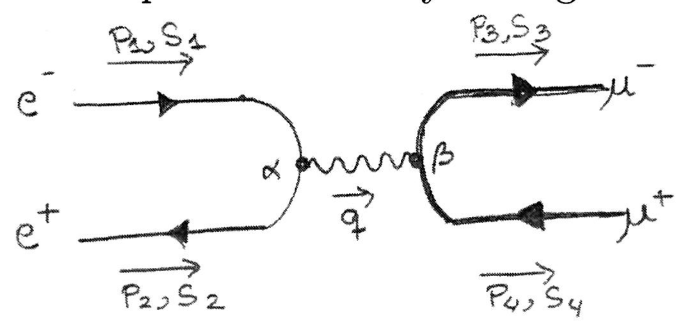
We predict

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2\text{ jets})\bigg|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\bigg|_{\text{COM}}$$

We compute the cross section for $e^+ e^- \to q\,\bar{q}$ from lowest order diagram in QED




If quarks are fermions that amplitude is totally analogous to the one for




$$\frac{d\sigma}{d\Omega}\bigg|_{\text{COM}} = \frac{\alpha^2}{4\,s}\left[1+\cos^2\theta\right]$$

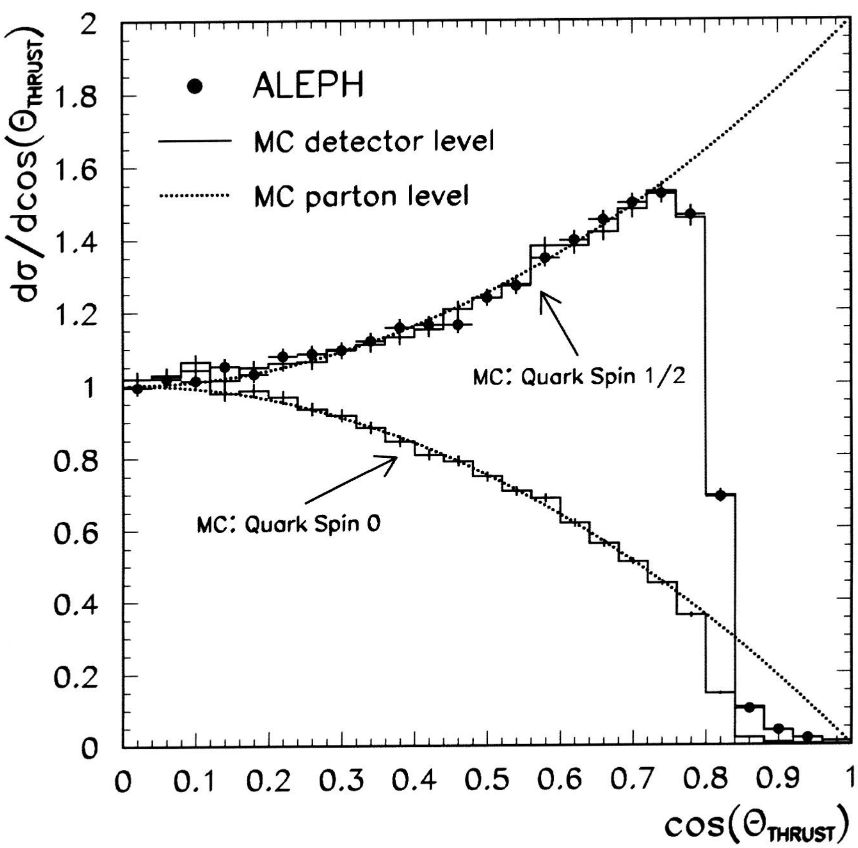
up to the number of colors and the charge of the quarks.

