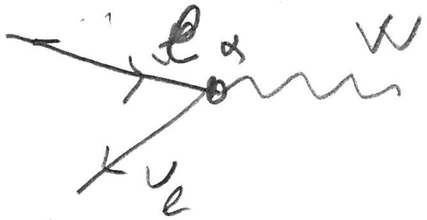
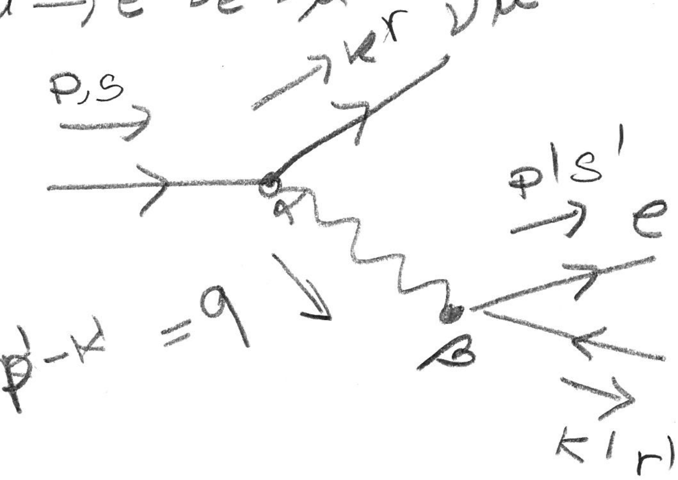
So the vertex for this interaction will be



$$-i\frac{g_W}{\sqrt{2}}\gamma^\alpha(g_V - g_A\gamma^5)$$

we are going to set $m_e \simeq m_{\nu} = m_{\bar\nu} \simeq 0$

$\mu^- \to e^- \bar\nu_e \nu_\mu$



$p = (m, 0)$

$k = (\omega, \vec{k})$

$k' = (\omega', \vec{k}')$

$p' = (E', \vec{p}')$

$$\mathcal{M} = \frac{g_W^2}{2}\left[\bar{u}^r_{\nu_\mu}(k)\gamma^\alpha(g_V - g_A\gamma_5)u^s_\mu(p)\right]\frac{g_{\alpha\beta} - \frac{q_\alpha q_\beta}{M_W^2}}{q^2 - M_W^2}$$

$$\left[\bar{u}^{s'}_e(p')\gamma^\beta(g_V - g_A\gamma_5)v^{s'}_\nu(k')\right]$$

$|q| \ll M_W$

$$\longrightarrow \simeq \underbrace{-\frac{g_W^2}{2M_W^2}}_{= G_F\frac{4}{\sqrt{2}}}\left[\bar{u}^r(k)\gamma^\alpha(g_V - g_A\gamma_5)u^s(p)\right]\left[\bar{u}^{s'}(p')\gamma_\alpha(g_V - g_A\gamma_5)v^{s}\right.$$

$$|\overline{\mathcal{M}}|^2 = \frac{1}{2}\sum_{ss'rr'} |M|^2 = 4G_F^2\, \mathrm{Tr}[\not k \gamma^\alpha (g_V - g_A\gamma_5)(\not p + m)\gamma^\beta (g_V - g_A\gamma_5)]$$

$$\times \mathrm{Tr}[\not p' \gamma_\alpha (g_V - g_A\gamma_5)\not k' \gamma_\beta (g_V - g_A\gamma_5)]$$

$$= 4G_F^2\Big\{\mathrm{Tr}[\not k\gamma^\alpha \not p \gamma^\beta \underbrace{(g_V-g_A\gamma_5)^2}_{=\,g_V^2+g_A^2-2g_Vg_A\gamma_5}]\, \mathrm{Tr}(\not p'\gamma_\alpha \not k'\gamma_\beta (g_V-g_A\gamma_5)^2]$$

$$= 4G_F^2\Big\{(g_V^2+g_A^2)^2\, \overbrace{\mathrm{Tr}(\not k\gamma^\alpha \not p \gamma^\beta)\,\mathrm{Tr}(\not p'\gamma_\alpha \not k'\gamma_\beta)}^{A}$$

$$+ 4g_V^2 g_A^2\, \overbrace{\mathrm{Tr}(\not k\gamma^\alpha \not p \gamma^\beta\gamma_5)\,\mathrm{Tr}(\not p'\gamma_\alpha \not k'\gamma_\beta\gamma_5)}^{B}$$

