Chapter 6

QED II: Predictions for processes with leptons and photons

1) Observables: Decay width and cross section
2) Cross section of $e^-\mu^- \to e^-\mu^-$: techniques
3) Some predictions at lowest order
   - $e^+e^- \to \mu^+\mu^-$ crossing symmetry
   - $e^+e^- \to e^+e^-$ (Babba scattering) identical fermions
   - $e^-\gamma \to e^-\gamma$ (compton scattering) real photon
   - $e^+e^- \to \gamma\gamma$ (pair annihilation)
4) Higher orders: renormalization, running coupling constant, anomalous magnetic moment

1) Observables

Decay width ≡ observable to quantify the probability of decay

$$A \to \underbrace{f_1 \ldots f_n}_{\equiv F} \qquad \Gamma = \frac{1}{\tau}$$

decay width

lifetime

Experimentally this is determined

$$\Gamma_{A\to F} = \frac{\#\text{decays per unit time (per volume)}}{\#\text{particles } A \text{ (per volume)}}$$

we predict

$$= \frac{\text{Prob}(A\to F)}{T \times V} \; \frac{\#\text{states } F \text{ (per } V)}{\#\text{particle } A \text{ (per } V)}$$

where $\text{Prob}(A\to F) = |T_{AF}|^2$

$$T_{Af} = (2\pi)^4 \delta^4(P_A - P_F)\, i\, M_{A\to F} \frac{1}{\sqrt{V^{N+1}}}$$

from normalization of 2E states per unit volume

We take F to be a set of N particles $i = 1 \ldots N$ momentum between $\vec{p}_i$ and $\vec{p}_i + d^3\vec{p}_i$

$$\#\text{states } F \text{ per volume } \frac{d^3p_1}{2E_1(2\pi)^3} V \ldots \frac{d^3p_n}{(2\pi)^3 2E_n} V$$

$$\#\text{particles } A \text{ in volume } V \; \frac{2E_A}{V} \qquad (\text{at rest } E_A = M_A)$$

clearly Γ_A depends on ref frame

Using that $\overbrace{}^{\text{integral representation of } \delta^4}$

$$\left[\delta^4(P_A - P_F)\right]^2 = \int \frac{d^4x}{(2\pi)^4} e^{i(P_A-P_F)x} \delta^4(P_A-P_F)$$

$$= \frac{\delta^4(P_A-P_F)}{(2\pi)^4} \int \underbrace{d^4x}_{V \times T}$$

So for A at rest

$$d\Gamma_{A\to f_1 \dots f_n} = \frac{1}{2M_A} |M_{AF}|^2 \overbrace{(2\pi)^4 \delta^4(P_A - P_1 \dots - P_n) \frac{d^3p_1}{(2E_1)(2\pi)^3} \dots \frac{d^3p_n}{(2E_n)(2\pi)^3}}^{\underset{|||}{dQ_N} \,\equiv\, \text{N-body phase space}}$$

$$\cancel{\frac{1}{V^{n+1}} \frac{V \times T}{V \times T} V^n \times V} \to 1$$

all V factor cancel as they should

For two particles in the final state dQ_2 can be integrated. Take A at rest

$$dQ_2 = \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \overbrace{\delta^4(P_A - P_1 - P_2)}^{\delta(M_A - E_1 - E_2)\,\delta^3(\vec{p}_1 - \vec{p}_2)}$$

we can integrate d^3p_2 with the $\delta^3(\vec{p}_1 - \vec{p}_2) \Rightarrow \vec{P}_1 = -\vec{P}_2$

$$\Rightarrow |\vec{p}_1| = |\vec{p}_2| \equiv p$$

and writing d^3p_1 in spherical coordinates

$$d^3p_1 = p^2\,dp\,\underbrace{d\cos\theta\, d\varphi}_{d\Omega} \quad = \text{solid angle}$$

$$dQ_2 = \frac{p^2\,dp}{(2\pi)^2(2E_1)(2E_2)} \overbrace{\delta(E_1+E_2-M_A)}^{f(p)}$$

where $E_1 = \sqrt{p^2+M_1^2}$ $\quad E_2 = \sqrt{p^2+M_2^2}$

using $\delta(f(x)) = \dfrac{\delta(x-x_0)}{\left|\dfrac{df}{dx}\right|_{x_0}}$ where $f(x_0)=0$

we get

$$\delta(E_1+E_2-M_A) = \frac{\delta(p-p_f)}{M_A\,p_f}\,E_1E_2$$

with $p_f = \dfrac{1}{2M_A}\sqrt{[M_A^2-(M_1-M_2)^2][M_A^2-(M_1+M_2)^2]}$

so $dQ_2 = \dfrac{p_f}{4M_A}\dfrac{1}{(2\pi)^2}\,d\Omega$

$$\Rightarrow d\Gamma_A = \frac{p_f}{32\pi^2 M_A^2}\,|M_{A\to f_1 f_2}|^2\,d\Omega$$

In general the integral over the angles can only be made if we know $|M|^2$.

But if A is at rest and we are not measuring its spin ($\equiv$ unpolarized measurement) or it is just a scalar ($s=0$) $\Rightarrow$ nothing points out in any direction in initial state $\Rightarrow |M_{A\to}|^2$ cannot depend on the direction of emission of f_1 and f_2

In this case we can integrate $\int d\Omega = 4\pi$

and $$\boxed{\Gamma_A = \frac{P_f}{8\pi M_A^2} |M_{A\to f_1 f_2}|^2}$$

at rest

Cross section : Observable that quantifies the probability of collision

$$a+b \to \underbrace{f_1 \dots f_n}_{F}$$

Experimentally to measure $\sigma(a+b\to F)$

1) Prepare a beam of particles "a" and a target of particles "b" (target can be a beam in opposite direction). This setup has some luminosity

$$\mathcal{L}_{ab} = \frac{N_a \times N_b}{A \times t}$$

particles a and B

time ....

crossarea of overlapp between beam and targe units of luminosity $= (\text{area})^{-1} \times (\text{time})^{-1}$

2) Count how many F states (N_F) come out per unit time.