So

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2\text{ jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}} = 3 \times \sum_q Q_q^2 \frac{d\sigma}{d\Omega}(e^+ e^- \to \mu^+ \mu^-)\Big|_{\text{COM}}$$

$$= 3 \times \sum_q Q_q^2 \frac{\alpha^2}{4\,s}\left[1+\cos^2\theta\right]$$

Comparing with data from ALEPH experiment




$\Theta_{\text{Thrust}} \equiv \theta_{\text{jet axis}}$

This confirms

– the picture of asymptotic free quarks

– that quarks are spin 1/2 particles like the muons

if quarks had spin 0

$\Rightarrow$ the amplitude would be proportional to $\sin\theta$

$$\Rightarrow \frac{d\sigma}{d\Omega}(e^+ e^- \to 2\text{ jets})\Big|_{\text{COM}} \propto \sin^2\theta = 1 - \cos^2\theta$$

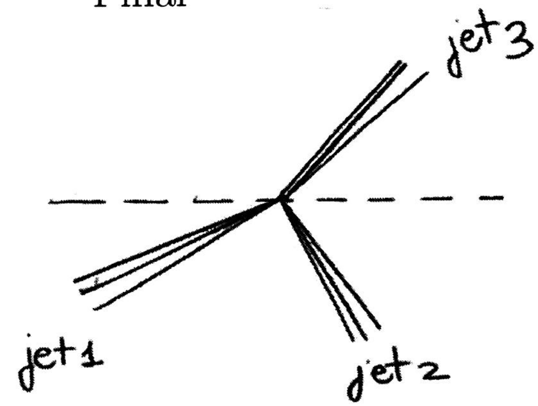
(dash-line)

