Chapter 2

"Relativistic Kinematics"

1) Lorentz transformations : implications
2) Four vector notation
3) Energy-momentum 4-vector
4) Examples

Chapter 3 Griffiths

1 Lorentz transformations: implications

Relativity principle $\equiv$ same laws of physics apply in any inertial system (inertial $\equiv$ moving at constant velocity) $\Rightarrow$ light ($\equiv$ em wave) must travel at same speed in any inertial system.

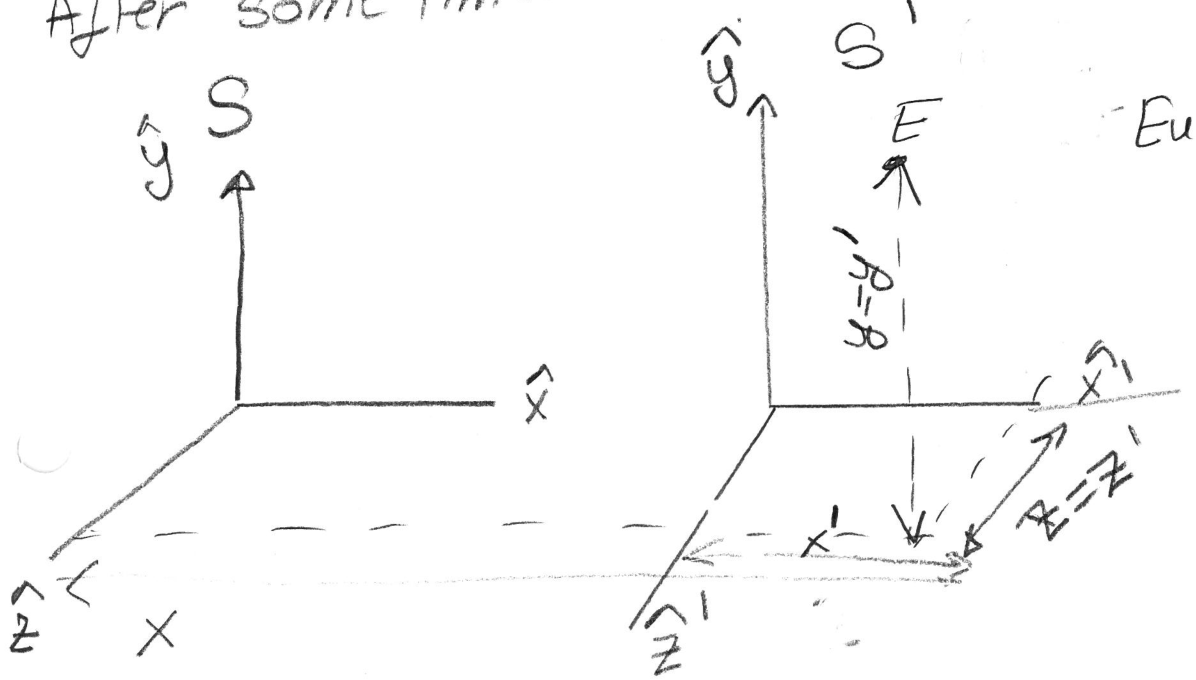
Take two systems of coordinates.

- S with spatial coordinates (x, y, z) and time coord t
- S' " " " (x', y', z') " " " t'

S' is moving wrt S at velocity $\vec{v} = v\hat{x}$

At $t = t' = 0$ $\quad x = x' = y = y' = z = z' = 0$

After some time



Event occurs at (x, y, z) and time t in S

That event is obsved in S' at x', y', z' at time t' (3)

with

$$x' = \gamma(x - vt) \overset{nu}{\equiv} \gamma(x - \beta t)$$
$$y' = y = y$$
$$z' = z = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = \gamma(t - \beta x)$$

with

$$\beta = \frac{v}{c} < 1$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 1$$

when $\beta \to 1$ $\gamma \to \infty$

or inversely

$$x = \gamma(x' + \beta t')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + \beta x')$$

check

$$x = \gamma(\gamma(x - \beta t) + \beta\gamma(t - \beta x))$$
$$= x$$

3 Most ubiquitous consequences in particle physics

(1) time dilatation: If at rest (S') a particle has a lifetime $\tau \equiv (t'_2 - t'_1)$ in a system in which is moving with velocity β (S) it lives longer

$$t_2 - t_1 = \gamma\tau + \gamma\beta\underbrace{(x'_2 - x'_1)}_{0 \text{ because it is at rest in } S'} = \gamma\tau > \tau$$

