We shall denote the longitudinal momentum fraction of parton a in hadron A by x_a and the parton density of a in A by $f_{a/A}(x_a)$. The cross section for producing a quark or lepton c in the inclusive reaction

$$A + B \to c + \text{anything}$$

is obtained by multiplying the subprocess cross section $\hat{\sigma}$ for

$$a + b \to c + \text{anything}$$

by $dx_a\, f_{a/A}(x_a)$ and $dx_b\, f_{b/B}(x_b)$, summing over parton and antiparton types a, b and integrating over x_a and x_b; also an average must be made over the colors of a and b. The resulting relation is

$$\sigma(AB \to cX) = \sum_{a,b} C_{ab} \int dx_a\, dx_b \cdot \left[f_{a/A}(x_a) f_{b/B}(x_b) + (A \leftrightarrow B \text{ if } a \neq b) \right] \hat{\sigma}(ab \to cX) .$$

In this formula $\hat{\sigma}$ is *summed over* initial and final colors; the initial color-averaging factor C_{ab} appears separately. The color-average factors for quarks and gluons are

$$C_{qq} = C_{q\bar{q}} = \tfrac{1}{9} , \qquad C_{qg} = \tfrac{1}{24} , \qquad C_{gg} = \tfrac{1}{64} .$$

In a Lorentz frame in which masses can be neglected compared with three-momenta, the four-momenta relations

$$a = x_a A \qquad \text{and} \qquad b = x_b B ,$$

lead to

$$\hat{s} = x_a x_b s = \tau s ,$$

where $\sqrt{\hat{s}}$ is the invariant mass of the ab system, $\sqrt{s}$ is the invariant mass of the AB system and we have introduced a convenient variable τ

$$\tau = x_a x_b \,.$$

Changing to x_a and τ as independent variables the cross section expression becomes

$$\sigma = \sum_{a,b} C_{ab} \int_0^1 d\tau \int_\tau^1 \frac{dx_a}{x_a} [f_{a/A}(x_a) f_{b/B}(\tau/x_a) + (A \leftrightarrow B \text{ if } a \neq b)] \; \hat{\sigma}(\hat{s} = \tau s).$$

Thus we can write

$$\frac{d\sigma}{d\tau} = \sum_{a,b} \frac{d\mathcal{L}_{ab}}{d\tau} \hat{\sigma}(\hat{s} = \tau s)$$

with

$$\frac{d\mathcal{L}_{ab}(\tau)}{d\tau} = C_{ab} \int_\tau^1 \frac{dx_a}{x_a} \left[f_{a/A}(x_a) f_{b/B}(\tau/x_a) + (A \leftrightarrow B \text{ if } a \neq b) \right] .$$

The quantity $d\mathcal{L}_{ab}/d\tau$ is called the *parton luminosity* since multiplying the parton cross section $\hat{\sigma}$ by $d\mathcal{L}/d\tau$ gives the particle cross section $d\sigma/d\tau$ in hadron collisions.

In hard scattering processes at high energy, where the only dimensional scale is $\hat{s}$, the subprocess cross section has the form

$$\hat{\sigma}(\hat{s}) = c/\hat{s} = c/(\tau s) \,,$$

where c is a dimensionless constant. Then, for scaling parton distributions (*i.e.* the f depend only on x), the quantity $s \, d\sigma/d\tau$ scales with τ, *i.e.* it depends on τ alone:

$$s \frac{d\sigma}{d\tau} = G(\tau)$$

with the form of the scaling function $G(\tau)$ depending on the parton distributions $f(x)$.