Next we can look at the process with thee hadron jets in the final state. In COM

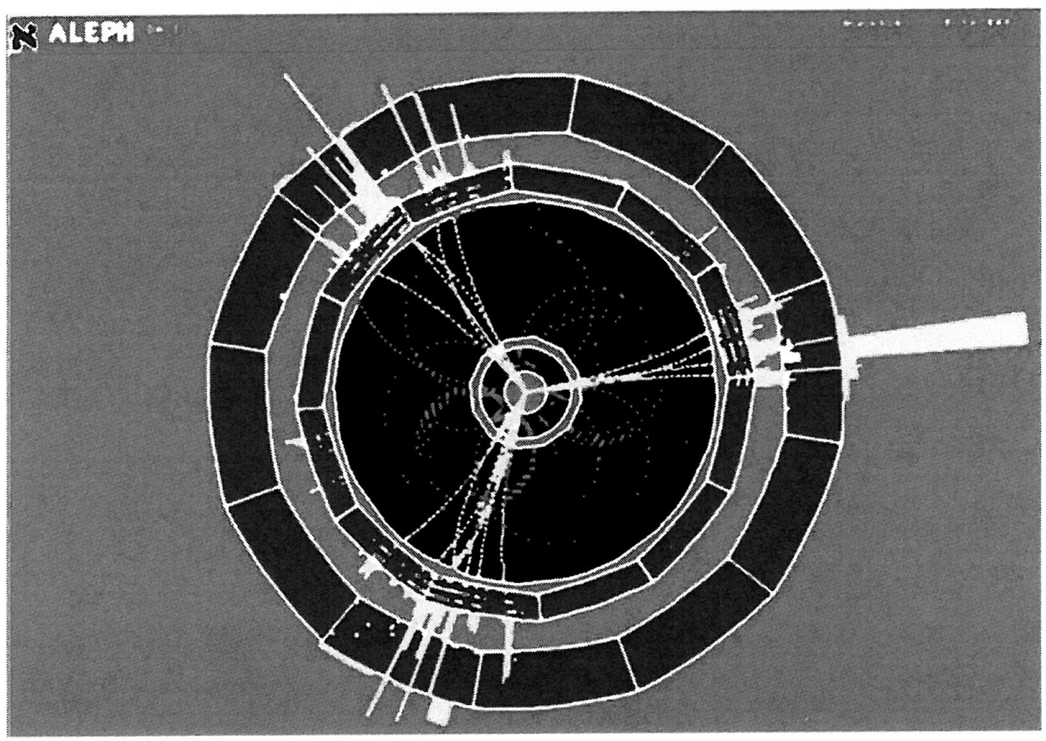
Initial

$e^- \longrightarrow \longleftarrow e^+$

Final

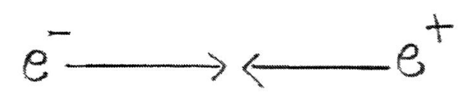


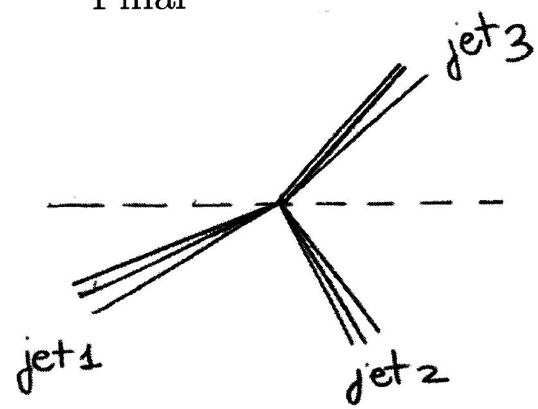

In the experiment



Next we can look at the process with thee hadron jets in the final state. In COM

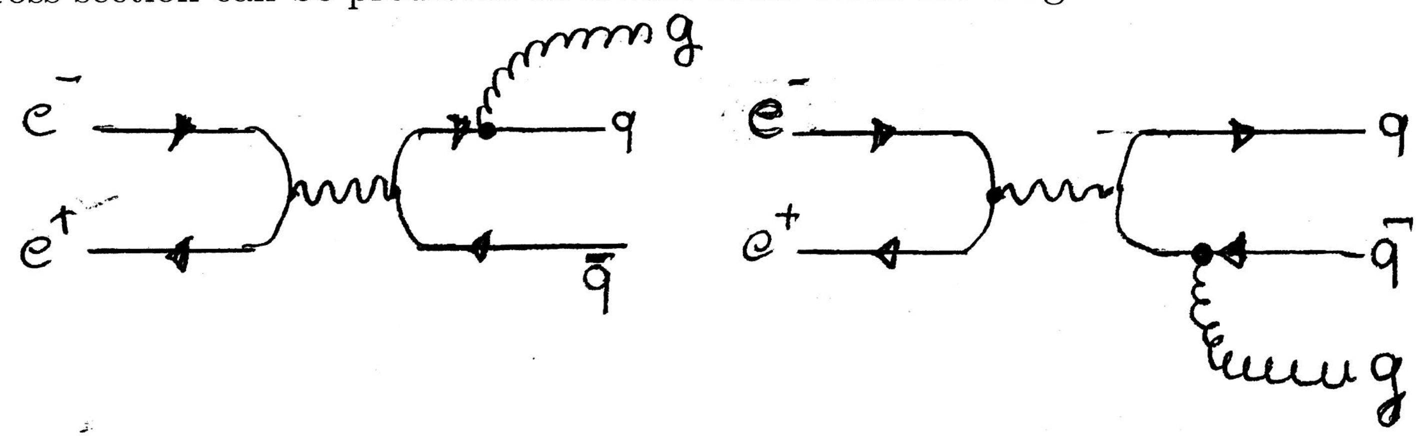
Initial Final




In QED+QCD the asymptotic free picture for this process is that the third jet comes from a gluon