For ultrarelativistic particle $\beta \to 1 \Rightarrow \gamma >> 1$

$\Rightarrow \Delta t >> \tau$

So in S the particle travels

$$d = \beta \Delta t = \beta\gamma\tau >> \beta\tau \leftarrow \text{naive non-relativistic estimate}$$

Example: muon (μ^-) was discovered at Earth surface when it was produced at top of atmosphere by collision of cosmic rays ($d = 10\,km$)

So it had to cross the atmosphere before decaying

At rest its lifetime is $\tau = 2.2 \times 10^{-6}\,s$

$\Rightarrow$ naively ($\equiv$ ignoring special relativity) one would think that since $v_{muon} < c \Rightarrow d_{muon} << c\tau = 660\,m$
$<< 10\,km$

But in the Earth system the muon is travelling at speed $v \Rightarrow$ lives a time $\Delta t = \gamma\tau = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\tau$

so it can travel a distance

$$L = v\Delta t = \gamma v \tau >> 660\,m \text{ if } \frac{v}{c} \simeq 1$$

fe if $v = (\overbrace{1-10^{-}}^{0.9999})c \Rightarrow \gamma = \frac{1}{\sqrt{1-(1-10^{-4})^2}} \simeq 71$

$\Rightarrow$ in Earth frame μ^- lives $\Delta t = 71 \times 2.2\times10^{-6}s$

$\simeq 0.15\,ms$

$\Rightarrow$ it can travel a distance $L = v\Delta t \simeq 71 \times 660\,m$

$= 47\,Km$

so it reaches the Earth surface before decaying

c2) Length contraction : an object that at rest

measures L_{rest} (say at S $L_{rest} = x_2 - x_1$) S is not imp

is a system S' wrt is moving at speed $\vec{v} = (-v\hat{x})$

is seen with length L' ("seen" $\equiv t'_1 = t'_2$)

$L' = x'_2 - x'_1$

$L_{rest} = x_2 - x_1 = \gamma(x'_2 - x'_1) + \gamma\beta(\overbrace{\cancel{t'_2 - t'_1}}^{0}) = \gamma L'$

$\Rightarrow L' = \frac{L_{rest}}{\gamma} < L_{rest} \Rightarrow$ contracted

Back to the muon in S' muon lives $\tau = 2.2 \times 10^{-6}$ s

at the atmosphere rest frame (S) the atmosphere measures $L_{atm,rest} = 10$ Km and it is moving towards the muon at $v = (1 - 10^{-4})c$

$\Rightarrow$ in S' the atmosphere thickness is

$$L'_{ATM} = \frac{1}{\gamma} L_{rest} = \frac{10 \text{Km}}{71} = 141 \text{ m} < v \cdot \tau = 660 \text{ m}$$

$\Rightarrow$ the muon can see the whole atmosphere passing by before decaying

c3) Velocities do not add linearly

An object moves with velocity $\vec{u}' = u \hat{x}$ in s'

$\Rightarrow$ in $\Delta t'$ in moves a distance $\Delta x' = u' \Delta t$

$\Rightarrow$ in S (s' moves wrt S at $\vec{v} = v \hat{x}$)

it has moved a distance in S

$$\left.\begin{aligned} \Delta x &= \gamma(\Delta x' + v \Delta t') \text{ in a time} \\ \Delta t &= \gamma(\Delta t' + \frac{v}{c^2} \Delta x') \end{aligned}\right\} \Rightarrow u = \frac{\Delta x}{\Delta t} = \frac{\Delta x' + v \Delta t'}{\Delta t' + \frac{v}{c^2} \Delta x'} = \frac{u' + v}{1 + \frac{v u'}{c^2}}$$

So if it moves at speed of light in $s' \Rightarrow u' = c \Rightarrow$ in S $= u = \frac{c + v}{1 + v/c} = c \Rightarrow$ also at speed of light

2) Four vector notation

We have seen that for systems with $v \sim c$ time and spatial coordinates mix when changing reference frame. To simplify the notation we introduce 4-vectors the contravariant position-time 4 vector has contravariant

