Chapter 4

"Relativistic wave equations for free particles

1) Relativistic scalar : Klein-Gordon Eq.

2) Relativistic spin 1/2 fermion : Dirac Equation
   Relativistic spinors

3) Spin 1 ≡ vectors $\begin{cases} \text{massless : Maxwell's Eq : photons} \\ \text{massive : Proacca Eq} \end{cases}$

4) Lagrangians for free particles

1 Relativistic scalar (scalar ⇒ spin $s=0$)

In non-relativistic QM the state of a scalar particle with mass M and velocity $\vec{v} = \frac{\vec{P}}{M}$ is described by a wave function $\phi(\vec{x}, t)$ which verifies

$$-\frac{|\vec{\nabla}|^2}{2M}\phi = i\frac{\partial \phi}{\partial t}$$ Schroedinger EQ

Notation $\nabla^i \equiv \frac{\partial}{\partial x^i} \equiv \partial_i$) $|\vec{\nabla}|^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

It can be obtained from the non-relativistic energy-momentum relation

$$E = \frac{|\vec{P}|^2}{2M}$$

substituting E and $\vec{P}$ by their operator $E \to P^0 = i\frac{\partial}{\partial t}$, $p^i \to P^i = i\frac{\partial}{\partial x_i} = -i\vec{\nabla}^i$

and $\rho = |\phi|^2 = \phi^*\phi$ gives the probability density

$\Rightarrow |\phi|^2 d^3x \equiv$ probability of finding particle in volume d^3x at time t

and normalization is $\int d^3x |\phi|^2 = 1$

From SE we can derive an eq for ρ

$$\frac{\partial \rho}{\partial t} = \phi \frac{\partial \phi^*}{\partial t} - \phi^* \frac{\partial \phi}{\partial t} = -\frac{i}{2M} \left(\phi \vec{\nabla}^2 \phi^* - \phi^* \vec{\nabla}^2 \phi \right)$$

$$= -\frac{i}{2M} \vec{\nabla} \left(\phi \vec{\nabla} \phi^* - \phi^* \vec{\nabla} \phi \right) \equiv -\vec{\nabla} \vec{J}$$

with $\vec{J} = \frac{i}{2M} \phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^*$

the eq $\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{J} = 0 \Rrightarrow$ continuity eq of fluid of density ρ and current $\vec{J}$

For plane waves $\phi = N e^{-i(Et - \vec{p}\vec{x})}$

$\rho = |N|^2$ and $\vec{J} = \frac{\vec{p}}{M} |N|^2 = \rho \vec{v}$ as expected

and the norm is $\int_{\text{unit vol}} d^3x \, \rho = |N|^2 = 1$

For a relativistic particle we can try the same but with $E^2 = |\vec{p}|^2 + M^2 \Rightarrow E = \pm\sqrt{|\vec{p}|^2 + M^2}$

(mathematically E can be positive or negative)

the eq is $-\frac{\partial^2 \phi}{\partial t} = -|\nabla|^2 \phi + M^2$

$$\Rightarrow \boxed{0 = \partial^\mu \partial_\mu \phi + M^2 \phi} \quad \text{with } \partial^\mu = \begin{pmatrix} \frac{\partial}{\partial t} \\ -i\vec{\nabla} \end{pmatrix}$$

$\equiv$ Klein-Gordon Eq

To give a probabilistic interpretation to ϕ we need to find the implied continuity eq

$$0 = \phi^*(Eq) - (Eq)^*\phi = \phi^*(\partial^\mu\partial_\mu \phi + \cancel{M^2}) - \phi(\partial^\mu\partial_\mu \phi^* + \cancel{M^2})$$

$$= \partial^\mu(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$$

which is a covariant form of a continuity eq $\partial^\mu J_\mu$ with $J^\mu = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix} = i \begin{pmatrix} \phi \partial^0 \phi - \phi \partial^0 \phi^* \\ -\phi \vec{\nabla} \phi + \phi \vec{\nabla} \phi^* \end{pmatrix}$

For plane wave sol $\phi = N e^{-ip_\mu x^\mu}$

$\Rightarrow \rho = 2|N|^2 E \Rightarrow$ solutions with $E = -\sqrt{|p|^2 + M^2}$ have negative density $\Rightarrow$ probabilistic interpretation seems to fail. Also notice ϕ: real $\Rightarrow \rho = 0$

the solution is found in QFT and introduction of antiparticles.