So the cross section can be predicted at lowest order from the diagrams



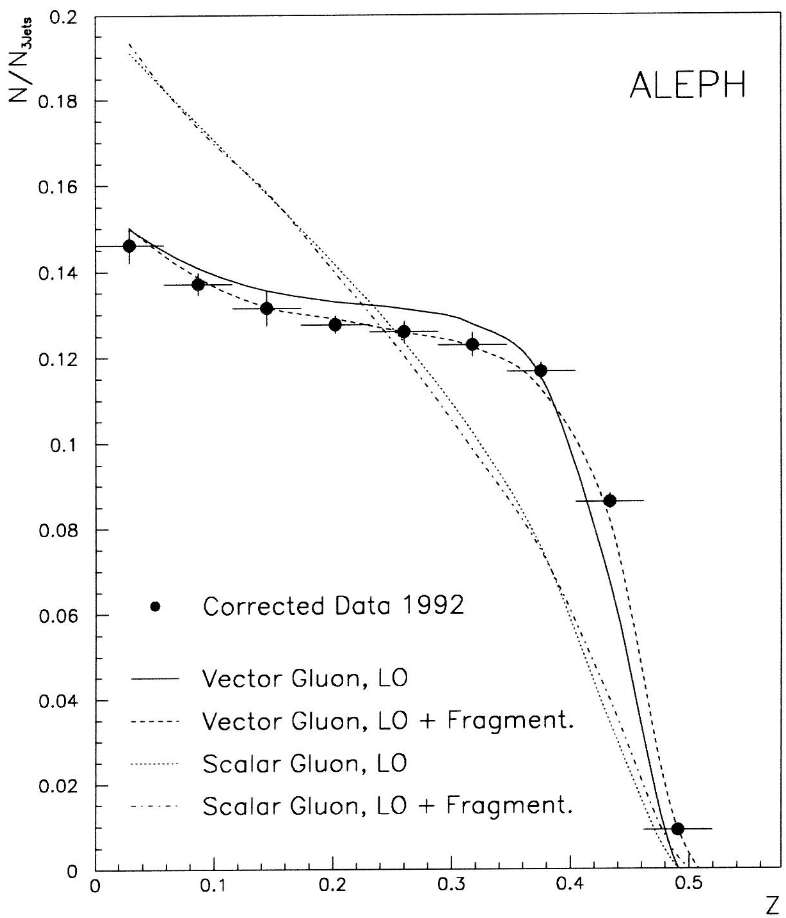

which involve the QCD vertex of quarks to gluons

$\Rightarrow$ comparing to data we can verify that gluons are vector particles and test gluon-$q$-$\bar{q}$ QCD vertex

$e^+e^- \to 3\ jets$ first observed at the PETRA collider in DESY in Hamburg, Germany in the late 70's.

More precise data from ALEPH: Ordering $E_{jet,1} \le E_{jet_2} \le E_{jet_3}$

Study the $Z = \frac{1}{3}\left(\frac{2\,E_{jet_2}}{\sqrt{s}} - \frac{2\,E_{jet_3}}{\sqrt{s}}\right)$ distribution of events compared to QCD prediction

