Chapter 5

QED I: Feynman amplitudes and Feynman diagrams

1) Non relativistic perturbation theory
2) Interaction of e^- with electromagnetic field
3) $\mu^- e^- \to \mu^- e^-$
4) Propagators
5) Feynman rules of QED
6) Lagrangians for QED as a gauge theory

1) Non-relativistic perturbation theory

Starting point are the solutions of Schröedinger eq for free particles

$$H_0 \phi_n = E_n \phi_n$$

$H_0(\vec{x}) \equiv$ time independen $\Rightarrow \phi = \phi(\vec{x})$

Normalised as $\int d^3x \, \phi_m^*(x)\phi_n(x) = \delta_{mn}$ $\Rightarrow$ 1 particle per unit volume

If we switch a potential $V(\vec{x}, t)$ the new Eq

$$[H_0 + V(\vec{x}, t)]\phi = i \frac{\partial \phi}{\partial t}$$

and its solutions can be expressed in terms of the basis made with $\phi_n(\vec{x})$

$$\Psi(x) = \sum_n a_n(t) \, \phi_n(\vec{x}) \, e^{-iE_n t}$$

Solve this eq $\equiv$ find the coefficients $a_n(t)$ which verify

$$i \sum_m \frac{da_m}{dt} \cdot \phi_m(\vec{x}) e^{-iE_m t} = \sum_n V(\vec{x}, t) \, a_n(t) \phi_n(\vec{x}) e^{-iE_n t}$$

(using that $H_0 \phi_n = E_n \phi_n$)

multiplying by ϕ_f^* and integrating over d^3x

$$i \frac{da_f}{dt} = \sum_n a_n(t) \int d^3x \, \phi_f^* e^{iE_f t} V(\vec{x}, t) \, \phi_n(\vec{x}) e^{-iE_n t}$$

which can be expressed in integral form integrals from some initial condi $a_i(-T/2) = 1$

$a_{f \neq i}(-T/2) = 0$

$$a_f(t) = \underset{\substack{\parallel \\ a_f^{(0)}(t)}}{\delta_{fi}} - i \int_{-T/2}^{t} dt' \sum_n a_n(t') \int d^3x\, \phi_f^*(\vec{x})\, e^{iE_f t'}\, V(\vec{x}, t')\, \phi_n(\vec{x})\, e^{-iE_n t'}$$

order zero
particle does not interact
with (V)

We can solve iteratively by introducing $a_f^{(0)}(t)$ in r.s.h.m

$$a_f^{(1)}(t) = a_f^{(0)}(t) - i \int_{-T/2}^{t} dt'\, d^3x\, \underbrace{\phi_f^*(\vec{x})\, e^{iE_f t'}}_{\phi_f^*(x)}\, \underbrace{V(\vec{x}, t)}_{V(\vec{x})}\, \underbrace{\phi_i(x)\, e^{-iE_i t'}}_{\phi_i(x)}$$

In particular a $T/2 \xrightarrow{\text{large}} -\infty$

$$T_{fi}^{(1)} \equiv a_{fi}^{(1)}(T/2) - \delta_{fi} = -i \int d^4x\, \underbrace{\phi_f^*(x) V(x) \phi_i(x)}_{\text{written in covariant form.}}$$

For a time indep potential the eq

$$a_f(t) = \delta_{fi} - i\int_{-T/2}^{t} dt' \sum_n e^{i(E_f - E_n)t'} a_n(t') \underbrace{\int d^3x\, \phi_f^*(\vec{x}) V(\vec{x}) \phi_n(\vec{x})}_{V_{fn}}$$

and 1st order solution

$$a_f^{(1)}(t) = a_f^{0}(t) - i\int_{-T/2}^{t} V_{fi}\, e^{i(E_f - E_i)t'} dt'$$

at second order

$$a_f = a_f^{(1)} \underbrace{- i\int_{-T/2}^{t} dt' \sum_n (-i) \int_{-T/2}^{t'} dt''\, e^{i(E_n - E_i)t''} V_{ni}^{0}\, e^{i(E_f - E_n)t'} V_{fn}}_{\equiv\, a_f^{(2)}(t)}$$

$$T_{fi}^{(1)} = -i V_{fi} \int_{-\infty}^{\infty} dt'\, e^{i(E_f - E_i)t} = -i(2\pi) V_{fi}\, \delta(E_f - E_i)$$

$$T_{fi}^{(2)} = (-i)^2 \sum_n V_{ni} V_{fn} \int_{-\infty}^{\infty} dt'\, e^{i(E_f - E_n)t'} \int_{-\infty}^{t'} e^{i(E_n - E_i)t''} dt''$$

To perform the integral in dt'' we introduce a small $i\epsilon t''$ and take the limit $\epsilon \to 0$

$$\int_{-\infty}^{t'} dt''\, e^{i(E_n - E_i - i\epsilon)t''} = \frac{i\, e^{i(E_n - E_i - i\epsilon)t'}}{E_i - E_n + i\epsilon}$$

on integrating $\displaystyle\int_{-\infty}^{\infty} dt' \, e^{i(E_f - E_n)} e^{i(E_n - E_i)t}$

$$= 2\pi \delta(E_f - E_i)$$

so at 2nd order

$$T^{(2)}_{fi} = (-i)^2 (2\pi i) \sum_n \frac{V_{fn} V_{ni}}{E_i - E_n + i\epsilon} \, \delta(E_f - E_i)$$