X^{μ} ← upper Lorentz index $\mu = 0, 1, 2, 3$

$$X \equiv \begin{pmatrix} x^0 = ct \overset{NC}{=} t \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix} \qquad \underset{\text{3 vector}}{\vec{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So in this notation the Lorentz transf. can be written in matrix form

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{4\times 4 \text{ matrix } \Lambda_{\text{boost } \hat{x}}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

or equivalently

$$X'^{\mu} = \sum_{\nu=0}^{3} \left(\Lambda_{\text{boost}\,\hat{x}}\right)^{\mu}{}_{\nu}\, X^{\nu} \equiv \left(\Lambda_{\text{boost}\,\hat{x}}\right)^{\mu}{}_{\nu}\, X^{\nu}$$

Einstein's convention ≡ repeated upper and lower indexes (≡ contracted) are summed from 0 to 3

Lorentz group is the group of transformations which leave the 4-norm invariant. It includes

- boost of velocity $\vec{\beta}$
- rotations in 3 dimension $\vec{x}' = (R)_{3\times3}\,\vec{x} \Rightarrow \Lambda_{rot} = \begin{pmatrix} 1 & 0\,0\,0 \\ 0 & \\ 0 & R \\ 0 & \end{pmatrix}$
- Parity $\vec{x}' = -\vec{x} \Rightarrow \Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
- Time reversal $x^{0\prime} = -x^0 \Rightarrow \Lambda_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

We define a contravariant Lorentz 4-vector to a 4-dimensional column object

$$\begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \equiv \begin{pmatrix} a^0 \\ \vec{a} \end{pmatrix}$$

which under any transf. of the Lorentz group transform as

$$a'^{\mu} = \Lambda^{\mu}{}_{\nu}\, a^{\nu}$$

and the corresponding covariant 4-vector

$$a_{\mu} = g_{\mu\nu} a^{\nu} = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3)$$
$$\equiv (a^0, -\vec{a})$$

and the scalar product of 2 4-vectors

$$a \cdot b = a^{\mu} b_{\mu} = a_{\mu} b^{\mu} = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$$
$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \vec{a}\cdot\vec{b}$$

the scalar product of ~~2~~ 4-vector is a Lorentz inv

$\equiv$ same value in any inertial frame

and the 4-norm of the 4-vector

$$a^2 = a^{0^2} - |\vec{a}|^2$$

unlike the norm of a 3-vector $|\vec{a}|^2 > 0$

for a 4-vect

$$a^2 > 0 \Rightarrow a^0 > |\vec{a}| \Rightarrow \text{time-like 4-vector}$$
$$a^2 < 0 \Rightarrow a^0 < |\vec{a}| \Rightarrow \text{space-like 4-vector}$$
$$a^2 = 0 \Rightarrow a^0 = |\vec{a}| \Rightarrow \text{light like 4-vector}$$

(3) Energy momentum 4-vector

In special relativity the motion of a particle of mass m and velocity $\vec{\beta} = \frac{d\vec{x}}{dx_0}$ is characterized by

not to forget

- its 3 momentum $\Rightarrow \vec{p} = m\gamma\vec{\beta} = E\vec{\beta}$
- " Energy $E = m\gamma$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = |\vec{\beta}| \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2}$$

$$\Rightarrow |\vec{p}|^2 = m^2\gamma^2\beta^2 = m^2\gamma^2 - m^2 = E^2 - m^2 \Rightarrow E^2 = |\vec{p}|^2 + m^2$$

Since $\begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}$ is a 4vector $\Rightarrow \begin{pmatrix} 1 \\ \frac{d\vec{x}}{dx^0} \end{pmatrix}$ is a 4vector

$$\Rightarrow m\gamma\begin{pmatrix} 1 \\ \vec{\beta} \end{pmatrix} = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \equiv p^0$$ is a 4-vector

we call it the energy-momentum 4-vector P

Its norm:

$$p^2 = p^{0\,2} - |\vec{p}|^2 = m^2 \Rightarrow$$ mass in invariant under Lorentz transf $\Rightarrow$ mass is the same in any inertial system

For a particle with $m=0 \Rightarrow E=|\vec{p}|=\beta E \Rightarrow \beta=1$

$\Rightarrow$ moves at speed of light.