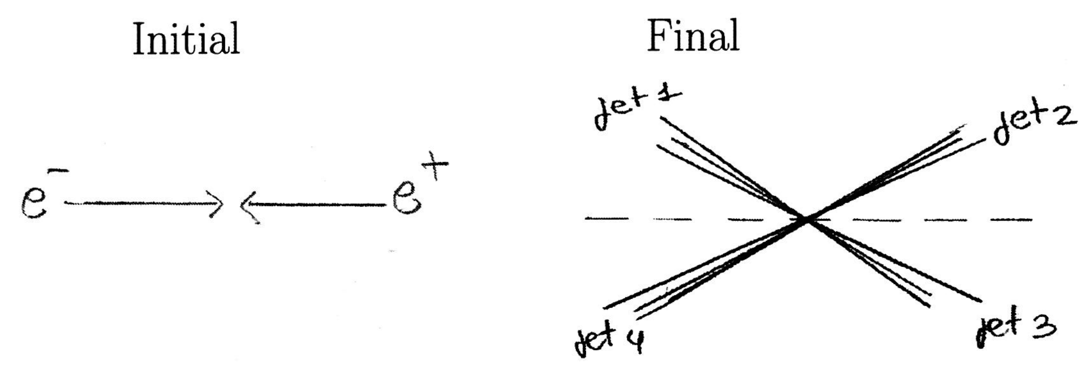



$Z$ distribution would be different if gluon had spin=0

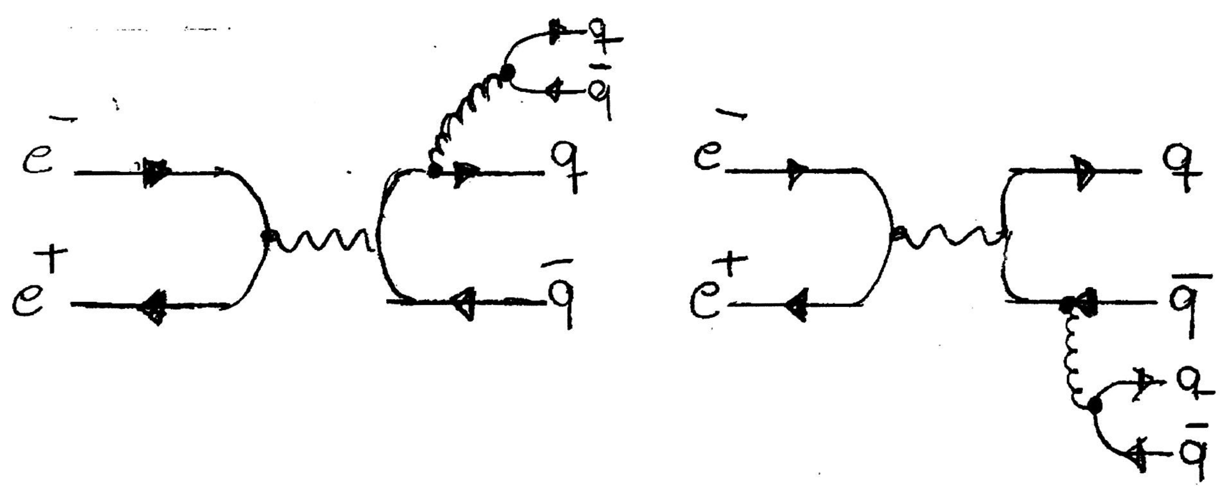
Data agrees with the QCD prediction

of a vector gluon-$q$-$\bar{q}$ coupling.

Next we can look at the process with four hadron jets in the final state. In COM




In QED+QCD the asymptotic free picture for this process the four jets can be either four quarks



For $e^-\mu^+ \to e^-\mu^+$ in QED

$$i\mathcal{M} = \frac{ie^2}{q^2}[\bar{u}_3\gamma^\alpha u_1][\bar{v}_2\gamma_\alpha v_4]$$

same with $g_s \to e$
$f_{col} \to 1$

In QED at low energies the Coulomb potential

between $e^-$ and $\mu^+$ is

negative $\Rightarrow$ attractive potential

$$V^{em}_{(r)} = -\frac{\alpha}{r}$$

From above comparison we expect

depends on r because

$$V^{strong}_{(r)} = -f_{col}\frac{\alpha_s(r)}{r} \qquad \alpha_s(q^2)$$

Let us evaluate $f_{col}$ for some colour conf of

$(u\bar{d})_{init}$ and $(u\bar{d})_{fin}$

triplet rep for q

for $\bar{q}$

colour octet

colour singlet

Composing $3_c \times \bar{3}_c = 8_c \oplus 1_c$

$u\bar{d}$ can be in a colour octet or a colour singlet

$c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ (29)

the colour singlet state is

$$C^{(1)} C^{(2)\dagger} = \frac{1}{\sqrt{3}} (r\bar{r} + b\bar{b} + g\bar{g}) = \frac{1}{\sqrt{3}} \sum_i c_i c_i^\dagger$$

$$C^{(3)\dagger} C^{4} = \frac{1}{\sqrt{3}} \sum_j c_j^\dagger c_j$$

$$\Rightarrow f^{\text{singlet}}_{\text{color}} = \frac{1}{4} \frac{1}{3} \sum_a \sum_j \sum_i \underbrace{\underbrace{c_j^\dagger \lambda^a c_i}_{\lambda^a_{ji}} \; \underbrace{c_i^\dagger \lambda^a c_j}_{\lambda^a_{ij}}}_{(\lambda^a)^2_{jj}}$$

$$= \frac{1}{12} \sum_a \mathrm{Tr}(\lambda^a)^2$$

$$= 4/3 > 0$$

so for colour singlet state the potential is attractive

For any non-singlet combination ie $\begin{matrix}12\\ r\bar{g}\end{matrix}$ $\begin{matrix}34\\ r\bar{g}\end{matrix}$