We can think of this perturbative expansion graphically

⇒ For each interaction vertex we get a factor V_{ni}

⇒ The intermediate (not initial nor final) state "n" is represented by a "propagator" $\dfrac{i}{E_i - E_n + i\epsilon}$

the intermediate state is "virtual" ≡ does not conserve energy since $E_i \neq E_n$

But energy conservation in full process is guaranteed by $\delta(E_i - E_f)$

(2) Interaction of e^- with em field

So far we have done non-relativistic perturbation expansion. But we have been able to write the first order in the expansion in a covariant form

$$T_{if} = -i\int d^4x\, \Phi_f^*(x)\, V(x)\, \Phi_i(x)$$

which we are going to use for relativistic systems

So let's start with e^- with 4 momentum p described by the 4-spinor

$$\psi(x) = u(p)\, e^{-ipx} \quad \text{"}p^\alpha x_\alpha\text{"}$$

which has the free eq DE: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

In classical em the motion of a particle of charge $-e$ in an em field A^μ is obtained with the substitution $p^\mu \to p^\mu + eA^\mu$

the QM version $i\partial^\mu \to i\partial^\mu + eA^\mu$

so the Dirac Eq in presence of A^μ

$$i\gamma^\mu \partial_\mu \psi - m\psi = -e\gamma^\mu A_\mu \psi \equiv \gamma^0 V \psi$$

introduce so $E \to E + V$.

with $V = -e\gamma^0\gamma^\mu A_\mu$

using this potential we can get the 1st order

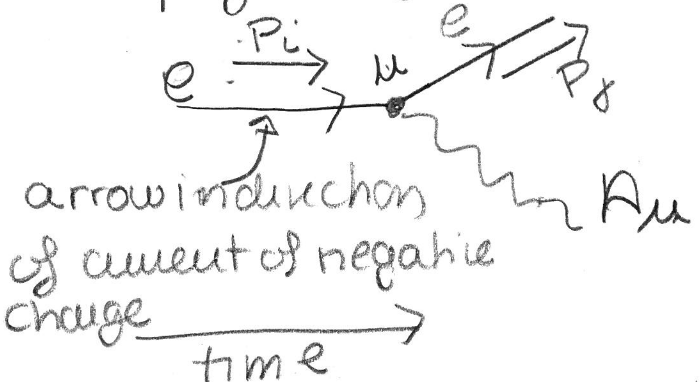
$$T_{fi} = ie \int \underbrace{\psi_f^{+}(x)\, \gamma^0}_{\bar{\psi}(x)} \gamma^\mu A_\mu \psi_i(x)\, d^4x \equiv -i \int d^4x\, j^\mu_{fi} A_\mu$$

so, $$j^\mu_{fi} = -e\, \bar{\psi}_f \gamma^\mu \psi_i = -e\, \underset{\bar{u}_f}{\bar{u}_f(p_f)}\, \gamma^\mu\, \underset{u_i}{u(p_i)}\, e^{i(p_f - p_i)x}$$

$$\Rightarrow T_{fi} = \int d^4x\, [ie\, \bar{u}_f \gamma^\mu u_i]\, e^{i(p_f-p_i)x}$$

is the electromagnet current for the electron $i \to f$ transition (notation $\gamma^\mu A_\mu \equiv \not{A}$)

Graphically we can represent this process



We can write T_{fi} starting with this graph by writing the fact

	factor
– incoming e^- with momentum p	$u(p)$
– outgoing e^- " "	$\bar{u}(p)$
– interaction vertex	$ie\gamma^\mu$

and writing these factors from right to left following the arrow of <u>negative</u> charge

For a positron we can write the corresponding equation for a spinor ψ^c with charge $+e$

so the equation is

$i\gamma_\mu \partial^\mu \psi^c - m\psi^c = e\gamma^\mu A_\mu \psi^c = -\gamma^0 V \psi$

So for e^- $[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\psi = 0$ $\equiv (1)$

for e^+ $[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi^c = 0$ $\equiv (2)$

But ψ contains both e^+ and e^- spinors so both eq must be equivalent - To find ψ^c in terms of ψ

we take $(1)^*$ and use that

$(1)^* = [-\gamma^{\mu *}(i\partial_\mu - eA_\mu) - m]\psi^* = 0$

if we define $C = i\gamma^2\gamma^0 \Rightarrow -C\gamma^0\gamma^{\mu *} = \gamma^\mu C\gamma^0$

$\Rightarrow C\gamma^0(1)^* = [\gamma^\mu(i\partial_\mu - eA_\mu) - m]\, C\gamma^0\psi^* = 0$

which is the positron DE with $\psi^c = C\gamma^0\psi^* = C\bar{\psi}^T$

C verifies $C = -C^{-1} = -C^+ = -C^T$)

so $\overline{\psi^c} = \psi^{c+}\gamma^0 = \psi^T\gamma^{0+}C^+\gamma^0 = \psi^T C = -\psi^T C^{-1}$

So to get the amplitude for a positron we can follow the same procedure as for e

but now $j_c^\mu = -e\,\bar{\psi}_f^c\,\gamma^\mu\psi_i^c = e\,\psi_f^T \underbrace{C^{-1}\gamma^\mu C}_{-\gamma^{\mu T}}\bar{\psi}_i^T$