$$\pm 2g_Vg_A(g_V^2+g_A^2)\Big[\overbrace{\mathrm{Tr}(\not k\gamma^\alpha \not p \gamma^\beta)\,\mathrm{Tr}(\not p'\gamma_\alpha \not k'\gamma_\beta\gamma_5)}^{C}$$

$$+ \underbrace{\mathrm{Tr}(\not k\gamma^\alpha \not p \gamma^\beta\gamma_5)\,\mathrm{Tr}(\not p'\gamma_\alpha \not k'\gamma_\beta)}_{D}\Big]\Big\}$$

$$A = 16\,[k^\alpha p^\beta + p^\alpha k^\beta - g^{\alpha\beta}(pk)][k'_\alpha p'_\beta + p'_\alpha k'_\beta - (k'p') g_{\alpha\beta}]$$

$$= 16\,[2(kk')(pp') + 2(kp')(pk') \cancel{- 4(k'p')(kp) + 4(k'p')(kp)}]$$

$$B = (-i4\epsilon^{\rho\alpha\sigma\beta}\, k_\rho p_\sigma)(-i4\epsilon_{\ell\alpha n\beta}\, p'^\ell k'^n)$$

$$= -16\,(k_\rho p_\sigma p'^\ell k'^n)(-\delta^\rho_\ell \delta^\sigma_n + \delta^\rho_n \delta^\sigma_\ell)\times 2$$

$$= 32\,[(kp')(pk') - (kk')(pp')]$$

C is the contraction of a tensor which is symmetric under $\alpha \leftrightarrow \beta$ and one which is antisymmetric under $\alpha \leftrightarrow \beta$ $\Rightarrow C = 0$

So altogether

$$|\overline{M}|^2 = 128\, G_F^2 \Big\{ \left((g_V^2+g_A^2)^2 + 4 g_V^2 g_A^2\right)(k p')(p k') + \underbrace{\left((g_V^2+g_A^2)^2 - 4 g_V^2 g_A^2\right)}_{(g_V^2-g_A^2)^2}(k k')(p p') \Big\}$$

notice that SM CC $\Rightarrow g_V = g_A = \frac{1}{2} \Rightarrow |\overline{M}|^2 = 128\, G_F^2 \left(\frac{1}{4}+\frac{1}{4}\right)$

$= 64\, G_F^2$ as it should

the decay width is

$$\Gamma = \frac{1}{2m} \int |\overline{M}|^2 d\phi_3$$

$$d\phi_3 = (2\pi)^4 \delta^4(p - p' - k - k') \frac{d^3p'}{(2\pi)^3 2E'} \frac{d^3k'}{(2\pi)^3 2\omega'} \frac{d^3k}{(2\pi)^3 2\omega}$$

$$p = k + k' + p'$$

using

$$k'p = m\omega'$$

$$kp' = \frac{1}{2}\left[(k+p')^2 - \overset{=0}{k^2} - \overset{=0}{p'^2}\right] = \frac{1}{2}(p-k')^2 = \frac{1}{2}p^2 - k'p$$

$$= \frac{1}{2}(m^2 - 2m\omega')$$

$$pp' = mE'$$

$$kk' = \frac{1}{2}\left[(k+k')^2 - \overset{=0}{k^2} - \overset{=0}{k'^2}\right] = \frac{1}{2}(p-p')^2 = \frac{1}{2}(m^2 - 2mE')$$

$$= \frac{1}{2}m^2 - mE'$$

So
$$\overline{|M|^2} = 64\, G_F^2 \left\{\left[(g_A^2+g_V^2)^2 + 4g_A^2 g_V^2\right] m^2\omega'(m-2\omega') + (g_A^2-g_V^2)^2 m^2 E'(m-2E')\right\}$$
← independent of angles

using that $d^3\bar{k} = 2\omega\,\delta(k^2)\,d^4k$

and integrating d^4k with δ^4

we get

$$d\Phi_3 = \frac{1}{(2\pi)^5}\,\frac{1}{4E'\omega'}\,\delta\left[(p-k'-p')^2\right] d^3p'\, d^3k'$$

let us define θ as the angle between $\vec{p}\,'$ and $\vec{k}\,'$

$$\Rightarrow (p-k'-p')^2 = m^2 - 2mE' - 2m\omega' + 2\omega' E'(1-\cos\theta)$$

so all angular variables can be integrated except θ

$$d^3p' d^3k' = \omega'^2 E'^2 (4\pi)(2\pi)\, d\cos\theta\, d\omega' dE'$$

and using that

$$\delta(m^2 - 2mE' - 2m\omega' + 2\omega' E'(1-\cos\theta))$$
$$= \delta\left(\cos\theta - \frac{m^2 - 2mE' - 2m\omega'}{2\omega' E'}\right) \frac{1}{2\omega' E'}$$