the ratio is the CS

$$N_F = \mathcal{L}_{ab}\, \sigma_{ab \to F}$$

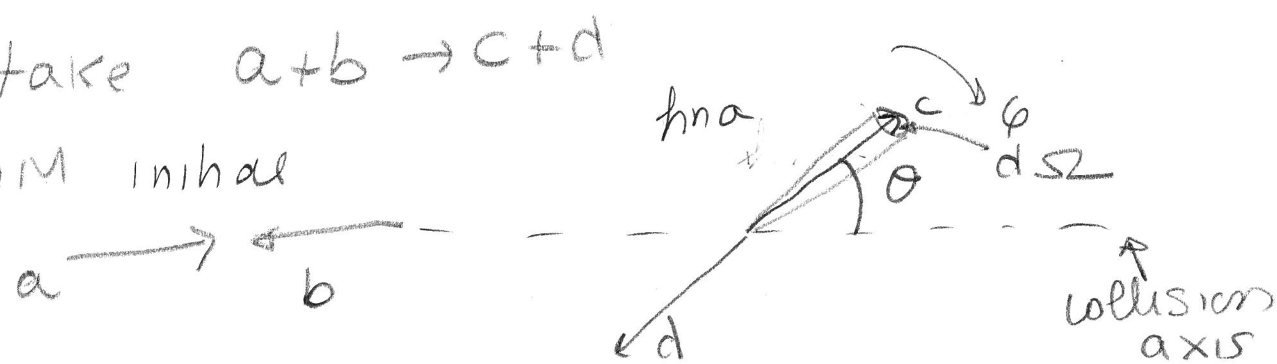
we count — we prepar — we infer

If all F's are counted irrespective of their energy and 3 momentum $\Rightarrow \sigma_{ab\to F}$ is independent of the reference frame

If we meassue F in some kinematic conf. (for example in some direction) we talk about "differential cs"

F.e take $a+b \to c+d$

in CoM initial



and we count only c's coming out within a cone of solid angle $d\Omega$ around direction θ

$\equiv N_c(\theta)$

We define

$$N_c(\theta) \equiv \mathcal{L}_{ab} \times \underbrace{\frac{d\sigma_{ab\to cd}(\theta)}{d\Omega}}_{\text{differential cs}}$$

clearly if we change reference frame θ changes

$\Rightarrow \frac{d\sigma}{d\Omega}$ depends on ref. frame

(Detour)

In a fix target experiment (bat rest)

- beam of particles a with transverse area A delivers some # of particles per unit time

$$\Rightarrow \frac{N_a}{A \times t} \equiv \phi^{beam}$$

- target is made of some material with some density ρ_{target} which will contain a total

$N_b \equiv N_{target}$ which can be known for ρ_{target}

$\propto \rho_{target} \times L \times A$ and dimension of target

L ← length of target

fix target

$$\mathcal{L}_{ab} = \Phi_{beam} \times N_{target}$$

In a collider experiment beams are made of bunches of particles which cross with some frequency f. Let us call $N_{1,2}$ = # particles per bunch in beam 1,2

$$\mathcal{L}_{ab}^{collider} = \frac{N_1 \times N_2}{A} \times f$$

f ← frequency of bunch crossing

A ← cross area of beams at collision point

Presently the most luminous HE collider is LHC colliding proton beams with $N_1 = N_2 = 10^{11}$ protons per bunch

bunch size $\sigma_{bunch} = 18\mu m \Rightarrow A_{bunch} = 4\pi(18\mu m)^2 = 4 \times 10^{-5} cm^2$

crossing every 25ns $\Rightarrow f = 4 \times 10^7 s^{-1}$

$$\mathcal{L}_{LHC} = \frac{4 \times 10^7 \times 10^{11} \times 10^{11}}{4 \times 10^{-5} cm^2} s^{-1} = 10^{34} cm^{-2} s^{-1}$$

Let us compare with the luminosity of a fix target experiment in which one of the beams is aimed at a target 1m long made of liquid hidrogen

$\rho_{target} \simeq 7.1 \times 10^{-6} \frac{kg}{cm^3}$

since in Hydrogen there is 1 proton per nucleus

$N_{target} \simeq \frac{\rho_{target}}{m_{prot}} \times A \times l_{target} = 4\times10^{-5} cm^2 \times 100 cm \times \frac{7.1\times10^{-6}}{1.6\times10^{-27} kg} \frac{kg}{cm^3}$

$= 2\times10^{20}$

$\Phi_{beam} = \frac{N_{per\,bunch}}{A} \times f_{bunch} = \frac{10^{11}}{4\times10^{-5} cm^2} 4\times10^7 s^{-1}$

$= 8\times10^{22} cm^{-2} s^{-1}$

$\mathcal{L}^{FT} = 1.6\times10^{43} cm^{-2} s^{-1} = 10^9 \mathcal{L}_{LHC}$

$\Rightarrow$ much more luminosity $\Rightarrow$ better statistics $\Rightarrow$
$\Rightarrow$ more precision

But remember $\sqrt{s}_{collide} > \sqrt{s}_{FT}$ with same E_{beam}

To predict $\sigma_{ab \to f_1 \dots f_n}$

we need to compute

$$\sigma_{a+b \to f_1 \dots f_n} = \frac{\#\text{scatterings } 1a+1b \to F}{\text{time}} \frac{1}{\mathcal{L}_{1a+1b}}$$

$$\frac{\#\text{scattering}}{T} = \frac{\text{Prob}(a+b \to F)}{T \times V} \times \#\text{states } F \text{ in } V$$

$$= (2\pi)^4 \delta^4(P^{TOT}_{INT} \times P^{TOT}_{FINA}) |M_{FI}|^2 \frac{V^n}{V^{n+2}} \frac{d^3p_1}{(2E)(2\pi)^3} \cdots \frac{d^3p_n}{(2E_n)(2\pi)^3}$$

now

$$\mathcal{L}_{1a+1b} \overset{\text{b at rest}}{=} \underbrace{\frac{N_{1a}}{A \times t}}_{2E\frac{\vec{v}_a}{V}} \times \underbrace{N_{1b}}_{\frac{2E_b}{V}} = \frac{\overbrace{2E_a \times 2E_b\, \vec{v}_a}^{\text{Flux}_{ab}}}{V^2}, \quad \vec{v}_a = \frac{|\vec{p}_a|}{E_a}$$