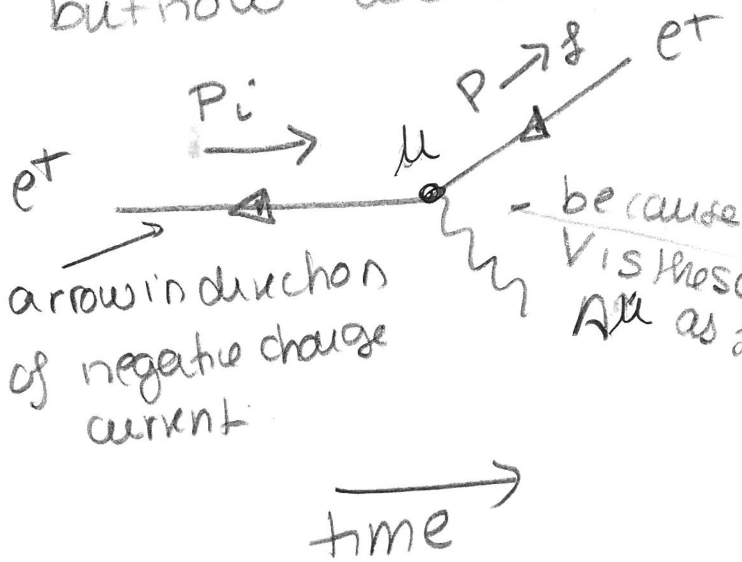
$= -\psi_f^T\,\gamma^{\mu T}\psi_i^T = (-)(-)\,\bar{\psi}_i\,\gamma^\mu\psi_f$

from exchange of fermions

So for antifermion spinors $\psi = v(p)\,e^{ipx}$

$T_{fi} = -i\int j_\mu^c A^\mu\, d^4x = \int d^4x\,(-)\left[ie\,\bar{v}_i\,\gamma^\mu v_f\, e^{i(p_f-p_i)x}\right.A_\mu$

and we could infer this from a diagram but now we must draw



arrow in direction of negative charge current

- because V is the same A^μ as for e^-

	factor
incoming e^+	$\bar{v}_i(p)$
outgoing e^+	$v_f(p)$
same vertex	$ie\gamma^\mu$

but notice there is a relative (-) sign between this w.r. to the e^- case

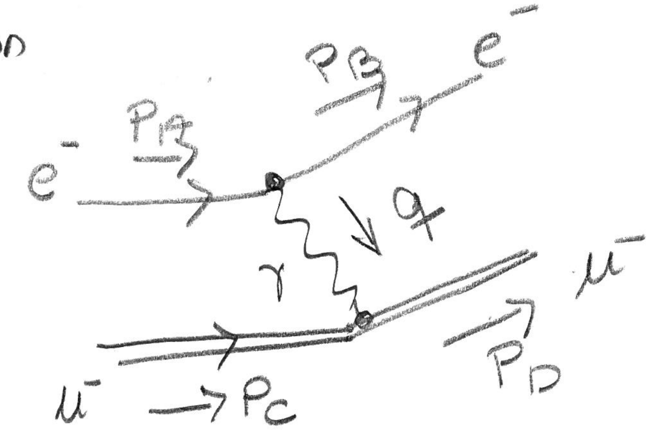
③ $e^- \mu^- \to e^- \mu^-$

Using what we derive for e^- in an em field let us now assume that the em field encountered by the e^- is due to muon μ^-

Using maxwell's eq (in Coulomb gauge)

$$\partial^\mu \partial_\nu A^\alpha \equiv J^\alpha_{muon}$$

Graphycally



$$J^\alpha_{muon} = -e \, \bar{u}_{D_{muon}} \gamma^\alpha u_{B_{muon}} \; e^{i\underbrace{(p_D - p_B)}_{= q} x}$$

since $\partial^\mu \partial_\nu e^{iqx} = -q^2 e^{iqx}$

the solution of ME is $A^\alpha(q^2) = -\frac{1}{q^2} J^\alpha_{muon}$

So
$$T_{fi} = -i \int J^\alpha_{elec} \left(-\frac{1}{q^2}\right) \underbrace{J_{\alpha, muon}}_{g_{\alpha\beta} J^\beta_{muon}} d^4x \, e^{i(p_A + p_B - p_B - p_D)x}$$

$q^2 = (p_D - p_B)^2$

$$= (ie)^2 \, \bar{u}_{C,e} \gamma^\alpha u_{A,e} \left(\frac{-i g_{\alpha\beta}}{q_2}\right) \bar{u}_{D,muon} \gamma^\beta u_{B,muon} \times \int d^4x \, \underbrace{e^{i(p_D - p_B + p_C - p_A)}}_{= (2\pi)^4 \delta^4 (p_A + p_B - p_C - p_D)}$$

We define the Feyman amplitude for process initial $\to$ final $\quad iM_{fi}$ so that

$$T_{fi} \equiv (2\pi)^4 \delta^4(P_{int} - P_{final}) \, (iM_{fi})$$

so for $e^- \mu^- \to e^- \mu^-$ / let us put all the helicity index also

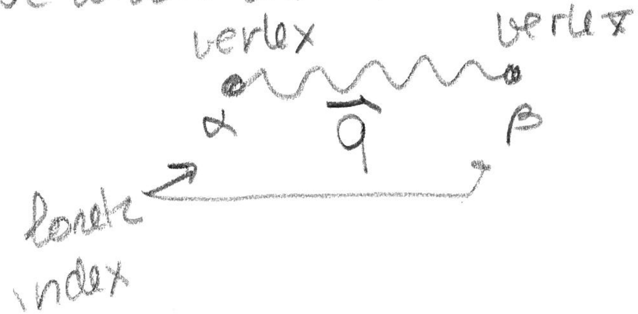
$$\begin{array}{cccc} P_A & P_B & P_C & P_D \\ S_a & S_b & S_c & S_D \end{array}$$

$$iM = (ie)\left[\bar{u}^{S_c}_{e}(P_C)\, \gamma^\alpha\, u^{S_A}_{e}(P_A)\right]\left[\frac{-i g_{\alpha\beta}}{q^2}\right](ie)\left[\bar{u}^{S_D}_{muon}(P_D)\, \gamma^\beta\, u^{S_B}_{mu}(P_B)\right]$$

$$q^2 \equiv (P_A - P_C)^2 = (P_B - P_D)^2$$

$\dfrac{-i g_{\alpha\beta}}{q^2} \equiv$ propagator for a "virtual" photon.
It is clearly virtual because $q^2 \neq 0$