Notice that if we take $N = \frac{1}{\sqrt{V}} \Rightarrow \int \rho \, d^3x = 2E$

$\Rightarrow$ # of relativistic state in a fix volume grows with E as expected because of the contraction of length induction of motion by a factor $\frac{1}{\gamma} = \frac{M}{E}$

(2) Relativistic spin 1/2 particles - Dirac Eq

Dirac realized that the origin of problem of K-G eq was that it was second order in time derivatives which allowed for $E<0$ solutions.

So he searched for an eq which was relativistic but linear in $\frac{\partial}{\partial t}$.

The most general form of that eq will be

α coefficients

$$p^0\psi \equiv i\frac{\partial\psi}{\partial t} = \sum_{j=1}^{3}\alpha_j\left(-i\frac{\partial\psi}{\partial x_j}\right) + \beta M\psi$$

$$\equiv \left[\sum \alpha_j P^j + M\beta\right]\psi$$

$\alpha_{1,2,3}$ and β are coefficients to be determined by the condition $(p^0)^2\psi = (|\vec{P}|^2 + M^2)\psi = (\sum_j (P^j)^2 + M^2)\psi$

squaring the Eq

$$(P^0)^2\psi = (\sum \alpha_j P^j + M\beta)(\sum_k \alpha_k P^k + M\beta)\psi$$

$$= \left[\sum \alpha_j^2 (P^j)^2 + \sum_{j\neq k}(\alpha_j\alpha_k + \alpha_k\alpha_j) P^j P^k + M(\alpha_j\beta + \beta\alpha_j) + M^2\beta^2\right]\psi$$

$$\Rightarrow \left.\begin{array}{l} \alpha_j^2 = 1 = \beta^2 \\ \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \\ \alpha_j\beta + \beta\alpha_j = 0 \end{array}\right\} \Rightarrow$$

α_j, β must be anticommuting objects $\Rightarrow$ matrices

the smallest dimension that 4 matrices can have and verify these conditions is 4×4

If α_i, β are 4×4 matrices

$\Rightarrow \psi$ must be a 4×1 ($\equiv$ column) object $\equiv$ Spinor

We define the 4 Dirac matrices

$\gamma^0 \equiv \beta \qquad \gamma^i = \beta\alpha^i \qquad i = 1,2,3$

the conditions above can be written compactly as

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2 g^{\mu\nu} \, \underset{\uparrow \text{ left implicit}}{\mathbb{I}_{4\times4}}$$

with $g^{00} = 1 \qquad g^{ii} = -1$ and $g^{0i} = 0 = g^{i\neq j}$

[notice $\beta^2 = \gamma^{0\,2} = \frac{1}{2} 2 g^{00} = 1$

$\alpha^{i\,2} = (\beta\gamma^i)^2 = \beta\gamma^i\beta\gamma^i = -\beta^2 \gamma^{i\,2} = -1 g^{ii} = 1$]

they verify $\gamma^{\mu +} = \gamma^0\gamma^\mu\gamma^0 = \begin{cases} \gamma^{0+} = \gamma^{0\,3} = \gamma^0 \\ \gamma^{i+} = -\gamma^i \end{cases}$

Dirac Matrices can have different numerical forms

= different representations. F.e

- Pauli-Dirac rep $\quad \gamma^0 = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & I_{2\times2} \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$

$\sigma_j \equiv$ Pauli matrices $\quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- chiral rep $\quad \gamma_0 = \begin{pmatrix} 0 & I_{2\times2} \\ I_{2\times2} & 0 \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$

Dirac Eq $\quad \gamma^0 \left[i\frac{\partial\psi}{\partial t} = \alpha_i^j \left(-i\frac{\partial\psi}{\partial x^j}\right) + M\beta\psi \right]$

$$= i\gamma^0 \frac{\partial\psi}{\partial x^0} + i\gamma^j \frac{\partial\psi}{\partial x^j} - M\psi = 0$$

$$\boxed{i\gamma^\mu \frac{\partial\psi}{\partial x^\mu} - M\psi = 0} \equiv \left[i\gamma^\mu \partial_\mu \psi - m\psi = 0 \right] \equiv (1)$$

$\Psi(x) \equiv 4\times 1$ spinor $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ but not a 4-vector