so again volume factors cancel and we get

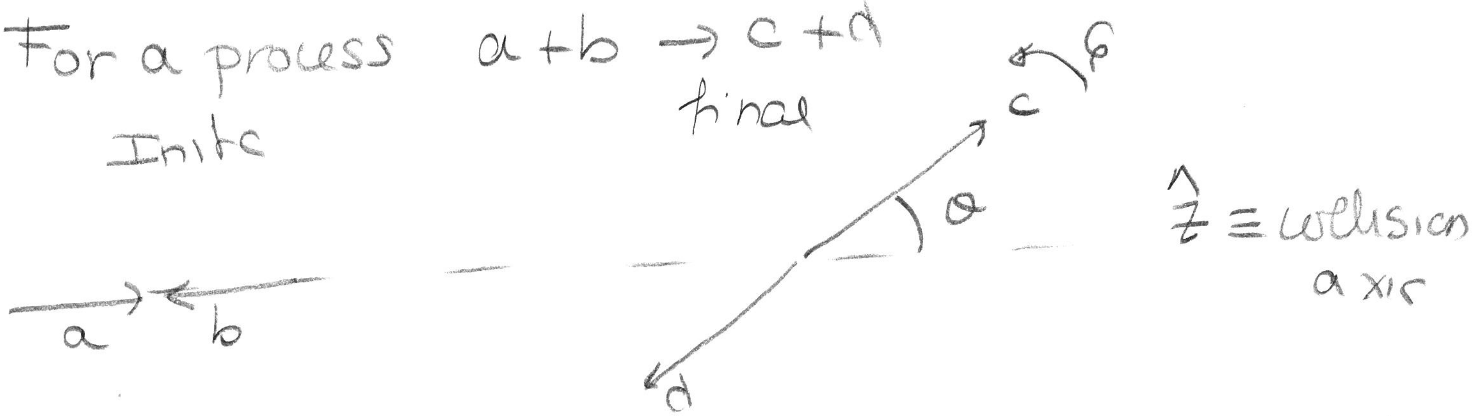
$$d\sigma_{ab \to f_1 \dots f_n} = \frac{|M_{FI}|^2}{\text{Flux}_{ab}}\, d\Phi_N$$

where

$$\text{Flux}_{ab} \overset{\text{b at rest}}{=} 4E_aE_b \frac{|\vec{p}_a|}{E_a} \overset{\text{in any ref frame}}{=} 4\sqrt{(p_ap_b)^2 - m_a^2 m_b^2}$$

In addition if in $f_1 \dots f_n$ there are some subset of "m" identical particles we need to include a factor $\frac{1}{m!}$ to avoid double counting.

For a process $\underbrace{a+b}_{\text{Initi}} \to \underbrace{c+d}_{\text{final}}$



we defined $s = (p_a+p_b)^2 = (p_c+p_d)^2$ and the integral in dQ_2 can be done as for $A \to p_1 p_2$ but with $M_A \to \sqrt{s}$. We get

$$|\vec{p}_a| = |\vec{p}_b| \equiv p_i = \frac{1}{2\sqrt{s}} \sqrt{(s-(m_a+m_b)^2)(s-(m_a-m_b)^2}$$

$$|\vec{p}_c| = |\vec{p}_d| \equiv p_f = \frac{1}{2\sqrt{s}} \sqrt{(s-(m_c+m_d)^2)(s-(m_c-m_d)^2)}$$

and $\text{Flux} = 4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2} = 4 p_i \sqrt{s}$

and $dQ_2 = \frac{1}{4\pi^2} \frac{p_f}{4\sqrt{s}} d\Omega$

$\simeq 1$ for $s \gg m_a^2, m_b^2, m_c^2, m_d^2$

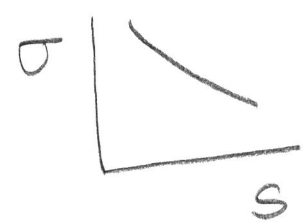
So
$$\left.\frac{d\sigma_{a+b\to c+d}}{d\Omega}\right|_{COM} = \frac{1}{64\pi^2 s}\underbrace{\frac{p_f}{p_i}}\left|\mathcal{M}_{a+b\to c+d}\right|^2$$

← diff cs one must specify the ref frame

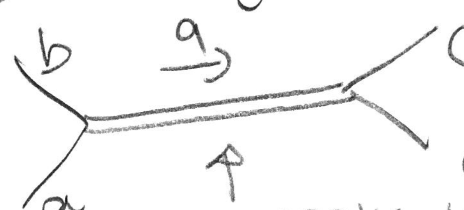
dimension $\left[\frac{d\sigma}{d\Omega}\right] = L^2 = E^{-2} \Rightarrow |\mathcal{M}_{a+b\to c+d}|$ is dimensionless

so in most cases the dominant dependence on "s" is in the prefactor (from the flux) $\frac{1}{s}$

$\Rightarrow$ generally cs decreases at higher energy



the exception occurs when we have a contribution to the amplitude of the form



massive intermediate state A

In this case
$$\mathcal{M}_{ab\to cd} \sim \frac{1}{\underset{q^2}{s} - M_A^2}$$
from the propagator

Which diverges when $s \to M_A^2$

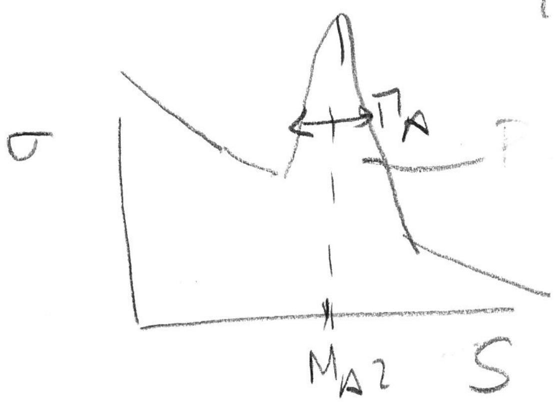
In this case we need to modify the propag.

$$\frac{1}{s - M^2 + i\epsilon} \longrightarrow \frac{1}{s - M_A^2 + i\Gamma_A M_A}$$

Γ_A ← total decay width of intermediate particle

In this case

$$\sigma(a+b \to c+d) \sim \frac{1}{(s-M_A^2)^2 + \Gamma_A^2 M_A^2}$$



peak of width Γ_A

Near $s = M_A^2$ there is an increase of the cs due to the production of the A state as a real physical particle.

The width of the peak measures the decay width of A.