We could have inferred M_{fi} from the diagram we wrote with the aditional rule that for



Factor

$$\frac{-i g_{\alpha\beta}}{q^2}$$

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④ Propagators

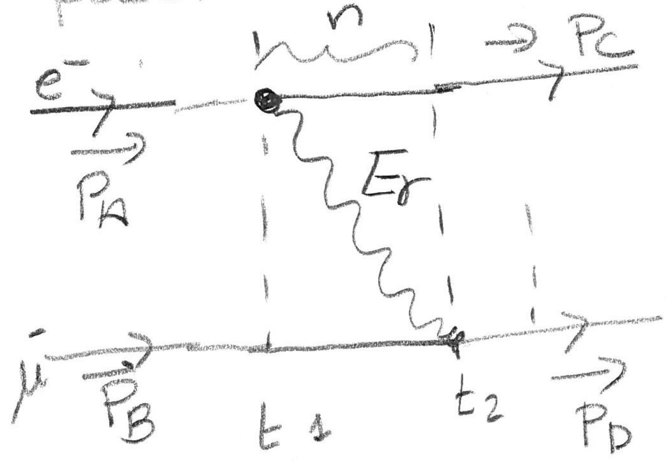
In non-relativistic perturbation theory we wrote

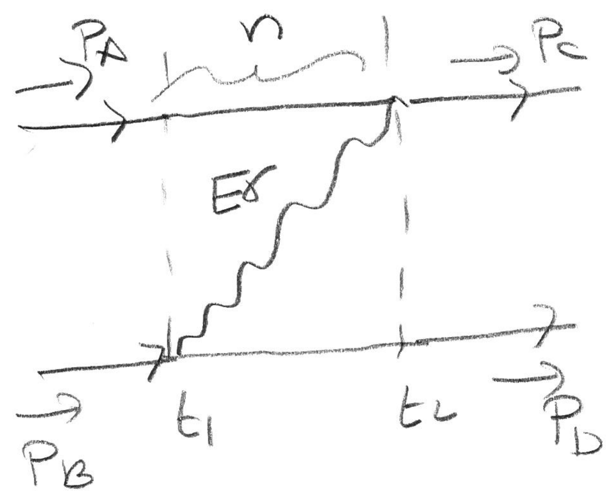
$$T_{fi} = 2\pi\delta(E_f - E_i)\left[-iV_{fi} + \sum_n (-iV_{fn}) \frac{i}{E_i - E_n} (-iV_{ni})\right]$$

and $\frac{i}{E_i - E_n}$ ≡ propagator in QM.

In QED we wrote for the photon $\frac{1}{q^2}$ · why?

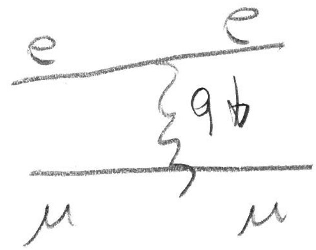
When we do QFT a Feynman diagram represents all "time ordered" diagrams with the same particle colours. Fe for $\mu^- e^- \to \mu^- e^-$ we could have two possible time sequences.





(1)

and both contribution are possible one with
E>0 in propagator propagating forward in time
and those with E<0 in " backward in time

So when we wrote 

we are adding both

Followin NR perturbation theory

In (1) $E_{n_1} = E_C + E_\gamma + E_B$

$E_i = E_A + E_B$

$= E_C + E_D$

In (2) $E_{n_2} = E_A + E_\gamma + E_D$

So $\frac{1}{E_i - E_{n_1}} + \frac{1}{E_i - E_{n_2}} = \frac{1}{E_A - E_C - E_\gamma} + \frac{1}{E_C - E_A - E_\gamma}$

$= \frac{-2E_\gamma}{(E_C - E_A)^2 - E_\gamma^2}$ ← related to state normalization

NR theory 3-momentum is conserved at vertices

$E_\gamma^2 = |\vec{p}_\gamma|^2 + m_\gamma^2 = |\vec{p}_A - \vec{p}_C|^2 + m_\gamma^2$ ← lets keep a mass for generality

$\Rightarrow (E_A - E_C)^2 = (p_A - p_C)^2 + |\vec{p}_A - \vec{p}_C|^2 = q^2 + E_\gamma^2 - m_\gamma^2$

$\Rightarrow \frac{1}{(E_A - E_C)^2 - E_\gamma^2} = \frac{1}{q^2 - m_\gamma^2}$

So the propagator for particles of mass m have always denominator

$\frac{1}{q^2 - m^2}$ $\quad q \equiv$ 4 momentum in propag obtained from 4-momentum conservation at vertices the numerator depends on the mass spin of the particle