We define the conjugate spinor $\overline{\Psi} \equiv \psi^{+}\gamma^{0} \equiv (1\times 4)$
$\equiv (\overline{\Psi}_1, \overline{\Psi}_2, \overline{\Psi}_3, \overline{\Psi}_4)$

and verifies $\boxed{i\,\partial_\mu \overline{\Psi} + M\overline{\Psi} = 0} \equiv (2)$

$$\overline{\Psi}(1) + (2)\,\overline{\Psi} = 0 = i\,\partial_\mu(\overline{\Psi}\gamma^\mu\psi) \equiv i\partial_\mu J^\mu$$

$\Rightarrow$ continuity eq of fluid with density

$$\rho = J^0 = \overline{\Psi}\gamma^0\psi = \psi^{+}\psi = |\Psi|^2 > 0$$

$\Rightarrow$ admits a QM probabilistic interpretation

Dirac Eq is a set of 4x4 differential linear Eq.

$\Rightarrow$ it admits 4 independent solutions.

For states of well defined 4-momentum ($\equiv$ plane waves)

we can define $\psi(x) \equiv u(p)\, e^{-ip_\mu x^\mu}$ — 4-spinor in momentum space

and introducing in (1) we get the Dirac Eq for $u(p)$

$$(\gamma^\mu p_\mu - M)\,u(p) = 0$$

4 linear algebraic system
$\Rightarrow$ 4 independent solutions

Equivalently for $\overline{\Psi}(x) \equiv \overline{u}(p)\, e^{+ip_\mu x^\mu}$
with $\overline{u}(p) = u^{+}(p)\,\gamma^0$ which verifies

$$\overline{u}(p)\cdot(\gamma^\mu p_\mu - m) = 0$$

Let us find the explicit form of these 4-solutions
in the chiral rep.

$$\gamma^\mu p_\mu = \gamma^0 E - \vec{\gamma}\vec{p} = \begin{pmatrix} 0 & E-\vec{\sigma}\vec{p} \\ E+\vec{\sigma}\vec{p} & 0 \end{pmatrix} u(p) = M u(p)$$

Let us start at rest $\vec{p}=0$

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} u(p) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} u(p)$$

2x1 objects

which has 4 independent solutions

$$u^{(1)} \equiv N \begin{pmatrix} \xi^{(1)} \\ \xi^{(1)} \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} \xi^{(2)} \\ \xi^{(2)} \end{pmatrix}$$

with $\xi^{(1)}\xi^{(2)+} = 0$

normalization

which verify $E\, u^{(s)} = M\, u^{(s)} \Rightarrow E = M > 0 \quad s=1,2$

and

$$u^{(3)} \equiv N \begin{pmatrix} \xi'^{(1)} \\ -\xi'^{(1)} \end{pmatrix} \qquad u^{(4)} \equiv N \begin{pmatrix} \xi'^{(2)} \\ -\xi'^{(2)} \end{pmatrix}$$

$\xi'^{(1)}\xi'^{(2)+} = 0$

which verify

$$E\, u^{(s)} = -M\, u^{(s)} \Rightarrow E = -M < 0 \text{ for } s=3,4$$

For $\vec{p} \neq 0$ the two spinors corresponding to $s=1,2$

$$u^{(s=1,2)}_{r}(p) = N \begin{bmatrix} \left\{ \sqrt{E-|\vec{p}|}\,\frac{1}{2}\left(1+\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) + \sqrt{E+|\vec{p}|}\,\frac{1}{2}\left(1-\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) \right\} \xi_p^s \\ \left\{ \sqrt{E-|\vec{p}|}\,\frac{1}{2}\left(1-\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) + \sqrt{E+|\vec{p}|}\,\frac{1}{2}\left(1+\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) \right\} \xi_p^s \end{bmatrix}$$

$$\equiv \bar{N} \begin{bmatrix} \sqrt{p\sigma}\; \xi_p^s \\ \sqrt{p\bar{\sigma}}\; \xi_p^s \end{bmatrix} \quad \text{for } s = 1,2$$

introducing them in the Eq $\Rightarrow E = \sqrt{|\vec{p}|^2 + M^2} > 0$

If we normalize the bispinors as $\xi^{(r)}\xi^{\dagger(s)} = \delta^{rs}$

then for these solutions

$$\underbrace{\int d^3x\, |\psi|^2}_{V} \equiv N\, V\, u^{s\dagger}(p)\, u^s(p) = 2EN^2V \Rightarrow N = \frac{1}{\sqrt{V}}$$