Invariance under translations (chapter 3) $\Rightarrow$

in any physical process the total Energy-momentum

4-vector is conserved

Initial $\longrightarrow$ Final

N particles $i=1..N$ $\longrightarrow$ M particles $j=1..M$

with $P_i^{init} = \begin{pmatrix} E_i^{int} = \sqrt{|\vec{p}_i^{\,Init}|^2 + m_i^2} \\ \vec{p}_i^{\,int} \end{pmatrix}$ $\quad P_j^{FIN} = \begin{pmatrix} E_j^{FN} = \sqrt{|\vec{p}_j^{\,F}|^2 + m_j^2} \\ \vec{p}_j^{\,FIN} \end{pmatrix}$

$$P_{TOT}^{INT} = \begin{pmatrix} \sum_{i=1}^{N} E_i^{int} \\ \sum_{i=1}^{N} \vec{p}_i^{\,int} \end{pmatrix} \equiv P_{TOT}^{FN} = \begin{pmatrix} \sum_{j=1}^{M} E_j^{FIN} \\ \sum_{j=1}^{M} \vec{p}_j^{\,FIN} \end{pmatrix}$$

so knowing the energy-momentum of initial particles $\Rightarrow$ relation among energy and 3-momenta of out final particles

$\equiv$ kinematic constraints

4 Examples

1 Two body decay : $A \rightarrow B \;\; C$

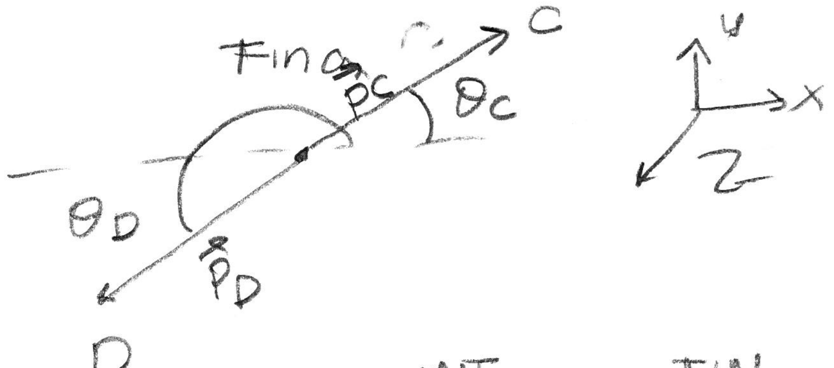
Let's start with A at rest ($\equiv R$ system)

Inihal	Final
A	C + D

4-momem

$$P_A^R = \begin{pmatrix} E_A^R = M_A \\ \vec{P}_A^R = 0 \end{pmatrix} \qquad P_{C,D}^R = \begin{pmatrix} E_{C,D}^R = \sqrt{M_{C,D}^2 + |\vec{P}_{C,D}^R|^2} \\ \vec{P}_{C,D}^R \end{pmatrix}$$

Inital



4-momentum conseuahon $\Rightarrow P^{INT} = P_A = P^{FIN} = P_C + P_D$

$\Rightarrow$ 4 equahons

$\mu=0$ $\quad E_A^R = M_A = E_C^R + E_D^R = \sqrt{M_C^2 + |\vec{P}_C|^2} + \sqrt{M_D^2 + |\vec{P}_D|^2}$

$\mu=1..3$ $\quad \vec{0} = \vec{P}_C^R + \vec{P}_D^R \Rightarrow \vec{P}_C^R = -\vec{P}_D^R \Rightarrow$ C, D produced back to back

$\Rightarrow |\vec{P}_C^R| = |\vec{P}_D^R| \equiv P_f$ $\qquad \Rightarrow \theta_D = \theta_C + \pi \equiv \theta_R$

$$M_A = \sqrt{P_f^2 + M_C^2} + \sqrt{P_f^2 + M_D^2} \Rightarrow \text{solvable for } P_f$$

$$P_f = \frac{1}{2M_A}\sqrt{(M_A^2 - (M_C + M_D)^2)(M_A^2 - (M_C - M_D)^2)}$$

Fully determined by masses involved

$$E_C^R = \sqrt{P_f^2 + M_C^2} = \frac{M_A^2 + M_C^2 - M_D^2}{2 M_A}$$

$$E_D^R = \sqrt{P_f^2 + M_D^2} = \frac{M_A^2 + M_D^2 - M_C^2}{2 M_A}$$

independent of emision angle θ_R

Notice that if $M_A < m_C + M_C \Rightarrow P_f$ imaginary

$\Rightarrow$ decay is not possible (≡ not kinematically allowed)

Lets look at the same process in a frame in which A is moving at $\vec{\beta} = \beta \hat{x}$

Let us call this frame "F".