Again in QM

$$T_{fi}^{(2)} = -2\pi i\, \delta(E_f - E_i) \sum_n \underbrace{-iV_{fn}}_{\langle f|V|n\rangle} \frac{i}{E_i - E_n} -iV_{ni}$$

with $H_0 |n\rangle = E_n |n\rangle$

and since $\sum_n |n\rangle\langle n| = I$

we can write this formally as

$$= (2\pi)\, \delta(E_i - E_n) \langle f| (-iV) \frac{i}{E_i - H_0} (-iV)|i\rangle$$

So now the propagator is an operator which is the inverse of the Schroedinger Eq

$$H\phi = E\phi \Rightarrow (H_0 + V)\phi = E\phi$$

$$\Rightarrow \underbrace{-i(E - H_0)}_{\text{inverse of prop.}}\phi = -iV\phi$$

So all we need to do to find propagator of any particle is to find the wave eq and invert it

Let us work with states of defined 4-momenta

- For scalar the KG eq for a plane wave

$$\underbrace{(\times i)\ (-q^2+m^2)}_{\substack{\parallel \\ (i\Delta(q))^{-1} \\ \parallel \\ \text{Scalar prop}}}\phi(q) = -V\phi(q) \quad (\times i)$$

$$\Rightarrow \quad i\Delta(q) = \frac{1}{i(-q^2+m^2)} = \frac{i}{q^2-m^2}$$

Graphically we will represent it as •- - -•

- For fermions we wrote

$$\underbrace{(-i)\ (\not p - m)}_{\substack{\parallel \\ (iS_F)^{-1} \\ \parallel \\ \text{Fermion prop}}} u(p) = \gamma^0 V u(p) \quad \times(-i)$$

$$\Rightarrow \quad iS_F = \frac{1}{-i(\not p - m)} \times \frac{\not p + m}{\not p + m} = \frac{i\ \not p + m}{p^2 - m^2}$$

graphycally

momentum in propag must go in same direct as arrow of charge

the numerator is

$$\not p + m \equiv \sum_{\text{helic}} u(p)\, \bar u(p)$$

so all virtual spin states propagate

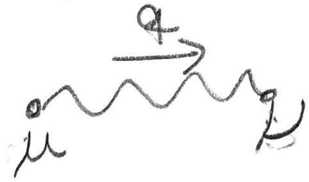
. For the photon: we have to work in some gauge. In the coloumb gauge

$$\overbrace{(-i)\, g^{\mu\lambda}(-q^2)}^{[iD^{\mu\lambda}]^{-1}}$$

$$g^{\mu\lambda}\,\partial^\alpha\partial_\alpha A_\lambda = j^\mu \Rightarrow (-i)\, g^{\mu\lambda}(-q^2)\, A_\lambda(q) = j^\mu(q) \quad (-i)$$

since $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$ we can invert

$$iD^{\mu\nu}(q) = \frac{g^{\mu\nu}}{iq^2} = -i\,\frac{g^{\mu\nu}}{q^2}$$

we represent it 

Notice that we have found that

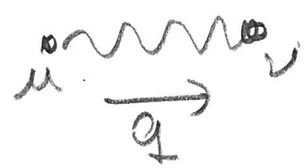
$$-g^{\mu\nu} = \sum_{r=0}^{3} \zeta_r\, \epsilon_r^\mu(p)\, \epsilon_r^\nu(p)$$

summed over all polarisation physical and unphysical. they all propagate in virtual photons

- Massive vector: Procca $\left[g^{\mu\lambda}(\partial^\alpha\partial_\alpha + M^2) - \partial^\mu\partial^\lambda\right] B_\lambda = j^\mu$

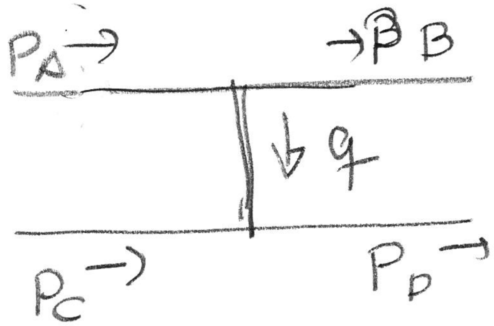
in momentum space $g^{\mu\lambda}(-q^2+M^2) + q^\mu q^\lambda$

So $$iD_V^{\mu\nu}(q^2) = \frac{i\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right)}{q^2 - M^2}$$

Represent 

So far we have push the problem of the poles at $E_n = E_i$. To solve it we added a $+i\epsilon$ in the denominator of propagators. in QFT the problem comes when $q^2 = m^2$ in the propag.

Notice that for any diagram like th energy momentum consv.



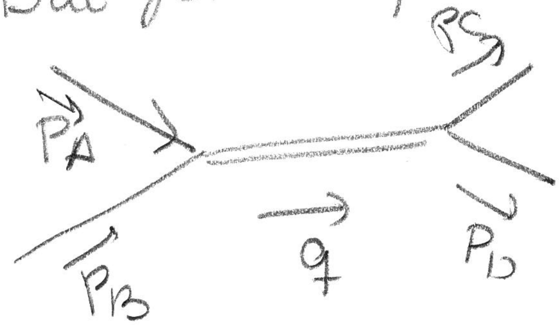
$$q^2 = (P_C - P_A)^2 < 0$$

So it is not possible

$$q^2 = m^2$$

$\Rightarrow$ for these cases the $+i\epsilon$ is not relevant

But for diagrams with propagator



$$q^2 = (P_A + P_B)^2 > 0$$

$\Rightarrow q^2 = m^2$ is possible (corresponds to product of a real particle

But in QED since $m_\gamma = 0$ this is not possible so we do not worry now.