For very fast decaying particles ($\equiv$ resonances) this is the way we measure their decay ~~t~~ lifetime by scanning $\sigma(s)$ around $s \simeq M^2$

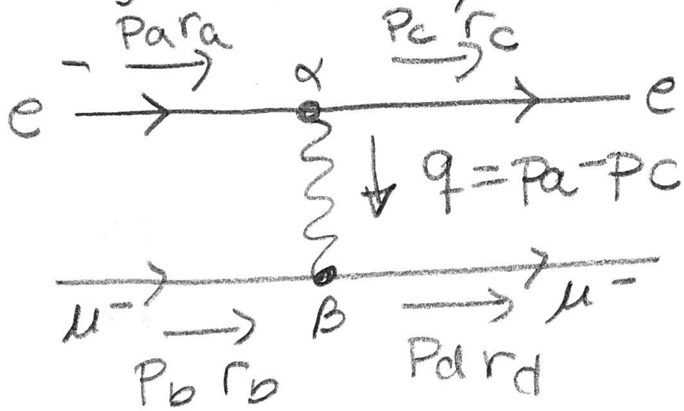
② Cross section $e^-\mu^- \to e^-\mu^-$

$e^-\ \mu^- \to e^-\ \mu^-$

$p_a\ p_b \qquad p_c\ p_d \qquad$ 4-momentum

$r_a\ r_b \qquad r_c\ r_d \qquad$ helicities

Only one diagram in QED



$r_a, r_b, r_c, r_d = 1,2$ or $(\pm)$

using F.R. $\quad \equiv \bar{u}^c_{elc}$

$$iM = \underbrace{\Big[\underbrace{\bar{u}^{r_c}_{elec}(p_c)}_{1\times 4}\ \underbrace{(ie\gamma^\alpha)}_{4\times 4}\ \underbrace{u^{r_a}_{elec}(p_a)}_{4\times 1}\Big]}_{1\times 1 \,\equiv\, \text{complex}\,\#}\ \underbrace{\frac{-ig_{\alpha\beta}}{(p_a-p_c)^2}}_{\text{complex}\,\#}\ \underbrace{\Big[\bar{u}^{r_d}_{mum}(p_d)(ie\gamma^\beta)u^{r_b}_{mum}(p_b)\Big]}_{\text{complex}}$$

in spinor space

$$= \frac{ie^2}{(p_a-p_c)^2}\left[\bar{u}^c_{ele}\gamma^\alpha u^a_{ele}\right]\left[\bar{u}^d_{mum}\gamma_\alpha u^b_{mum}\right] \equiv i M_{r_a r_b r_c r_d}$$

this is the amplitude for a given set of helicities

If we do not meassure the helicities of initial or final particles what we meassure is the sum over final helicities and average over initial helicities

Each helicity amplitude corresponds to a different physical process so the sum is incoherent. So the unpolarized c.s. for

$$a + b \to f_1 \cdots f_n$$

is proportional to the unpolarised squared ampli[tude]

$$|\overline{M}|^2 \equiv \frac{1}{(2S_a+1)} \frac{1}{(2S_b+1)} \sum_{r_a r_b} \sum_{r_1 \cdots r_n} |M_{r_a r_b r_1 \cdots r_n}|^2$$

$2S+1 \equiv$ # helicity states of particle with spin S_a (assumed massive)

for $a, b = e, \mu$ $\quad S_a = S_b = 1/2$

So for $e\mu \to e\mu^-$

$$|\overline{M}|^2 = \frac{e^4}{(p_a - p_c)^2} \underbrace{\overbrace{\frac{1}{2} \sum_{r_a r_c} [\bar{u}^{r_c}_{ele}(p_c) \gamma^\alpha u^{r_a}_{ele}(p_a)] [\bar{u}^{r_c}_{ele}(p_c) \gamma^\beta u^{r_a}_{ele}(p_a)]^*}^{L^{\alpha\beta}_{elec}} \frac{1}{2} \sum_{r_b r_d} [\bar{u}^{r_d}_{muon}(p_d) \gamma_\alpha u^{r_b}_{muon}(p_b)] [\bar{u}^{r_d}_{muon}(p_d) \gamma_\beta u^{r_b}_{muon}(p_b)]^*}_{L_{\alpha\beta, muon}}$$

To evaluate this square analytically we employ a set of standard techniques

a) $[\bar{u}^{rc}(p_c)\gamma^\beta u^{ra}(p_a)]^*$ is a number $\Rightarrow * = \dagger$

$$\overset{\parallel}{[\bar{u}^{rc}(p_c)\gamma^\beta u^{ra}(p_a)]^\dagger} = u^{ra\dagger}(p_a)\underbrace{\gamma^{\beta\dagger}\gamma^{0\dagger}}_{=\gamma^0\gamma^\beta} u^{rc}(p_c)$$

$$\overset{\parallel\parallel}{u^{rc\dagger}(p_c)\gamma^0}$$

$$= \bar{u}^{ra}(p_a)\gamma^\beta u^{rc}(p_c)$$

b) $[\bar{u}^c\gamma^\alpha u^a][\bar{u}^a\gamma^\beta u^c]$ is a # $\Rightarrow$ $= Tr(\#)$

$$L^{\alpha\beta}_{elec} = \frac{1}{2}\sum_{ra\,rc} Tr\{[\bar{u}^c\gamma u^a][\bar{u}^a\gamma u^c]\}$$

c) Trace is cyclic $\quad Tr(ABC) = Tr(CAB)$

$$L^{\alpha\beta}_{elec} = \frac{1}{2}\sum_{ra\,rc} Tr[u^c\bar{u}^c\gamma^\alpha u^a\bar{u}^a\gamma^\beta]$$

d) Tr and Σ commute

$$L^{\alpha\beta}_{el} = \frac{1}{2} Tr\left\{\sum_{rc}(u^c\bar{u}^c)\gamma^\alpha\sum_{ra}(u^a\bar{u}^a)\gamma^\beta\right\}$$

e) completness $\quad \sum_r u^r(p)\bar{u}^r(p) = \not{p} + m$

$$L^{\alpha\beta}_{elec} = \frac{1}{2} Tr[(\not{p}_c + m_e)\gamma^\alpha(\not{p}_a + m_e)\gamma^\beta]$$

Trace and contraction theorems for γ matrices

1) $\mathrm{Tr}\, I = 1$

2) $\mathrm{Tr}(\text{odd \# of } \gamma\text{'s}) = 0$ ← not γ^5

3) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu} \Rightarrow \mathrm{Tr}(\not{a}\not{b}) = 4(ab)$

4) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4(g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta})$

5) $\mathrm{Tr}\, \gamma^5 = 0$

6) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^5) = 4 i \epsilon^{\mu\nu\alpha\beta}$