$\equiv 1$ (unit volume)

For $\vec{p} \neq 0$ the two spinors with $s=3,4$

$$u^{(3,4)} = \begin{bmatrix} \left\{ \sqrt{-E-|\vec{p}|}\,\frac{1}{2}\left(1-\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) + \sqrt{-E+|\vec{p}|}\,\frac{1}{2}\left(1+\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) \right\} \xi_p'^{(1,2)} \\ -\left\{ \sqrt{-E-|\vec{p}|}\,\frac{1}{2}\left(1+\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) + \sqrt{-E+|\vec{p}|}\,\frac{1}{2}\left(1-\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\right) \right\} \xi_p'^{(1,2)} \end{bmatrix}$$

and introducing them in the Eq

$$E = -\sqrt{|\vec{p}|^2 + M^2} < 0$$

We interpret these $E<0$ particle spinors $u^{3,4}$ as spinors of the antiparticle. We define the antiparticle spinors $v^{(1)}, v^{(2)}$

$$v^{(1)}(E>0, \vec{p}) \equiv u^{(4)}(-E, -\vec{p})$$

$$v^{(2)}(E>0, \vec{p}) \equiv u^{(3)}(-E, -\vec{p})$$

So

$$v^{(s)} = \begin{bmatrix} \left\{ \sqrt{E-|\vec{p}|}\, \frac{1}{2}\left(1+\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right) + \sqrt{E+|\vec{p}|}\, \frac{1}{2}\left(1-\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right) \right\} \eta_p^{s} \\ -\left\{ \sqrt{E+|\vec{p}|}\, \frac{1}{2}\left(1-\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right) + \sqrt{E+|\vec{p}|}\, \frac{1}{2}\left(1+\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right) \right\} \eta_p^{s} \end{bmatrix}$$

$$\eta_p^{(1)} = \xi_p'^{(2)} \qquad \eta_p^{(2)} = \xi_p'^{(1)}$$

so for $v^{(1,2)}$ $\quad E = \sqrt{|\vec{p}|^2 + m^2} > 0$

So $u^{(1,2)}$ and $v^{(1,2)}$ represent two states of a particle and 2 states of their antiparticle all with 4-momentum $\left(E = \sqrt{|\vec{p}|^2+m^2}, \vec{p}\right)$

What differenciates $u^{(1)}$ ~~from~~ $u^{(2)}$ (and $v^{(1)}$ from $v^{(2)}$)

Notice that $\frac{1}{2}\frac{\vec{\sigma}\vec{p}}{|\vec{p}|} \equiv$ helicity operator in QM

$\equiv$ projection of spin operator $\vec{S} \equiv \frac{\vec{\sigma}}{2}$ over 3-momentum

If we chose ξ_p^1 so that $\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\xi_p^{(1)} = \xi_p^{(1)} \equiv \xi_p^{+}$

ξ_p^2 " $\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\xi_p^{(2)} = -\xi_p^{(2)} \equiv -\xi_p^{-}$

then $u_p^{(1)} \equiv u_p^{(+)}$ represents a fermion

$\overset{\vec{s}}{\Rightarrow}$
$\xrightarrow[\vec{p}]{}$ possitive helicity u-spinor

and $u_p^{(2)} \equiv u_p^{(-)}$ represents a fermion

$\overset{\vec{s}}{\Leftarrow}$
$\xrightarrow[\vec{p}]{}$ negative helicity u-spinor

In the chiral representation

$$u_p^{(1)}(p) \equiv u^{+}(p) = \begin{pmatrix} \sqrt{E-|\vec{p}|}\,\xi_p^{+} \\ \sqrt{E+|\vec{p}|}\,\xi_p^{+} \end{pmatrix} \xrightarrow[(E \simeq |p|)]{\substack{\text{ultrarelativistic} \\ E \gg M}} \begin{pmatrix} 0 \\ \sqrt{2E}\,\xi_p^{+} \end{pmatrix}$$

$$u^{(2)}(p) \equiv u^{-}(p) = \begin{pmatrix} \sqrt{E+|\vec{p}|}\,\xi_p^{-} \\ \sqrt{E-|\vec{p}|}\,\xi_p^{-} \end{pmatrix} \xrightarrow{E \gg M} \begin{pmatrix} \sqrt{2E}\,\xi_p^{-} \\ 0 \end{pmatrix}$$