5 Feynman rules for QED

For any process $|I\rangle \to |F\rangle$ we can obtain the amplitude

convention

$$T_{fi} \equiv (2\pi)^3 \delta^4(P_I^{TOT} - P_F^{TOT}) \, (-) \, i \, \mathcal{M}_{FI}$$

where $\mathcal{M}_{FI} = \sum_n \mathcal{M}_{FI}^{(n)}$

where $\mathcal{M}_{FI}^{(n)}$ can be obtained from all diagrams connecting $|I\rangle$ to $|F\rangle$ containing "n" vertices.

same flavour and continuing arrow

the $\mathcal{M}_{FI}^{(n)}$ of each diagram is obtained following these rules

(0) Assign arrows to all fermion lines following the flux of negative charge

Label all external 4-momentum and helicities.

Draw an arrow for all the for momenta

For fermion propagator the momentum arrow must have the same direction as the "charge" arrow

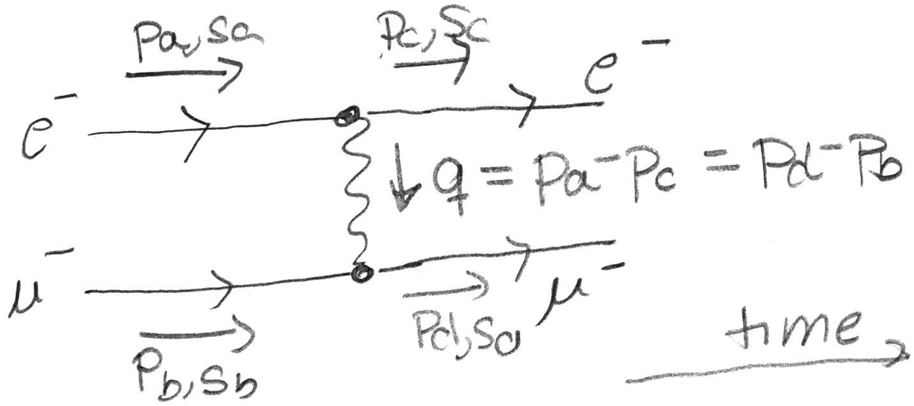
Obtain the value of all momentum in propagator by energy momentum conservation in the vertices

Assign a lorentz index to each vertex

Take $\quad e^- \mu^- \longrightarrow e^- \mu^-$

$\quad\quad p_a \; p_b \quad\quad p_c \; p_d$

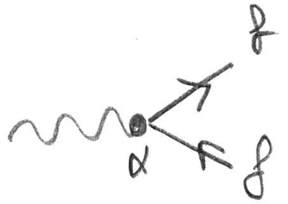
Rule (0) $\quad s_a \; s_b \quad\quad s_c \; s_d$



$\Rightarrow q^2 = (p_a - p_c)^2$
$= (p_d - p_b)^2$
$\equiv t$ ← another Mandelstam variable more later

Factor

1) For each vertex



$-ieQ_f \gamma^\alpha$

2) For each external line $\xrightarrow{p, s}$

incoming f	$u^s(p)$
outgoing f	$\bar{u}^s(p)$
incoming $\bar{f}$	$\bar{v}^s(p)$
outgoing	$v^s(p)$
incoming γ	$[\epsilon_s(p)]_\alpha$
outgoing γ	$[\epsilon_s^*(p)]_\alpha$

3) Internal fermion line $\xrightarrow{q}$

$$\frac{i(\not{q} + m_f)}{q^2 - m_f^2 + i\epsilon}$$

4) Internal photon line $\alpha \xrightarrow{q} \beta$

$$\frac{-i g_{\alpha\beta}}{q^2 + i\epsilon}$$

5) For each fermion line in the diagram write these factor from right to left following arrow of charge

$$i\mathcal{M}_{e^-\mu^-\to e^-\mu^-} = \left[\bar{u}^{s_c}_{e}(p_c)(ie\gamma^\alpha)u^{s_a}_{e}(p_a)\right]\frac{-ig_{\alpha\beta}}{(p_a-p_c)^2}\left[\bar{u}^{s_d}_{mu}(p_d)(ie\gamma^\beta)u^{s_b}_{(p_b)}\right]$$

$$\Rightarrow \mathcal{M}_{e^-\mu^-\to e^-\mu^-} = \frac{-e^2}{(p_a-p_c)^2}\bar{u}_c\gamma^\alpha u_a\left[\bar{u}_d\gamma_\alpha u_b\right]$$

6) Diagrams differing in exchange of

- identical initial fermion
- " " anti "
- " final "
- " " anti "
- "
- initial $f \longleftrightarrow$ final $\bar{f}$
- initial $\bar{f} \longleftrightarrow$ " f