Contractions

7) $\gamma^\mu \gamma_\mu = 4$

8) $\gamma^\mu \gamma^\alpha \gamma_\mu = -2 \gamma^\alpha$

9) $\gamma^\mu \gamma^\alpha \gamma^\rho \gamma_\mu = 4 g^{\alpha\rho}$

10 $\gamma^\mu (\gamma^\alpha \gamma^\beta \gamma^\rho) \gamma_\mu = -2 \gamma^\rho \gamma^\beta \gamma^\alpha$

With this

$$L^{\alpha\beta}_{elec} = \frac{1}{2} [\mathrm{Tr}(\not{p}_c \gamma^\alpha \not{p}_a \gamma^\beta] + 4 m_e^2 g^{\alpha\beta}]$$

$$= 2[P_c^\alpha P_a^\beta + P_c^\beta P_a^\alpha - (P_c P_a) g^{\alpha\beta}] + 4 m_e^2 g^{\alpha\beta}]$$

Equivalently

$$L_{\alpha\beta,\, muon} = 2[P_{D,\alpha} P_{b,\beta} + P_{d,\beta} P_{b\alpha} - (P_D P_a) g_{\alpha\beta})$$

$$+ 4 m_\mu^2 g_{\alpha\beta}]$$

and couhach

$$|\overline{M}|^2 = \frac{4e^4}{(p_a - p_c)^2}\Big[2(p_a p_d)(p_b p_c) + 2(p_a p_b)(p_c p_d) - 2m_e^2(p_b p_d) - 2m_\mu^2(p_a p_c) + 4m_e^2 m_\mu^2\Big]$$

In terms of Maudelstan variable:

$$s \equiv (p_a + p_b)^2 = (p_c + p_d)^2 = m_e^2 + m_\mu^2 + 2(p_a p_b)$$
$$= m_e^2 + m_\mu^2 + 2(p_c p_d)$$

$$t \equiv (p_a - p_c)^2 = (p_b - p_d)^2 = 2m_e^2 - 2(p_a p_c)$$
$$= 2m_\mu^2 - 2(p_b p_d)$$

$$s = (p_a - p_d)^2 = (p_b - p_c)^2 = m_e^2 + m_\mu^2 - 2(p_a p_d)$$
$$\doteq m_e^2 + m_\mu^2 - 2(p_c p_b)$$

neglecting m_e, m_μ (assme $E_i \gg m$)

$$|M|^2 = 2\frac{e^4(s^2+u^2)}{t^2}$$

from photon progpas in "t-channel"

this is written in terms of Lorentz invariant quan-

In coM

$m_e = 0$ $\quad$ $m_\mu = 0$

$$E_a = |\vec{p}_a| = \frac{\sqrt{s}}{2} \equiv |\vec{p}_b| = E_b \equiv p_i$$

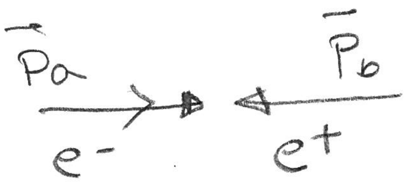
$$E_c = |\vec{p}_c| = \frac{\sqrt{s}}{2} = |\vec{p}_d| = E_d \equiv p_f$$

$m_\mu = 0$

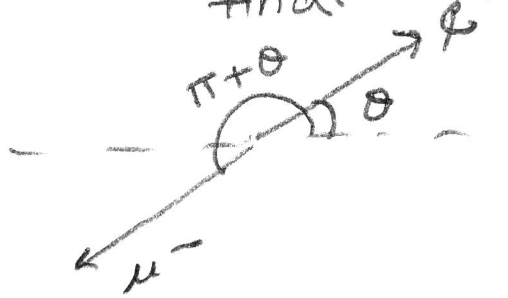
so the expression in terms of Maudelstan variables is valid in any ref frame

COM

Initial



Final



$\hat{z} \equiv$ collision axis

$\theta = 0 \Rightarrow$ outgoing e^- in direction of incoming e^-
" μ^- " " μ^-

in this frame

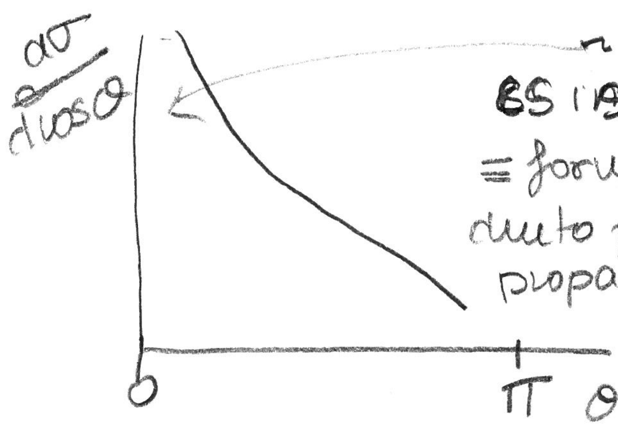
$$t \simeq -2(p_a p_c) = -2E_aE_c(1-\cos\theta) = -\frac{s}{2}(1-\cos\theta) = -s\sin^2\frac{\theta}{2}$$

$$u \simeq -2(p_a p_d) = -2E_aE_d(1+\cos\theta) = -\frac{s}{2}(1+\cos\theta) = -s\cos^2\theta/2$$

Altogether

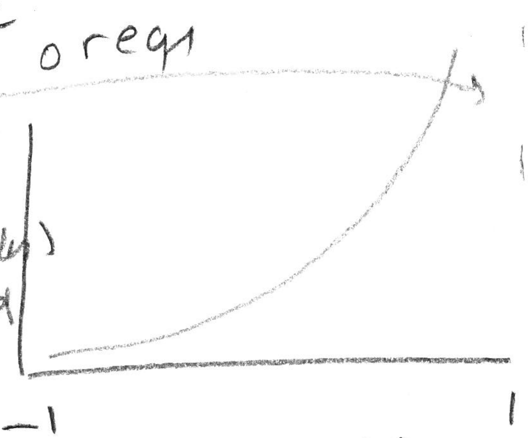
$$\left.\frac{d\sigma}{d\Omega}(e^-\mu^-\to e^-\mu^-)\right|_{COM} = \overbrace{\frac{e^4}{32\pi^2 s}}^{\alpha^2/2s} \cdot \frac{(1+\cos^4\theta/2)}{\sin^4\theta/2}$$

$$\left.\frac{d\sigma}{d\cos\theta}\right|_{COM} = 2\pi\alpha^2 \frac{(1+\cos^4\theta/2)}{\sin^4\theta/2}$$



is ∞ at $\theta=0$
$\equiv$ forward peak
due to photon ($\equiv$massless)
propag in t channel

or eq.



you can infer this from the Feynman diag just seeing a massless propag in "t"-channel

So the most probable direction of emission of e^- is in the direction of incoming e^-

So $$\sigma = \int_{-1}^{1} \frac{d\sigma}{d\cos\theta}\, d\cos\theta = \infty$$

this CS is infinite because the em interaction between the e^- and μ^- has ∞ range (which is the same as saying that $m_{photon} = 0$)
the process has the same initial and final states (≡ elastic scattering) so the e^- and μ^- are "feeling each other" from $-\infty$ to $+\infty$ in time.