we get

$$d^3\phi = \frac{1}{(2\pi)^3} \frac{1}{4} \delta(\cos\theta - \quad) d\omega' dE' d\cos\theta$$

we have to integrate $dE' d\omega'$ making sure the

$$-1 \leq \cos\theta \leq 1 \quad \Rightarrow \quad 1 \leq \frac{m^2 - 2mE' - 2m\omega'}{2\omega' E'} \leq 1$$

$$\tfrac{1}{2}m - E' \leq \omega' \leq \frac{m}{2}$$
$$\Rightarrow \quad 0 \leq E' \leq m/2$$

So

$$\frac{d\Gamma}{dE'} = \frac{64 G_F^2}{(2\pi)^3\, 8} m \int_{\frac{1}{2}m - E'}^{m/2} d\omega' \Big\{ \left[(g_V^2 + g_A^2)^2 + 4 g_V^2 g_A^2 \right] \omega'(m - 2\omega')$$

$$+ (g_V^2 - g_A^2)^2 E'(m - E') \Big\}$$

$$= \frac{G_F^2}{\pi^3} m \Big\{ (g_V^2 - g_A^2)^2 E'(m-2E') E'$$

$$+ \left[(g_V^2 + g_A^2)^2 + 4 g_V^2 g_A^2 \right) \frac{m E'^2}{6} \left(3 - \frac{4E'}{m} \right) \Big\}$$

$$\text{So } \Gamma = \int_0^{m/2} \frac{d\Gamma}{dE} = \frac{G_F^2 m}{\pi^3} \Big\{ (g_V^2 - g_A^2) \frac{m^4}{\cancel{1}96}$$

$$+ \left[(g_V^2 + g_A^2)^2 + 4 g_V^2 g_A^2 \right] \frac{m^4}{96} \Big\}$$

So for $g_V = g_A$ the energy distribution of e^-

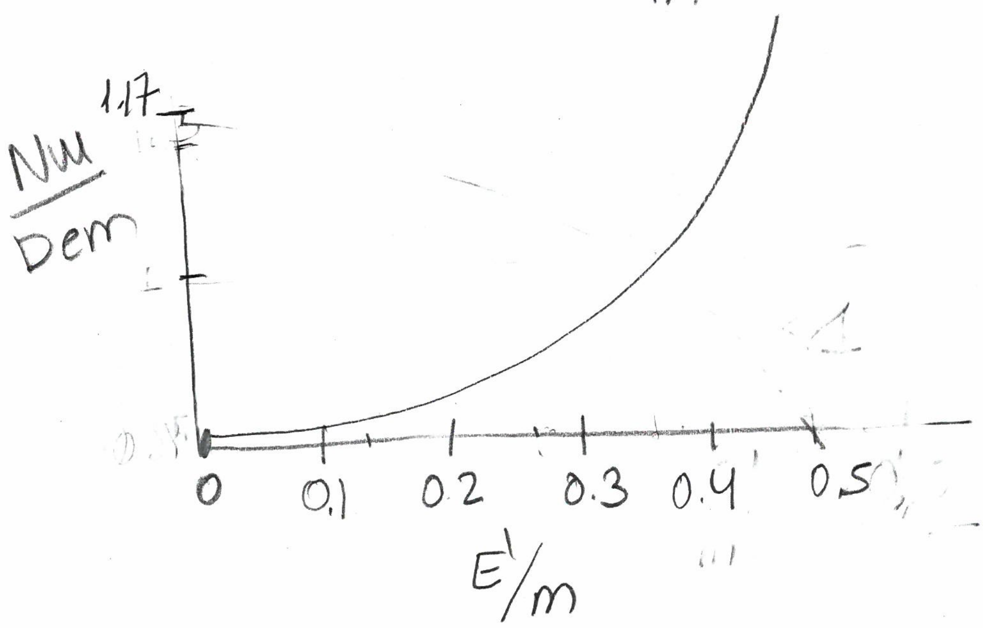
$$\frac{1}{\Gamma}\frac{d\Gamma}{dE'} = \frac{16 E'^2}{m^3}\left(3 - \frac{4E'}{m}\right) \equiv Num$$

for $g_A = 0$ the energy distribution of e^-

$$\frac{1}{\Gamma}\frac{d\Gamma}{dE'} = \left[E'^2(m-2E') + \frac{E'^2}{6}(3m-4E')\right]\frac{196}{6m^4}$$