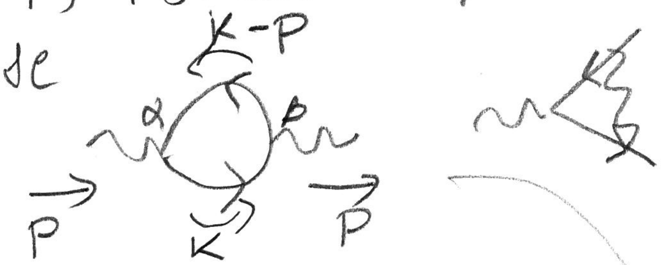
relative (−) sign

7) For each loop in the diag — factor

$$\int \frac{d^4k}{(2\pi)^4}$$

momentum in the loop not determined by conservation in vertices

ie



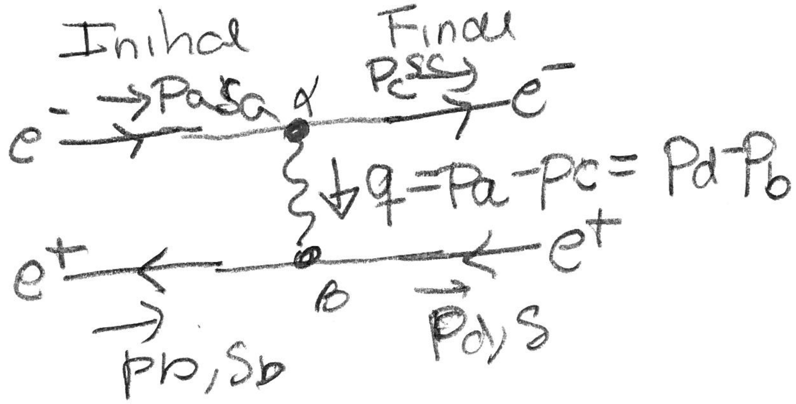
8) If the loop is fermionic — (−) Tr[internal fermion line]

$$(-)\,\mathrm{Tr}\frac{[(\not{k}-\not{p}+m)\gamma^\alpha(\not{k}+m)\gamma^\beta]}{[(k-p)^2-m^2][k^2-m^2]}$$

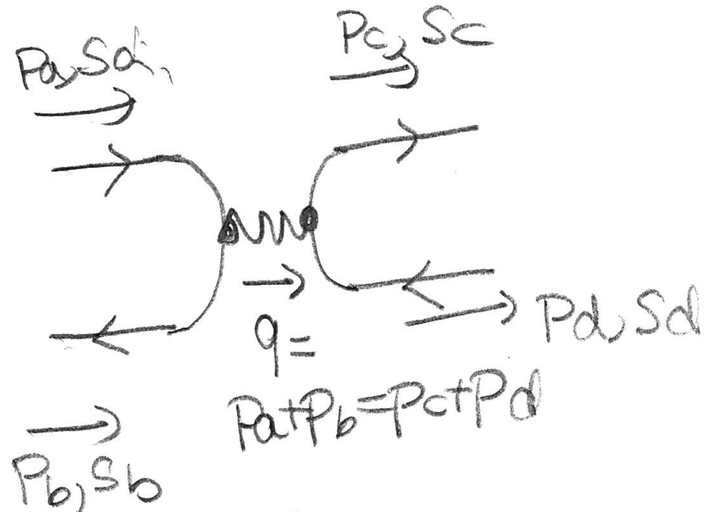
Exampl $e^- e^+ \to e^- e^+$

$p_a \; p_b \quad p_c \; p_d$ 4-momentum

$s_a \; s_b \quad s_c \; s_d$ helicities



(a)



(b)

$$iM_a = \left[\bar{u}^{s_c}(p_c)(ie\gamma^\alpha)\, u^{s_a}(p_a)\right] \frac{-ig_{\alpha\beta}}{(p_a-p_c)^2} \left[\bar{v}^{s_b}(p_b)(ie\gamma^\beta)\, v^{s_d}(p_d)\right]$$

$$= \frac{ie^2}{(p_a-p_c)^2}\left[\bar{u}_C \gamma^\alpha u_A\right]\left[\bar{v}_B \gamma_\alpha v_D\right]$$

$$iM_b = (-)\left[\bar{v}^{s_b}(p_b)\, ie\gamma^\alpha\, v^{s_a}(p_a)\right] \frac{-ig_{\alpha\beta}}{(p_a+p_b)^2} \left[\bar{u}^{s_c}(p_c)(ie\gamma^\beta)\, v^{s_d}(p_a)\right]$$

$(-)$ because $a \leftrightarrow b \Rightarrow$ Initial $e^+ \leftrightarrow$ final e^-

$$= -\frac{ie^2}{(p_a+p_b)^2}\left(\bar{v}_b \gamma^\alpha u_a\right)\left(\bar{u}_c \gamma_\alpha v_d\right)$$

5 QED as an abelian gauge theory

An internal continuos transf transforms

$$\psi_i(x) \longrightarrow \psi_i'(x) = \sum_j \underbrace{U(\theta)_{ij}}_{\text{unitary matrix}} \psi_j(x)$$

$U(\theta) \in$ Group $\quad \theta \equiv$ parameter of transf.

Simplest case $\quad U(\theta) = e^{i\theta} \quad (G \equiv U(1))$

If θ is independent of x $\Rightarrow$ global $U(1)$ transf

If $\theta(x)$ $\Rightarrow$ local ($\equiv$ gauge) $U(1)$ transf

In modern particle physics all fundamental interactions among elementary particles are understood as emerging from invariance under some gauge transformations

In such gauge theories the Lag is fully determined by

- symmetry group G
- the matter contents and its transf. properties under G

Let's start by the simplest case of a theory with matter made of a fermion "f" with w.f $\psi(x)$

The free lagrangian

$$\mathcal{L}_{0F} = \bar{\psi}(x)\left[i\gamma^{\mu}\partial_{\mu}\psi(x)\right] - m\,\bar{\psi}(x)\psi(x)$$