$$C^{(1)} C^{(2)\dagger} = c_1 c_3^\dagger$$
$$C^{(3)\dagger} C^{(4)} = c_1^\dagger c_3$$
$$\Rightarrow f_{\text{colour}} = \frac{1}{4} \sum_a \lambda^a_{11} \lambda^a_{33} = -\frac{1}{6} < 0$$

so the potential in repulsive & Same for any non-singlet configuration

So the mesons ($\equiv$ bound state of strong int) $q\bar{q}'$

can only be formed if constituent quarks and

# (6) Bound states of strong interactions and $SU(3)_{FLAVOUR}$

Starting with the Lagrangian of QCD in terms of quarks one cannot describe perturbatively the physics of the bound states (hadrons) because this is the regime in which $\alpha_S$ is very large and perturbative expansion does not work.

But using the global symmetries of $\mathcal{L}_{QCD}$ as guide one can infer some of the features of bound states, in particular those made of the light quarks $u, d, s$.

If we neglect the mass differences between these 3 quarks, the QCD Lag

$$\mathcal{L}_{QCD} = \sum_{\substack{ij \\ \text{colour}}} \left\{ \bar{u}_i \left[ i\gamma^\mu D_{\mu ij} - m\delta_{ij} \right] u_j + \bar{d}_i \left[ i\gamma^\mu D_{\mu ij} - m\delta_{ij} \right] d_j \right.$$

$$\left. + \bar{s}_i \left[ i\gamma^\mu D_{\mu ij} - m\delta_{ij} \right] s_j \right.$$

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s \sum_{a=1}^{8} \frac{\lambda^a_{ij}}{2} G^a_\mu$$

which can be written as

$$\mathcal{L}_{QCD} = \sum_{ij} (\bar{u}_i, \bar{d}_i, \bar{s}_i)\left[i\gamma^\mu (D_{ij})\, I_{3\times3} - M_q\,\delta_{ij}\right]\begin{pmatrix} u_j \\ d_j \\ s_j \end{pmatrix}$$

$$M_q = \begin{pmatrix} m & & \\ & m & \\ & & m \end{pmatrix}$$

This Lagrangian is invariant under a global rotation in the flavour space

3×3 unit matrix $\subset SU(3)_{FLAV}$

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow U \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

we can think of $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ as a triplet $3_F$ of the global $SU(3)_F$ and correspondingly $\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$ is a $\bar{3}_F$

Consequently for hadrons made of these quarks their wave functions

$$\psi^{had} = \psi_{spatial} \times \psi_{spin} \times \underset{\substack{\uparrow \\ \text{singlet}}}{\psi_{col}} \times \psi_{flav}$$

$\psi_{flavour}$ will be in some irrep of $SU(3)_F$ obtained from composition of $3_F$ and $\bar{3}_F$ of its constituents

For example for a meson ($\equiv$ bound state $q\bar{q}'$)

we need to compose

$$3_F \times \bar{3}_F = 8_F \times 1_F \Rightarrow$$ 1 octet and 1 singlet $\equiv$ 9 mesons (for a given $\psi_{spatial}$ and $\psi_{spin} \Rightarrow$ gives S and P)

and according to $SU(3)_F$ we can build their flavour

$\psi_{flav}$

In this limit in which $m_u = m_d = m_s$ all hadrons in a given $SU(3)_F$ representation have same mass.