For v spinors we can chose $\eta^{(1,2)}$ also as eigenstates of $\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}$ but we must take into account that angular momentum operator for antiparticles comes with a $(-)$ sign

So the positive helicity state $\eta^{+} = \eta^{1} \equiv \xi^{-}$

negative " " $\eta^{-} = \eta^{2} \equiv -\xi^{+}$

with this

$$v^{(+)}(p) \equiv v^{(1)}(p) = \begin{bmatrix} \sqrt{E+|\vec{p}|}\,\xi_p^{-} \\ -\sqrt{E-|\vec{p}|}\,\xi_p^{-} \end{bmatrix} \xrightarrow{E \gg M} \begin{bmatrix} \sqrt{2E}\,\xi_p^{-} \\ 0 \end{bmatrix}$$

$$v^{(-)}(p) \equiv v^{(2)}(p) = \begin{bmatrix} \sqrt{E-|\vec{p}|}\,\xi_p^{+} \\ -\sqrt{E+|\vec{p}|}\,\xi_p^{+} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \sqrt{2E}\,\xi_p^{+} \end{bmatrix}$$

Let us introduce a matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

In the chiral rep $\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}$ (in Dirac Pauli $\gamma^5 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$)

It verifies $\{\gamma^5, \gamma^\mu\} = 0$ $\gamma^{5+} = \gamma^5$, $(\gamma^5)^2 = I$

We define the chiral lefthanded and chiral right handed projectors

$$P_L \equiv \frac{1}{2}(1-\gamma^5) \overset{\text{in chiral rep}}{=} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} ; \quad P_R \equiv \frac{1}{2}(I+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

So

$$u_L^+ \equiv P_L u^+ = \begin{pmatrix} \sqrt{E-|\vec{p}|}\,\xi_p^+ \\ 0 \end{pmatrix} \xrightarrow[E \gg M]{\text{ultra relativ.}} 0$$

$$u_R^+ \equiv P_R u^+ = \begin{pmatrix} 0 \\ \sqrt{E+|p|}\,\xi_p^+ \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \sqrt{2E}\,\xi_p^+ \end{pmatrix} = u^+$$

$$u_L^- \equiv P_L u^- = \begin{pmatrix} \sqrt{E+|p|}\,\xi_p^- \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \sqrt{2E}\,\xi_p^- \\ 0 \end{pmatrix} = u^-$$

$$u_R^- \equiv P_R u^- = \begin{pmatrix} 0 \\ \sqrt{E-|\vec{p}|}\,\xi_p^- \end{pmatrix} \longrightarrow 0$$

So for an ultrarelativistic u spinor
positive helicity u-spinor $\simeq$ chiral right-handed
negative " u " $\simeq$ " left-handed

For v-spinors

$$v_L^+ \equiv P_L v^+ = \begin{pmatrix} \sqrt{E+|\vec{p}|}\,\xi_p^- \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \sqrt{2E}\,\xi_p^- \\ 0 \end{pmatrix} = v^+$$

$$v_R^+ \equiv P_R v^+ = \begin{pmatrix} 0 \\ -\sqrt{E-|\vec{p}|}\,\xi_p^- \end{pmatrix} \longrightarrow 0$$

$$v_L^- \equiv P_L v^- = \begin{pmatrix} \sqrt{E-|\vec{p}|}\,\xi_p^+ \\ 0 \end{pmatrix} \longrightarrow 0$$

$$v_R^+ = P_R v^- = \begin{pmatrix} 0 \\ \sqrt{E+|\vec{p}|}\,\xi_p^+ \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \sqrt{2E}\,\xi_p^+ \end{pmatrix} = v^-$$

So for ultrarelativistic v-spinors

positive helicity v-spinor $\simeq$ chiral left-handed

negative " v- " $\simeq$ " right-handed

BEWARE: many book (H&M, P&S) call "left-handed" to a fermion of negative helicity irrespective of weather it is a fermion or an anti fermion. And "right-handed" fermion to a fermion with positive helicity

So for them an ultrarelativistic "left-handed" anti fermion has a chiral-right handed" v-spinor. which is confusing.