$\mathcal{L}_{0F}$ is invariant under a global U(1) transf

$$\psi'(x) = e^{-i\theta_f}\psi(x) \Rightarrow \partial_{\mu}\psi' = e^{-i\theta_f}\partial_{\mu}\psi$$

$$\bar{\psi}'(x) = e^{i\theta_f}\psi(x)$$

$$\Rightarrow \mathcal{L}'_{0F} = \bar{\psi}' i\gamma^{\mu}\partial_{\mu}\psi' - m\psi'\psi = \bar{\psi}\underbrace{e^{i\theta}e^{-i\theta}}_{1} i\gamma^{\mu}\partial_{\mu}\psi - m\underbrace{e^{-i\theta}e^{i\theta}}_{1}\bar{\psi}\psi$$

$$= \mathcal{L}_{0F}$$

But for a local transf. Lets write $\theta(x) = Q_f\alpha(x)$

$$\psi'(x) = e^{-iQ_f\alpha(x)}\psi(x) \Rightarrow \partial_{\mu}\psi' = e^{-iQ_f\alpha}\partial_{\mu}\psi - iQ_f(\partial_{\mu}\alpha)\psi e^{-iQ_f\alpha}$$

$$\bar{\psi}'(x) = e^{iQ_f\alpha(x)}\psi(x)$$

$$\Rightarrow \mathcal{L}'_{0F} = \mathcal{L}_{0F} + \underbrace{Q_f\partial_{\mu}\alpha\;\bar{\psi}(x)\gamma^{\mu}\psi(x)}_{\text{this term breaks U(1) gauge inv.}}$$

We can build an invariant gauge Lag for ψ if we substitute $\partial_\mu \psi$ by $D_\mu \psi$ which ues

$$(D_\mu \psi)' = e^{-iQ_f \alpha} D_\mu \psi \qquad (*)$$

so that $(\bar\psi \gamma^\mu D_\mu \psi)' = \bar\psi \gamma^\mu D_\mu \psi$ arbitrary constant

Let us call $\underset{\text{Lorentz 4 vector}}{D_\mu \psi} - \underset{\text{Lorentz 4-vector}}{\partial_\mu \psi} \equiv i e Q_f \underset{\text{Lorentz 4-vectors}}{A_\mu(x)}$

So the condition $(*)$

$$(D_\mu \psi)' = \underbrace{(\partial_\mu + ieQ_f A'_\mu) e^{-iQ_f\alpha}\psi}_{(1)} = e^{-iQ_f\alpha}(\partial_\mu + ieQ_f A_\mu)\psi$$

$$(1) = -iQ_f \partial_\mu \alpha\, e^{-iQ_f\alpha} \psi + e^{-iQ_f\alpha} \partial_\mu \psi + ieQ_f A'_\mu e^{-iQ_f\alpha} \psi$$

$$(2) = e^{-iQ_f\alpha} \partial_\mu \psi + e^{-iQ_f\alpha}\, ieQ_f A_\mu$$

$$(1) = (2) \implies -iQ_f \partial_\mu \alpha + ieQ_f A'_\mu = ieQ_f A_\mu$$

$$A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \alpha$$

So a theory described by a La

$$\mathcal{L}_F = \bar\psi \gamma^\mu \partial_\mu \psi - e\bar\psi \gamma^\mu A_\mu \psi - m\bar\psi\psi$$

is $U(1)$ gauge invariant if any physical observable described by A_μ or A'_μ are the same

But $A'_\mu = A_\mu + \partial_\mu \chi$ is the invariance that we found in the ME's for the em 4-vector potential. the physical fields $\vec{E}, \vec{B}$ obtained with A and A' are the same

So we are going to identify the A_μ in $D_\mu - \partial_\mu$ as the vector potential of an em field.

Once we quantize the theory $A_\mu(x)$ is the wave function of the photon. We call it the gauge boson of the U(1) transf.

the full Lag should also contain the pieces for the free photons (which of course is gauge inv by itself).

$$\mathcal{L}_{0A} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Notice that if we had introduced a mass for A^μ

$$-\frac{1}{2} m_A^2 A^\mu A_\mu \neq -\frac{1}{2} m_A^2 A'^\mu A'_\mu$$

So a mass for the gauge boson breaks GI

$\Rightarrow$ gauge bosons are massless

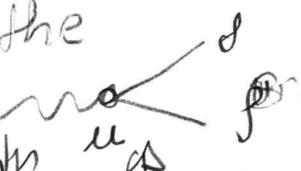
Altogether the U(1) gauge invariant Lag we have constructed is

$$\mathcal{L}_{QED} = \bar{\Psi}(i\gamma^\mu(\partial_\mu + ieQ_f A_\mu))\Psi - m\bar{\Psi}\Psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

$$= \underbrace{\bar{\Psi}\, i\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi}_{\mathcal{L}_{0F}} \underbrace{- \frac{1}{4} F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{0A}} \underbrace{- eQ_f\,\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\mathcal{L}_{I}^{QED}}$$

↑ interaction

describes the interaction



the form of the interaction can be read from $\mathcal{L}_I$
$-eQ_f\gamma^\mu$
(the FR adds "i")

In summary

- we started with matter made of f
- we know how this fermion transforms under a U(1) transf (≡ we know Q_f)
- we look for a Lag which is invariant under a gauge U(1) transf ⇒ need to introduce a vector field A_μ which has the transf properties of the vector 4-potential of em ≡ photon

The resulting Lag describes a theory with a fermion, a photon and their interactions ≡ QED

We only need one A_μ because dimension of U(1) is one