We can use $P_{C,D}^F$ from $P_{C,D}^R$ using that they are 4-vectors so under the Lorentz boost from R to F they transform as

$$P_{C,D}^F = \Lambda_{\text{boost}\,-\beta\hat{x}}\, P_{C,D}^R$$

↑ in F A flies in R at rest $\Rightarrow$ F ≡ S, R ≡ S'

$$\begin{pmatrix} E_C^F \\ P_{C,x}^F \\ P_{C,y}^F \\ P_{C,z}^F \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_C^R \\ P_{C,x}^R = P_f \cos\theta_R \\ P_{C,y}^R = P_f \cos\theta_R \\ P_{C,z}^R = 0 \end{pmatrix}$$

$$\Rightarrow E_C^F = \gamma(E_C^R + \beta P_f \cos\theta_R)$$

$$P_{C,x}^F = \gamma(P_f \cos\theta_R + \beta E_C^R)$$

$$P_{C,y}^F = P_{C,y}^R = P_f \sin\theta_R$$

For D the expressions are the same with change

$$\sin\theta_R \to \sin(\pi + \theta_R) = -\sin\theta_R$$

$$\cos\theta_R \to \cos(\pi + \theta_R) = -\cos\theta_R$$

So the emission angle of C (- D) in the F frame

$$\tan\theta_C^F = \frac{P_{C,y}^F}{P_{C,x}^F} = \frac{P_f \sin\theta_R}{\gamma(P_f \cos\theta_R + \beta E_C^R)}$$

$$\tan\theta_D^F = -\frac{P_f \sin\theta_R}{\gamma(-P_f \cos\theta_R + \beta E_D^R)}$$

Take $\cos\theta_R > 0$ always possible to chose C, D that way

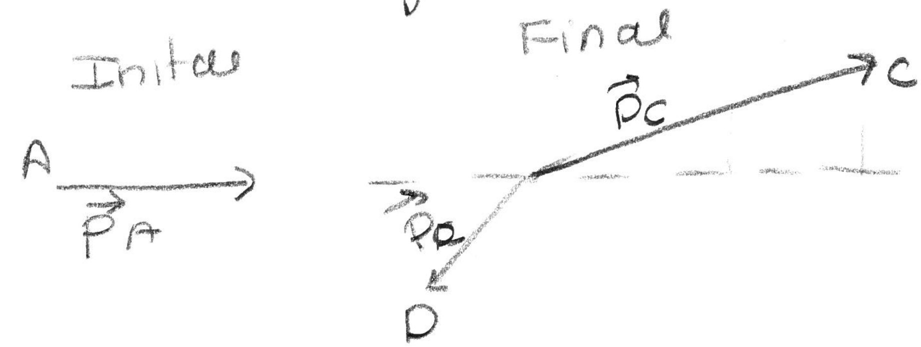
if $\beta \to 1 \Rightarrow \gamma$ very large

• $\tan\theta_C^F \simeq \frac{1}{\gamma}\,\frac{\sin\theta_R}{\cos\theta_R + \underbrace{\beta\,\frac{E_C^R}{P_f}}_{>0}} < \frac{1}{\gamma}\tan\theta_R \ll \tan\theta_R$

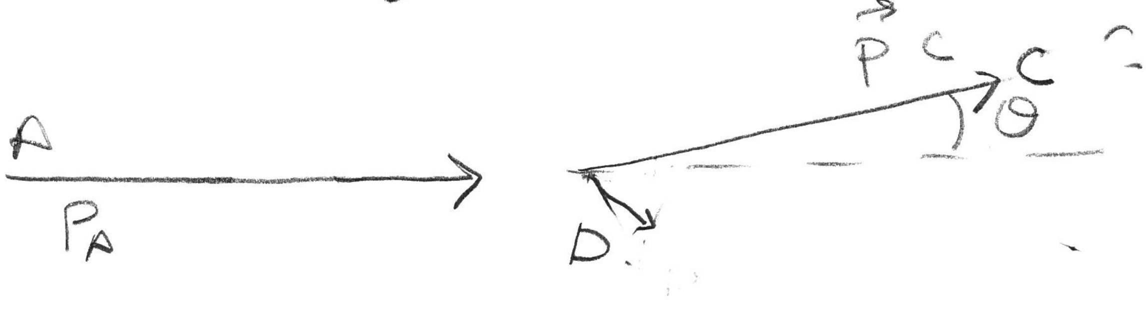
$\Rightarrow$ C is emitted closer to A flight direction