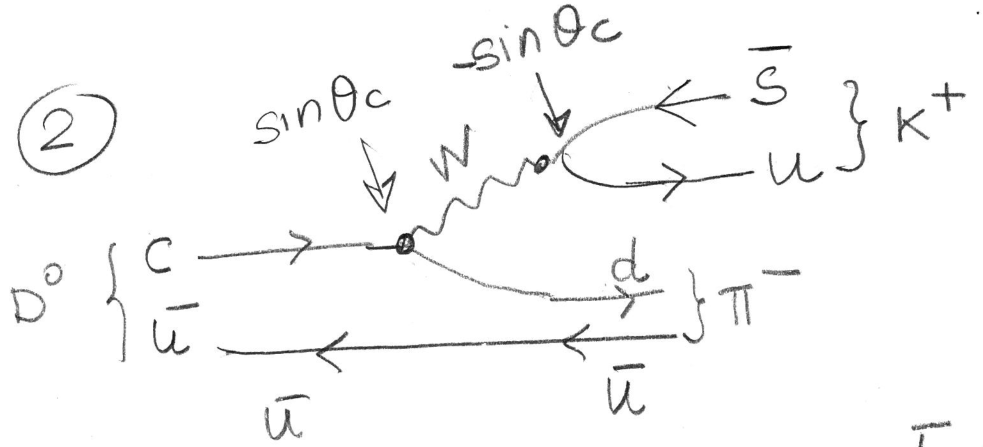
$$= \frac{16}{6m^3}E'^2\left(9 - 16\frac{E'}{m}\right) \equiv Den$$

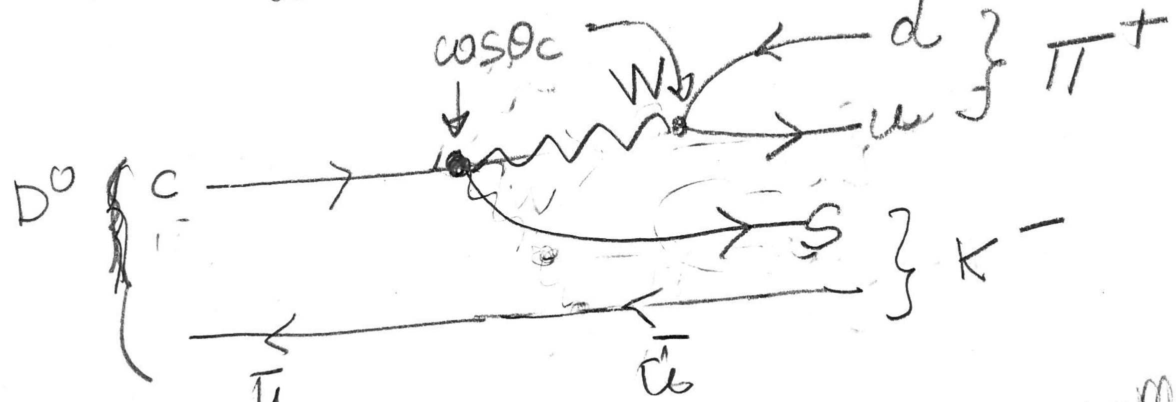
$$\frac{Num}{Den} = \frac{6\left(3 - 4\frac{E'}{m}\right)}{\left(9 - 16\frac{E'}{m}\right)}$$



So the interactions with $g_A = 0$ produces less energetic e^-

2





because $m_{K^+} = m_{K^+}$ and $m_{\pi^+} = m_{\pi^-}$ so kinematic factors are the same

$$\text{So } R \equiv \frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)}$$

$$= \frac{|M|^2 (D^0 \to K^+\pi^-)}{|M|^2 (D^0 \to K^-\pi^+)} = \frac{\sin^4\theta_c}{\cos^4\theta_c} \simeq \frac{0.22^4}{(1-0.22^2)^2} \simeq 0.0026$$

$$\text{looking at data} = \frac{(1.5 \pm 0.07)\times 10^{-4}}{(3.947 \pm 0.03)\times 10^{-2}} = 0.0038 \pm 0.0002$$

which is in the same ballpark

Including the full CKM

$$R = \frac{|V_{us}|^2 |V_{cd}|^2}{|V_{cs}|^2 |V_{ud}|^2} = \frac{0.224^2 \times 0.221^2}{0.975^2 \times 0.974^2} = 0.00227$$