But notice that this ∞ is not observable because to observe this process we need a minimum deflection of the e^- (or μ^-) from its incoming direction
So we never observe exactly $\theta = 0$.

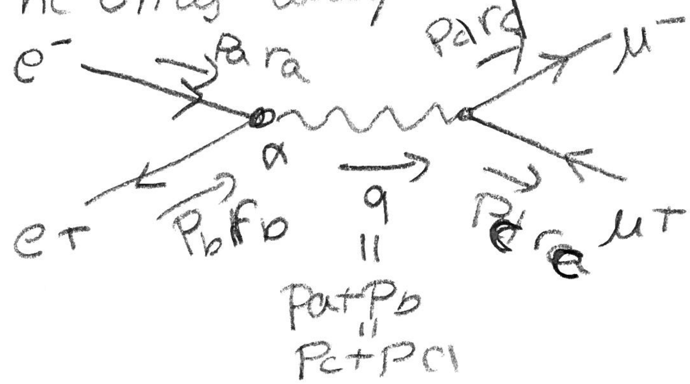
3 Some process at lowest order

a) $e^- e^+ \to \mu^- \mu^+$

$p_a \; p_b \quad p_d \; p_c$

$r_a \; r_b \quad r_d \; r_c$

the only diagram in QED



comparing with $e^- \mu^- \to e^- \mu^-$ $\qquad$ $e^- e^+ \to \mu^+ \mu^-$

initial $e^-(p_a)$ $\longrightarrow$ initial $e^-(p_a)$

final $\mu^-(p_d)$ $\longrightarrow$ final $\mu^-(p_d)$

Initial $\mu^-(p_b)$ $\longrightarrow$ final $\mu^+(p_c)$ $\equiv \mu^-(-p_c)$

final $e^-(p_c)$ $\longrightarrow$ initial $e^+(p_b)$ $\equiv e^-(-p_b)$

$s=(p_a+p_b)^2 \longrightarrow (p_a-p_c)^2 = t$

$t=(p_a-p_c)^2 \longrightarrow (p_a+p_b)^2 = s$

$u=(p_a-p_d)^2 \longrightarrow (p_a-p_d)^2 = u$

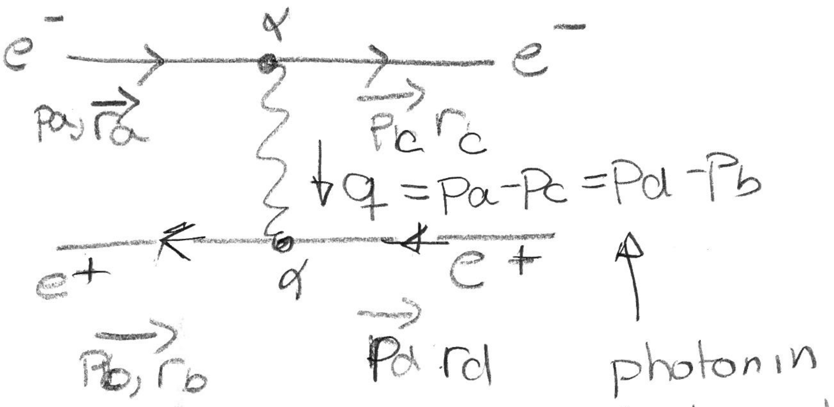
So $\overline{|M|}^2_{e^+e^- \to \mu^+\mu^-} = \dfrac{2e^4}{s^2}(t^2+u^2)$

b) $e^- e^+ \to e^- e^+$ (Bhabba scattering)

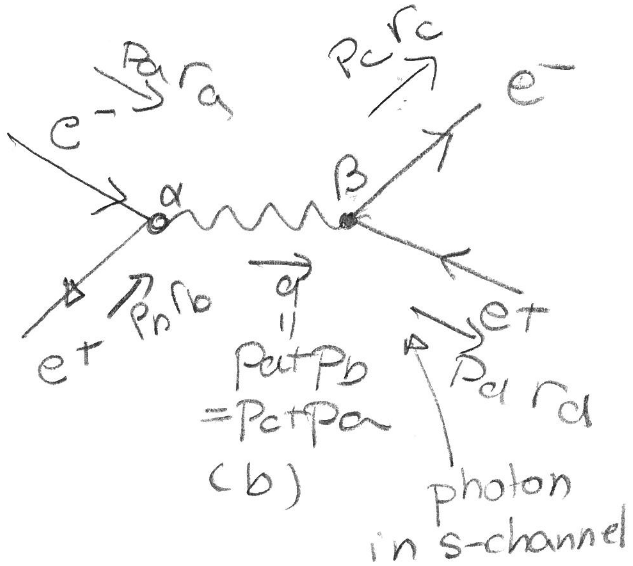
$p_a \; p_b \quad p_c \; p_d$

$r_a \; r_b \quad r_c \; r_d$

two diagrams



(a)



(b)

$$\mathcal{M}_a = \frac{e^2}{t} (\bar{u}^c \gamma^\alpha u^a)(\bar{v}^b \gamma_\alpha v^d)$$

relative sign

$$\mathcal{M}_b = -\frac{e^2}{s} (\bar{v}^b \gamma^\alpha u^a)(\bar{u}^c \gamma_\alpha v^d)$$

neglecting mass

$$\overline{|\mathcal{M}|^2} = \frac{e^2}{4} \Big\{ \frac{1}{t^2} \mathrm{Tr}(\not{p}_c \gamma^\alpha \not{p}_a \gamma^\beta)(\not{p}_b \gamma_\alpha \not{p}_d \gamma_\beta) \leftarrow (a)^2$$

$$+ \frac{1}{s^2} \mathrm{Tr}(\not{p}_c \gamma^\alpha \not{p}_d \gamma^\beta)(\not{p}_b \gamma_\alpha \not{p}_a \gamma_\beta) \leftarrow (b)^2$$

$$+ \frac{2}{st} \mathrm{Tr}[\not{p}_c \gamma^\alpha \not{p}_a \gamma^\beta \not{p}_b \gamma_\alpha \not{p}_d \gamma_\beta) \leftarrow 2\mathrm{Re}(ab^*) \Big\}$$