• $\tan\theta_D^F = \frac{1}{\gamma}\,\frac{-\sin\theta_R}{-\cos\theta_R + \beta\,\frac{E_C^R}{P_f}}$

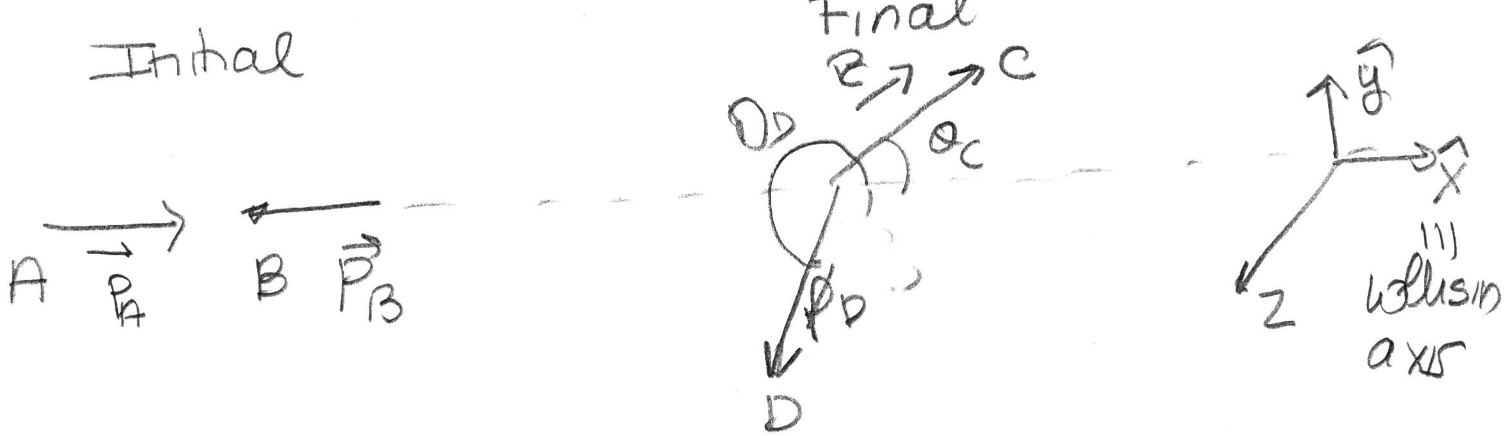
$\to$ if $\cos\theta_R > \beta\,\frac{E_C^R}{P_f} \Rightarrow P_{Dx}^F < 0 \Rightarrow$ D still emitted backwards wrt A

$\Rightarrow$ 

$\to$ If $\cos\theta_R < \beta\,\frac{E_C^R}{P_f} \Rightarrow P_{ex}^R > 0 \Rightarrow$ D also emitted forward



Example 2: $2 \to 2$ scattering $\quad A+B \to C+D$



In any reference frame

4-momenta $\rightarrow$ $P_{TOT}^{init} = P_{TOT}^{Final}$

$$P_A + P_B = P_C + P_D$$

$$\begin{pmatrix} E_A = \sqrt{|\vec{p}_A|^2 + M_A^2} \\ \vec{p}_A \end{pmatrix} + \begin{pmatrix} E_B = \sqrt{|\vec{p}_B|^2 + M_B^2} \\ \vec{p}_B \end{pmatrix} = \begin{pmatrix} E_C = \sqrt{|\vec{p}_C|^2 + m_C^2} \\ \vec{p}_C \end{pmatrix} + \begin{pmatrix} E_D = \sqrt{|\vec{p}_D|^2 + m_D^2} \\ \vec{p}_D \end{pmatrix}$$

$$\left.\begin{aligned} E_A + E_B &= E_C + E_D \\ \vec{p}_A + \vec{p}_B &= \vec{p}_C + \vec{p}_D \end{aligned}\right\} \Rightarrow 4 \text{ constraints}$$