Since in reality $m_u \simeq m_d$ and $m_s$ is a bit larger

the hadrons with only u and d quarks and quark antiquark have more similar masses ($SU(2)_{FLAV} \equiv$ ISOSPIN)

If we look at the PDB we find 9 pseudoscalar

mesons ($S=0$, $P=-1$) with similar masses ($\pi^{\pm 0}$, $K^\pm$; $K^0 \bar{K}^0$, $\eta$, $\eta'$)

and 9 vector mesons ($S=1$, $P=-1$) with

($\rho^{\pm 0}$, $K^{*\pm}$ $K^{*0}$ $\bar{K}^{*0}$ $\omega$, $\phi$)

(slides)

In this model if meson $(q_1 \bar{q}_2)$

$$M_{meson} = \underbrace{m_1 + m_2}_{\text{constituent quark mass}} + \underbrace{A}_{\text{constant to be fitted to data}} \frac{\langle \vec{S}_1 \cdot \vec{S}_2 \rangle}{m_1 m_2}$$

spins of quark

$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle$ can be computed from the spin wave funct of the meson (composition of $SU(2)_{spin}$ of $q$ and $\bar{q}$)

one finds

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \frac{S(S+1)}{2} - \frac{3}{4} = \begin{cases} -\frac{3}{4} & \text{for } S=0 \text{ (pseudoscalar)} \\ \frac{1}{4} & \text{for } S=1 \text{ (vectors)} \end{cases}$$

(slides)

For baryons ($\equiv$ bound state of 3 fermions)

$\Rightarrow$ wave function must be totally antisymmetric under exchange of any 2 of them

For light baryons ($\ell=0$) $\Rightarrow \psi_{spat}$ is totally symmetric

We have seen that $\psi_{col}^{baryon}$ is totally anti-symmetric

$\Rightarrow \psi_{spin} \times \psi_{FLAV}$ must be totally symmetric composing for flavor

$$3_F \times 3_F \times 3_F = 10_F + 8_{FS} + 8_{FA} + 1_F$$

$\psi_{FLAV}$ totally symmetric

$\psi_{FLAV}$ partly symmetric partly anti-sym

$\psi_{FLAV}$ totally anti-sym

for spin

$$2_S \times 2_S \times 2_S = 4_S + 2_{SS} + 2_{SA}$$

$S = 3/2$

$\psi_{spin}$ totally symmetric

$\psi_{spin}$ partly symmetric

" anti-sym

So possible combinations are

$10_F \times 4_S \Rightarrow$ decuplet of $S = 3/2$ baryons

$$\left.\begin{array}{l} 8_{FS} \times 2_{SS} \\ 8_{FA} \times 2_{SA} \end{array}\right\} \Rightarrow \text{octet of } S = 1/2 \text{ baryons}$$

(slides)

- For the pseudoscalar mesons

| particles | $S_{\text{strangeness}}$ | Mass (MeV) | Q | $I$ | $I_3$ | $\psi_{flavour}$ |
|---|---|---|---|---|---|---|
| $K^0$, $K^+$ | 1 | 496 | 0,+1 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $d\bar{s}$, $u\bar{s}$ |
| $\pi^-, \pi^0, \pi^+$ | 0 | 138 | -1,0,+1 | 1 | -1,0,+1 | $d\bar{u}$, $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $u\bar{d}$ |
| $\eta$ | 0 | 776 | 0 | 0 | 0 | $\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$ |
| $\eta'$ | 0 | 957 | 0 | 0 | 0 | $\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ |
| $K^-$, $\overline{K}^0$ | -1 | 496 | -1,0 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $s\bar{u}$, $s\bar{d}$ |

- $K^+, K^0, \overline{K}^0, K^-, \pi^+, \pi^0, \pi^- \eta$ are the octet and $\eta'$ is the singlet
- For the vector mesons

| Particles | $S$ | Mass(MeV) | Q | $I$ | $I_3$ | $\psi_{flavour}$ |
|---|---|---|---|---|---|---|
| $K^{*0}$, $K^{*+}$ | 1 | 896 | 0, +1 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $d\bar{s}$, $u\bar{s}$ |
| $\rho^-, \rho^0, \rho^+$ | 0 | 776 | -1,0,+1 | 1 | -1,0,+1 | $d\bar{u}$, $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $u\bar{d}$ |
| $\omega$ | 0 | 783 | 0 | 0 | 0 | $\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$ |
| $\phi$ | 0 | 1020 | 0 | 0 | 0 | $s\bar{s}$ |
| $K^{*-}$, $\overline{K}^{*0}$ | -1 | 896 | -1,0 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $s\bar{u}$, $s\bar{d}$ |

- $K^{*+}, K^{*0}, \overline{K}^{*0}, K^{*-}, \rho^+, \rho^0, \rho^-$ are 7 members of the octet
- $\omega$ and $\phi$ are an admixture of the singlet and the neutral component of the octet.