From $e^-\mu^- \to e^-\mu^-$ $\quad (a)^2 = 2e^4\,\frac{s^2+u^2}{t^2}$

$e^+e^- \to \mu^+\mu^-$ $\quad (b) = 2e^4\,\frac{t^2+u^2}{s^2}$

to evaluate the interference we use that

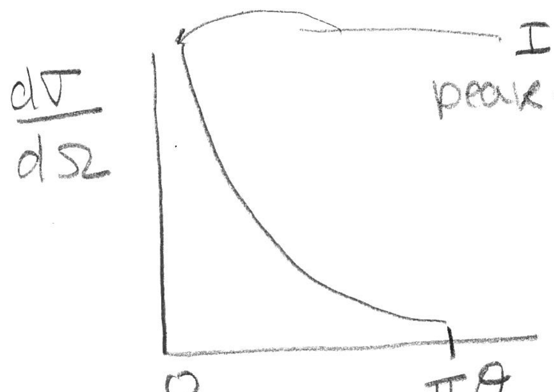
$$\gamma^\mu\gamma^\alpha\gamma_\mu = -2\gamma^\alpha$$
$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4g^{\alpha\beta}$$
$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\beta\gamma^\alpha$$

$$Tr(\not p_c\gamma^\alpha\not p_a\gamma^\beta\not p_b\gamma_\alpha\not p_d\gamma_\beta) = -2Tr(\not p_c\not p_b\gamma^\beta\not p_a\not p_d\gamma_\beta)$$
$$= -8(p_a p_d)\,Tr(\not p_c\not p_b) = -32(p_a p_d)(p_c p_b) = -8u^2$$

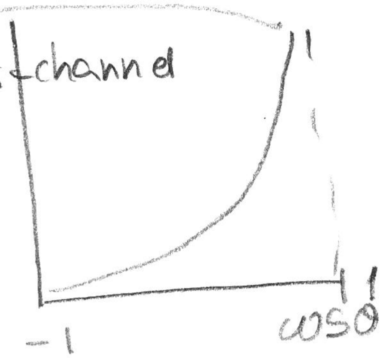
So altogether

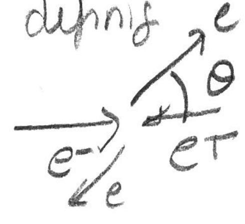
$$\frac{d\sigma}{d\Omega}(e^+e^-\to e^+e^-) = \frac{e^4}{32\pi^2 s}\left[\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2} + \frac{2u^2}{ts}\right]$$

$$\Big|_{COM} = \frac{\overset{\frac{e^2}{4\pi} =}{\alpha^2}}{2s}\left[\frac{1+\cos^4\theta/2}{\sin^4\frac{\theta}{2}} + \cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2} - \frac{2\cos^4\theta/2}{\sin^2\theta/2}\right]$$



(=0=)
Infinite forward peak due to γ propag in t-channel
or equiv

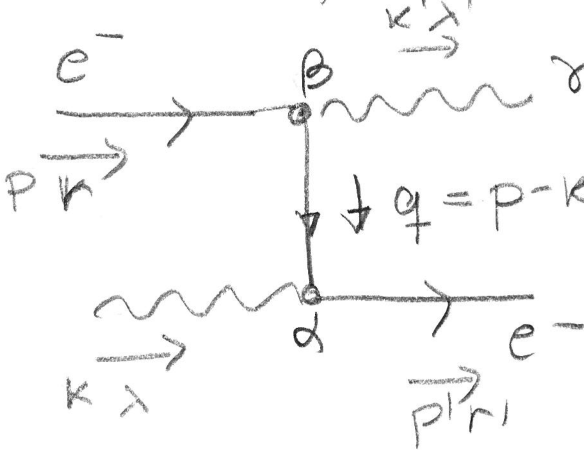


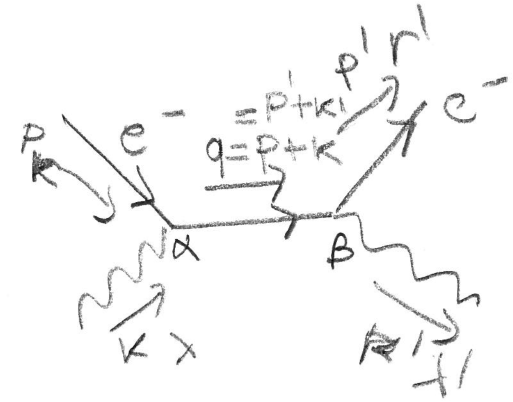


c) Compton scattering $e^- \gamma \to e^- \gamma$

$$\begin{array}{cccc} p & k & p' & k' \\ r & \lambda & r' & \lambda' \end{array}$$

Two diagrams





(a)
$q^2 = (p-k)^2 = u$

(b)
$q^2 = (p+k)^2 = s$

$$i\mathcal{M}_a = \bar{u}^{r'}(p')\, ie\gamma^\alpha\, i\frac{(\not{p}-\not{k}')+m}{(p-k')^2-m^2}\, ie\gamma^\beta u^r(p) \left[\epsilon_\lambda(k)\right]_\alpha \left[\epsilon^*_{\lambda'}(k')\right]_\beta$$

$$= -e^2\, \bar{u}'\gamma^\alpha \frac{(\not{p}-\not{k}')+m}{(p-k')^2-m^2}\gamma^\beta\, \epsilon_\alpha\, \epsilon'^*_\beta$$

$$i\mathcal{M}_b = -e^2\, \bar{u}'\gamma^\alpha \frac{(\not{p}+\not{k})+m}{(p+k)^2-m^2}\gamma^\beta\, \epsilon_\alpha\, \epsilon'^*_\beta$$

helicity av. squared amplitude

$$\overline{|\mathcal{M}|^2} = \frac{1}{2} \times \frac{1}{2} \sum_{\substack{\lambda\lambda' \\ rr'}} |\mathcal{M}_a + \mathcal{M}_b|^2$$