In this class we will call chiral left and right-handed to the mathematical objects coming from spinors to the application of the chiral projection $P_{L/R} = \frac{1}{2}(I \mp \gamma_5)$

and we call positive/negative helicity spinor to the physical projection of its spin over 3-momentum

Notice that chiral left and right definitions are independent of reference frame

But helicity depends on ref frame. So if using the "labeling" of H&M and P&S leads to apparent paradoxes

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Spinors are 4 dimensional objects but they are not 4-vectors. A 4 vector a^μ transforms under a Lorentz transformation Λ as $a'^\mu = \Lambda^\mu{}_\nu a^\nu$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'}$$

A spinor $\psi'(x') = S_\Lambda \psi(x)$, $x' = \Lambda x$

S_Λ → matrix representing Λ transf in spinor space

Since
$$i\gamma^\mu \frac{\partial \psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0$$
$$i\gamma^\mu \frac{\partial \psi(x)}{\partial x} - m\psi(x) = 0$$
$$\Rightarrow S_\Lambda^{-1}\gamma^\mu S_\Lambda = \Lambda^\mu{}_\nu \gamma^\nu$$

So $\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}$ is a 4 vector

So under parity
$$\left\{\begin{array}{l} S_P^{-1}\gamma^0 S_P = \gamma^0 \\ S_P^{-1}\gamma^i S_P = -\gamma^i \end{array}\right\} \Rightarrow S_P = \gamma^0$$
$$\Rightarrow \psi_P(x) = \gamma^0\psi(x)$$

So under parity
$$\overline{\psi}_P = \psi_P^+\gamma^0 = \psi^+\gamma^{0+}\gamma^0 = \overline{\psi}\gamma^0$$

$$\Rightarrow \overline{\psi}_P\psi_P = \overline{\psi}\psi$$

with this we can deduce the Lorentz transf of bilinear of spinors

bilinear	# components	under P	
$\bar\psi\psi$	1	$\bar\psi\psi$	$\Rightarrow$ scalar
$\bar\psi\gamma^\mu\psi$	4	$\bar\psi\gamma^0\gamma^\mu\gamma^0\psi \begin{cases}\bar\psi\gamma^0\psi \\ -\bar\psi\gamma^i\psi\end{cases}$	$\Rightarrow$ 4vector
$\bar\psi\gamma^5\psi$	1	$\bar\psi\gamma^0\gamma^5\gamma^0\psi = -\bar\psi\psi$	$\Rightarrow$ pseudo scalar
$\bar\psi\gamma^\mu\gamma^5\psi$	4	$\bar\psi\gamma^0\gamma^\mu\gamma^5\gamma^0\psi \begin{cases}-\bar\psi\gamma^0\psi \\ \bar\psi\gamma^i\psi\end{cases}$	$\Rightarrow$ pseudo vector (or axial vector)

3 Spin 1 ($\equiv$ vectors)

3.1 Massless ; let us take the photon. In classical e/m the $\vec{E}$ and $\vec{B}$ fields generated by charge density ρ and current $\vec{J}$ verify Maxwell's

$$\vec{\nabla}\vec{E} = \rho \quad (1) \qquad \vec{\nabla}\wedge\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \quad (2) \qquad \vec{\nabla}\vec{B} = 0 \quad (3) \qquad \vec{\nabla}\wedge\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J} \quad (4)$$

Defining the strength tensor
$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The 4 Maxwell's eq can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu \qquad \text{with } J^\nu = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$$

and since $F^{\mu\nu}$ is antisymmetric under $\mu \leftrightarrow \nu$

$$\Rightarrow \partial_\nu\partial_\mu F^{\mu\nu} = 0 = \partial_\nu J^\nu = 0 \Rightarrow \text{continuity eq.}$$

$$(3) \Rightarrow \vec{B} \equiv \vec{\nabla}\wedge\underset{\text{vector pot}}{\vec{A}} \Rightarrow (2)\ \vec{\nabla}\wedge\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} \equiv -\vec{\nabla}\underset{\text{scalar pot}}{\phi}$$

we define a 4-vector pot $A^\mu \equiv \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix}$

and with the def of $F^{\mu\nu}$ in terms of $\vec{E}, \vec{B}$ and of $\vec{E}, \vec{B}$ in terms of A^μ one gets that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

So Maxwell's eq in vacuum ($J^\mu=0$) are

$$\partial_\mu F^{\mu\nu}=0 \Rightarrow \boxed{\partial_\mu\partial_\nu A^\mu-\partial_\mu\partial_\nu A^\mu=0}$$

$\equiv$ system of 4 second order diff eq.