Mass formula for the mass of a meson compose of a quark 1 and and antiquark 2 as

$$\mathrm{M(meson)} = m_1 + m_2 + A \frac{\langle \vec{S}_1.\vec{S}_2 \rangle}{m_1\, m_2}$$

with

$$\langle \vec{S}_1.\vec{S}_2 \rangle = \frac{s(s+1)}{2} - \frac{3}{4} = \begin{cases} -\frac{3}{4} \text{ for pseudoscalar mesons (s = 0)} \\ \frac{1}{4} \text{ for vector mesons (s = 1)} \end{cases}$$

For the values $m_u = m_d = 310$ MeV $m_s = 483$ MeV and $A = (2m_u)^2 \times 160$ MeV

| Meson | Calculated | Observed |
|---|---|---|
| $\pi$ | 140 | 138 |
| K | 484 | 496 |
| $\eta$ | 559 | 549 |
| $\rho$ | 780 | 776 |
| $\omega$ | 780 | 783 |
| $K^*$ | 896 | 892 |
| $\phi$ | 1032 | 1020 |

So this model could explain the masses of all the light meson (except the $\eta'$) to about 1% accuracy.

- Baryons with Spin $s = \frac{1}{2}$: 8 particles

| particles | $S$ | Mass (MeV) | Q | $I$ | $I_3$ | quark content |
|---|---|---|---|---|---|---|
| $n$, $p$ | 0 | 939 | 0,+1 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $udd$, $uud$ |
| $\Sigma^-, \Sigma^0, \Sigma^+$ | -1 | 1193 | -1,0,+1 | 1 | -1,0,+1 | $dds$, $uds$, $uus$ |
| $\Lambda^0$ | -1 | 1114 | 0 | 0 | 0 | $uds$ |
| $\Xi^-$, $\Xi^0$ | -2 | 1318 | -1,0 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $dss$, $uss$ |

- Baryons with Spin $s = \frac{3}{2}$: 10 particles

| particles | S | Mass (MeV) | Q | $I$ | $I_3$ | quark content |
|---|---|---|---|---|---|---|
| $\Delta^-,\Delta^0,\Delta^+,\Delta^{++}$ | 0 | 1232 | -1,0,+1,+2 | $\frac{3}{2}$ | $-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$ | $ddd$ $udd$, $uud$, $uuu$ |
| $\Sigma^{*-}, \Sigma^{*0}, \Sigma^{*+}$ | -1 | 1384 | -1,0,+1 | 1 | -1,0,+1 | $dds$,$uds$, $uus$ |
| $\Xi^{*-}$, $\Xi^{*0}$ | -2 | 1533 | -1,0 | $\frac{1}{2}$ | $-\frac{1}{2}, +\frac{1}{2}$ | $dss$, $uss$ |
| $\Omega^-$ | -3 | 1672 | -1 | 0 | 0 | $sss$ |

Again, the masses of all baryons in a given multiplet are not all the same because the $SU(3)_{\text{flavour}}$ symmetry is broken by the different quark masses.

But they are more similar within each multiplet so the potential between the quarks must be spin dependent.

$$\mathrm{M(baryon)} = m_1 + m_2 + m_2 + A' \left( \frac{\langle \vec{S}_1.\vec{S}_2 \rangle}{m_1\, m_2} + \frac{\langle \vec{S}_1.\vec{S}_3 \rangle}{m_1\, m_3} + \frac{\langle \vec{S}_2.\vec{S}_3 \rangle}{m_2\, m_3} \right)$$

For the baryon decuplet in all spin configurations one finds

$$\langle \vec{S}_i.\vec{S}_j \rangle = \frac{1}{4}$$

The mass of all baryons in the decouplet can well explained with $A' = (2m_u)^2\, 50$ MeV.