We define the COM system as one where

$$\vec{p}_A + \vec{p}_B = 0 \Rightarrow |\vec{p}_A| = |\vec{p}_B| \equiv P_{in}$$

$$\Rightarrow \vec{p}_C + \vec{p}_D = 0 \Rightarrow |\vec{p}_C| = |\vec{p}_D| = P_f$$

C, D come out back to back $\Rightarrow \theta_D = \theta_C + \pi$

COM is the LAB system for symmetric colliders

i.e LEP e^+e^- $E_{e^+} = E_{e^-} \simeq 45$ GeV

Let us define the Mandelstam variable S

$$S \equiv (p_A + p_B)^2 = (E_A + E_B)^2 - |\vec{p}_A + \vec{p}_B|^2$$
$$= (p_C + p_D)^2 = (E_C + E_D)^2 - |\vec{p}_C + \vec{p}_D|^2$$

S is the norm of a 4-vector $\Rightarrow$ same value in any inertial frame. In COM

$$S = \sqrt{M_A^2 + P_i^2} + \sqrt{M_B^2 + P_i^2}$$
$$= \sqrt{M_C^2 + P_f^2} + \sqrt{M_C^2 + P_f^2}$$

if $\sqrt{S} \gg M_A, M_B$
M_C, M_D

solving

in COM

$$P_i = \frac{1}{2\sqrt{S}} \sqrt{(S + (M_A + M_B)^2)(S + (M_A - M_B)^2)} \to \frac{\sqrt{S}}{2}$$

$$P_f = \frac{1}{2\sqrt{S}} \sqrt{(S + (M_C + M_D)^2)(S + (M_C - M_D)^2)} \to \frac{\sqrt{S}}{2}$$

$$E_A = \frac{S + M_A^2 - M_B^2}{2\sqrt{S}} \longrightarrow \frac{\sqrt{S}}{2}$$

$$E_B = \frac{S + M_B^2 - M_A^2}{2\sqrt{S}} \longrightarrow \sqrt{S}/2$$

$$E_C = \frac{S + M_C^2 - M_D^2}{2\sqrt{S}} \longrightarrow \frac{\sqrt{S}}{2}$$

$$E_D = \frac{S + M_D^2 - M_C^2}{2\sqrt{S}}$$

If $\sqrt{s} < (M_C + M_D) \Rightarrow P_f$ imaginary $\Rightarrow$

$\Rightarrow$ collision $A+B \to C+D$ is not kinematically allowed.

Notice P_f, E_D, E_C in COM similar to the expression of P_f, E_D, E_C in $A \to C\,D$ in A rest frame with $\sqrt{s} \leftrightarrow M_A$

In general if we want to produce some states with total mass $M_{fin} = \sum_i M_{Fi}$

in a collision $A+B \to F_1 \dots F_N$

one needs $\sqrt{s} \gg M_{FIN}$

In a collider experiment with symmetric beam-beam collisions (take $E_{beam} \gg M_{beam}$)

$$E_{beam} = \frac{\sqrt{s}}{2} \gg \frac{M_{FIN}}{2}$$

In a fix target experiment (A on B at rest) $E_B = M_B$, $\vec{P}_B = 0$

$$S = (E_A + M_B)^2 - |\vec{P}_A|^2 \simeq E_A^2 + 2 E_A M_B + M_B^2 - E_A^2 + M_A^2$$

$$\simeq 2 E_A M_B = 2 E_{beam} M_B$$

$$\Rightarrow E_{beam} \gg \frac{M_{FIN}^2}{2 M_B}$$

To produce heavy particles we are better off with a collider experiment because we need to accelerate less the beams.

$$\text{i.e } m_H = 126 \text{ eV}$$

In e^+e^- collider $E_{e^+} = E_{e^-} \gtrsim 63 \text{ GeV}$

in $e^- p_{\text{Proton}}$ Fixtarget $\quad E_{e^-} > \frac{(126 \text{ GeV})^2}{2 \times 1 \text{ GeV}} \gtrsim 8000 \text{ GeV}$

particle at rest $\quad$ $\simeq m_p$