In particle physics we identify A^μ with the "wave-function" of the photon (ie given same $F^{\mu\nu}$)

Problem is that given $\vec{E}$ and $\vec{B}$ which are the physical observables A^μ is not unique. Te take so A^μ, so $\vec{B}=\vec{\nabla}\wedge\vec{A}$ $\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\phi$

we get the same $\vec{B}$ and $\vec{E}$ with $A'^\mu=A^\mu+\partial^\mu\chi$ for any χ function

You can check that $\vec{\nabla}\wedge\vec{A}=\vec{\nabla}\wedge A'$ and $-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\phi=-\frac{\partial\vec{A}'}{\partial t}-\vec{\nabla}\phi'$

or more easily

$$F'^{\mu\nu}=\partial^\mu A'^\nu-\partial^\nu A'^\mu=\partial^\mu A^\nu-\partial^\nu A^\mu+\underbrace{\cancel{\partial^\mu\partial^\nu\chi-\partial^\nu\partial^\mu\chi}}_{0}$$

$$=F^{\mu\nu}$$

using an specific form of $\chi\equiv$ fixing gauge

using this freedom of choice ($\equiv$ gauge invariance) we can choose to work in a gauge in which

$$\partial^\mu A_\mu=0 \quad \text{and} \quad A^0=0$$

this is the transverse (≡ Coulomb) gauge Lorentz

In this gauge the wave eq for the photon is

$$\boxed{\partial^\mu \partial_\mu A^\nu = 0}$$

polarization vector ≡ wave funct of photon in momentum space

the plane wave solution $A^\mu(x) = \epsilon^\mu(p)\, e^{-ip^\alpha x_\alpha}$

Introducing this in the eq

$$0 = p^\mu p_\mu \left(\epsilon^\nu e^{-ip^\alpha x_\alpha}\right) = m_\gamma^2 \left(\epsilon^\nu e^{-ip^\alpha x_\alpha}\right)$$

⇒ photon massless

$\epsilon(p)$ is a 4 vector which can be written in a basis $\epsilon_r(p)$ $r = 0, 1, 2, 3$

$$\epsilon^\mu(p) = \sum_{r=0}^{3} \alpha_r\, \epsilon_r^\mu(p)$$

we can chose the basis

$$\epsilon_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \epsilon_{1,2} = \begin{pmatrix}0\\ \vec{\epsilon}_{1,2}\end{pmatrix} \text{ with } \vec{p}\cdot\vec{\epsilon} = 0$$

transverse polarization vector

and $\epsilon_3 = \begin{pmatrix}0\\ \frac{\vec{p}}{|\vec{p}|}\end{pmatrix}$ longitudinal pol. vector

4 product

They verify $\epsilon_r^*(p)\cdot\epsilon_s(p) = -\xi_r \delta_{rs}$ $\quad \xi_0 = -1$, $\xi_{1,2,3} = 1$

completness $\sum_r \xi_r\, \epsilon_r^{*\mu}(p)\, \epsilon_r^\nu(p) = -g^{\mu\nu}$

In the coloumb gauge

$$\left.\begin{array}{l}\partial^\mu A_\mu = 0 \Rightarrow P^\mu \epsilon_\mu = 0 \equiv E\alpha_0 - |\vec{p}|\alpha_3 \\ A^0 = 0 \Rightarrow \epsilon^0 = 0 \Rightarrow \alpha_0 = 0\end{array}\right\} \Rightarrow \alpha_3 = \alpha_0 = 0$$

$\Rightarrow$ only ϵ_1 and ϵ_2 are physical $\equiv$ only 2 polarisations (as it should be because $m_\gamma = 0$) and the two polarisations are transverse to the direction of the photon

