Elementary Particle Physics: Assignment # 4

Due Feb 26 11 am

(1) The Dirac equation for a relativist spin 1/2 particle is: $i\hbar\frac{\partial}{\partial t}\psi = H$

with Hamiltonian operator $H = \vec{\alpha}.\vec{P} + m\,\beta$ where $P^j = -i\frac{\partial}{\partial x^j}$ is the 3-momentum operator, and α and β matrices and their commutation relations are given in the lectures.

1.1 Show that each of the components of the 3-momentum operator P^j commutes with the Hamiltonian in and therefore they are conserved, while none of the orbital angular momentum operator ($\vec{L} = \vec{x} \times \vec{P}$) components commute with the Hamiltonian (*Comment: notice that this means that total 3-momentum is conserved and its value can be used together with the energy to characterize a state*). [Hint: Remember $[x_i, P_j] = i\,\delta_{ij}$, $[P_i, P_j] = 0$,$[x_i, x_j] = 0$]

1.2 Show that the combination $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ with $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ commutes with the Hamiltonian (again you have to show that each of the components of this vector commutes with H) (*Comment: notice that this means that total angular momentum is also conserved and its value be used together with energy to characterize the state*). σ are the Pauli matrices given in class.

1.3 Show that none of the components of $\vec{J}$ commutes with the 3-momentum components (*Comment: therefore total angular momentum cannot be used to characterize a state of well defined energy and 3-momentum*).

1.4 Show that the helicity operator, $\vec{J}.\vec{P}$, does commute with the 3-momentum, (*Comment: therefore the value of the helicity can be used to characterize the state together with its energy-momentum*).

(2) In the chiral representation the 4-spinors for a fermion with momentum $\vec{p} = |\vec{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with positive and negative helicity are:

$$u^{1,2}(\vec{p}) = u^{\pm}(\vec{p}) = \begin{pmatrix} \sqrt{E \mp |\vec{p}|}\;\xi_p^{\pm} \\ \sqrt{E \pm |\vec{p}|}\;\xi_p^{\pm} \end{pmatrix} \quad \text{with} \quad \xi_p^{+} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \quad \xi_p^{-} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

and the corresponding 4-spinors for the anti-fermion are: $v^{1,2}(\vec{p}) = v^{\pm}(\vec{p}) = \pm\begin{pmatrix} \sqrt{E \pm |\vec{p}|}\;\xi_p^{\mp} \\ -\sqrt{E \mp |\vec{p}|}\;\xi_p^{\mp} \end{pmatrix}$

Using these expresions evaluate by direct calculation

$$\bar{u}^s(\vec{p})v^r(-\vec{p}) \quad \text{for the four helicity combinations}$$

(3) Suppose you apply a gauge transformation with gauge function

$$\chi(x) = i\,\kappa\,\mathrm{e}^{-ip^\alpha x_\alpha}$$

(were κ is an arbitrary constant) to the plane wave 4-vector potential

$$A^\nu(x) \equiv \epsilon^\nu(p)\,\mathrm{e}^{-ip^\alpha x_\alpha}$$

3.1) Show that this gauge transformation has the effect of modifying

$$\epsilon^\mu \to \epsilon^\mu + \kappa\,p^\mu$$

3.2) Show that if we chose $\kappa = \frac{-\epsilon^0}{p_0}$ we obtain the a polarization vector in the Coulomb gauge, ie

$$\epsilon^0 = 0 \qquad \text{and} \qquad \vec{\epsilon}.\vec{p} = 0$$

.