3.2 For a massive vector of mass M, let us call V^μ its wave function.

the wave eq can be obtained from ME with substitution $\partial^\mu\partial_\mu \to \partial^\mu\partial_\mu + M^2$

$$\boxed{\partial^\mu\partial_\mu V^\nu + M^2 V^\nu - \partial^\nu\partial_\mu V^\mu = 0} \equiv \text{Procca Eq.}$$

taking $\partial_\nu\left[\partial^\mu\partial_\mu V^\nu + M^2 V^\nu - \partial^\nu\partial_\mu V^\mu\right] = 0 \Rightarrow \partial_\nu V^\nu = 0$

but now $\partial_\nu V^\nu$ is a physical condition not a gauge fixing the plane wave solutions $V^{\mu}(x) = \epsilon^{\mu}(p)\, e^{-ip^{\alpha}x_{\alpha}}$

condition $\partial_{\mu} V^{\mu} = 0 \Rightarrow$ one polarization can be eliminated $\Rightarrow$ massive vector has 3 independent polarization states which correspond to $m = -1, 0, 1$ states of a $s=1$ massive particle

Two of the are transverse to ~~this~~ direction

$$\epsilon_{1,2} = \begin{pmatrix} 0 \\ \vec{\epsilon}_{1,2} \end{pmatrix} \quad \text{with } \vec{\epsilon}_1 \vec{p} = \vec{\epsilon}_2 \vec{p} = 0 \quad \text{like the phot}$$

and the 3rd one is longitudinal and to unity

$$\epsilon_3^{\mu} = \begin{pmatrix} \frac{|\vec{p}|}{M} \\ \frac{E}{M} \frac{\vec{p}}{|\vec{p}|} \end{pmatrix}$$

$$\sum_{r=1}^{3} \epsilon_r^{\mu} \epsilon_r^{\nu *} = -g^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{M^2}$$

4 Lagrangians for free particles

We have introduced the equations for relativistic scalars, fermions and vectors.

these eq: can be derived from the principle of minimal action for a corresponding Lagrangian which is a function of these wave functions

Let us call ϕ_a and $\partial_\mu \phi_a$ the wave function of some particle "a"

Let us call $\mathcal{L}(x) \equiv \mathcal{L}(\phi_a, \partial_\mu \phi_a)$ the Lagrangian density

$\Rightarrow$ the Lagrangian $L(t) = \int d^3x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$

$\Rightarrow$ the action $S = \int dt\, L(t) = \int d^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$

Principle of minimal action $\Rightarrow \phi_a$ must evolve along trajectories which minimize $S \Rightarrow \delta S = 0$

$$\delta S = \delta \left[\int d^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a) \right] \qquad \partial_\mu(\delta\phi_a) = \delta(\partial_\mu\phi_a)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right] \underset{\text{integrals by part}}{=}$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a + \underbrace{\cancel{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu \phi_a} \delta\phi_a \right)}}_{\text{total derivative}} \right]$$

So $\delta S = 0 \Rightarrow \boxed{\dfrac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \dfrac{\mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0}$ $\equiv$ Euler-Lagrange Eq.

- For $s=0$ scalar ϕ (complex so two fields ϕ, ϕ^*)

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi$$

taking $\phi_a = \phi^*$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} = \partial_\mu \phi$$

E-L Eq

$$\Rightarrow \boxed{\partial^\mu \partial_\mu \phi + m^2 \phi = 0}$$

K-G eq

- For $S=0$ but ϕ real only one field

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2\right] \Rightarrow \text{KG eq}$$

- For $s=1/2$ two wave funct. ψ and $\bar{\psi}$

$$\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = -m\psi + i\gamma^\mu \partial_\mu \psi \xrightarrow{\text{E-L}} \boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0}$$

Dirac Eq

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

• For $s = 1$ massless and real A^μ

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad \text{with } F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}$$

$$\overset{\text{E-L}}{\Downarrow} \Rightarrow \boxed{\partial_\mu F^{\mu\nu} = 0}$$

$\equiv$ Maxwell's

• For $s = 1$ massive and complex V^μ, $V^{\mu\dagger}$

$$\mathcal{L} = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu\dagger} + m_V^2 V^\mu V_\mu^\dagger$$

$$\frac{\partial \mathcal{L}}{\partial V^{\mu\dagger}} = m_V^2 V^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu V^\nu)} = -F_{\mu\nu}$$

$$\overset{\text{E-L}}{\Downarrow} \Rightarrow \boxed{\partial_\mu F^{\mu\nu} + M^2 V^\nu = 0}$$

$\equiv$ Procca Eq

• For $s = 1$ massive but real

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_V^2 V_\mu V^\mu$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} + m^2 V^\nu = 0}$$