2 e^- ~~hel~~ $\qquad$ 2 γ pol

this can be simplified using gauge invariance which implies that we should get the same result usig $\epsilon^\mu(k)$ or $\epsilon^\mu(k) + \underset{\uparrow}{c}\, k^\mu$
constant
and als will $\epsilon^\mu(k')$ or $\epsilon^\mu(k') + c k'^\mu$

So if we write

$$\mathcal{M} \equiv T^{\alpha\beta} \left(\epsilon^*_{\lambda'}(k')\right)_\beta \left(\epsilon_\lambda(k)_\alpha\right)$$

then $k_\alpha T^{\alpha\beta} = k'_\beta T^{\alpha\beta} = 0$ (important test)

this property allows to sum over the photon polarization using completeness relation because

$$\sum_{\lambda=1}^{2} \epsilon_\lambda^{*\mu} \epsilon_\lambda^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{|\vec{k}|^2}$$

and the second term dos not contribute so

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{rr'} T^{\alpha\beta} T^*_{\alpha\beta}$$

neglecting mass

$$\stackrel{\downarrow}{=} \frac{e^4}{4} \Big\{ \frac{1}{u^2} \mathrm{Tr}[\not{p}' \gamma^\alpha (\not{p} - \not{k}') \gamma^\beta \not{p} \gamma_\beta (\not{p} - \not{k}') \gamma_\alpha] \leftarrow (a)^2$$

$$+ \frac{1}{s^2} \mathrm{Tr}[\not{p}' \gamma^\beta (\not{p} + \not{k}) \gamma^\alpha \not{p} \gamma_\alpha (\not{p} + \not{k}) \gamma_\beta] \leftarrow (b)^2$$

$$+ \frac{2}{su} \mathrm{Tr}[\not{p}' \gamma^\alpha (\not{p} - \not{k}') \gamma^\beta \not{p} \gamma_\alpha (\not{p} + \not{k}) \gamma_\beta] \leftarrow \text{interf.}$$

surprisingly the interference

$$\mathrm{Tr}[\not{p}'\gamma^\alpha(\not{p}-\not{k}')\gamma^\beta\not{p}\gamma_\alpha(\not{p}+\not{k})\gamma_\beta]$$

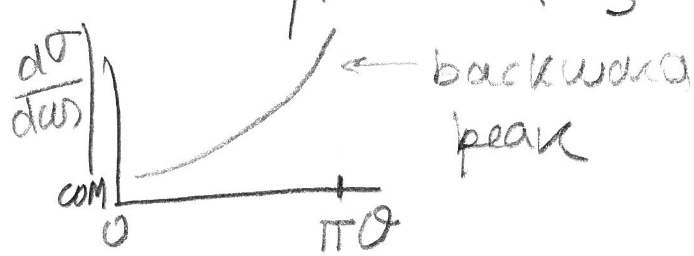
$$=-2\,\mathrm{Tr}[\not{p}'\not{p}\gamma^\beta(\not{p}-\not{k}')\gamma(\not{p}+\not{k})\gamma_\beta]$$

$$=32\,(p'p)\underbrace{(p-k')(p+k)}_{pk-pk'-kk'=2(s+t+u)=0}$$

So the final result

$$\overline{|M|^2}=2e^4\left(-\frac{u}{s}-\frac{s}{u}\right)$$

← peak at $\theta=\pi$ from e^- propag neglecting m_e



So most probable direction of photon emission is back ≡ laser back scattering

If one does the calculation in a system with a very high energy e^- hits a very low energy γ (practice exam exercise) one finds that the out going γ bounces back carrying most of the incoming e^- energy ≡ method for accelerating γ's

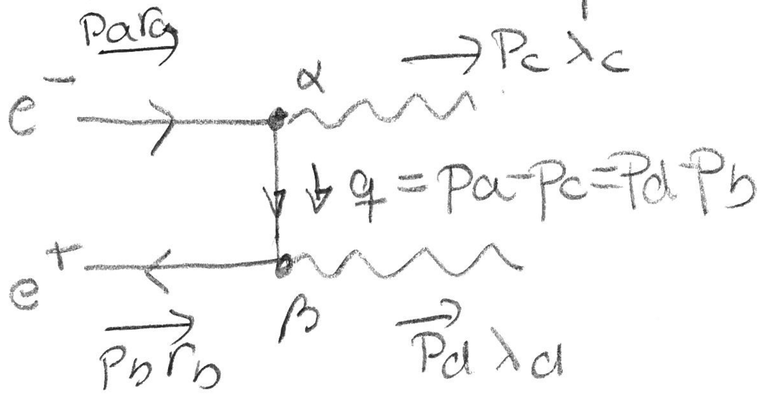
the peak at $\theta=\pi$ is not ∞ once m_e is included and one gets $\sigma(e^-\gamma\to e^-\gamma)\neq\infty=\frac{2\pi\alpha^2}{s}\ln\frac{s}{m_e^2}$

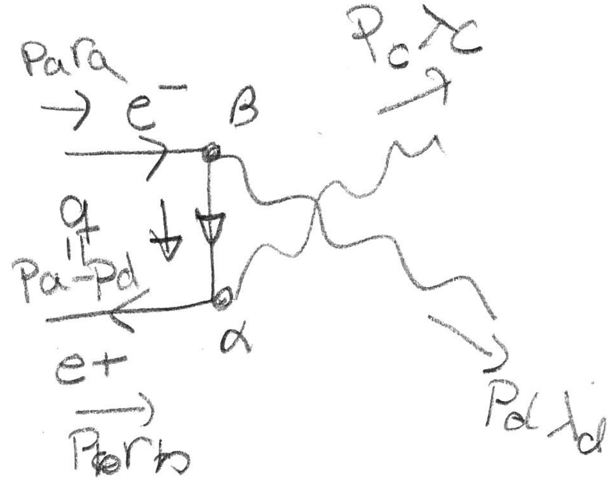
a) $e^+e^- \to \gamma\gamma$ pair annihilation

$p_a \, p_b \quad p_c \, p_d$

$r_a \, r_b \quad \lambda_c \, \lambda_d$

two tree level diagrams





(a)

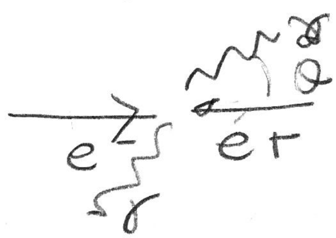
$$q^2 = (p_a - p_c)^2 = t$$

(b)

$$q^2 = (p_a - p_d)^2 = u$$

$$\overline{|M|^2} = 2e^4\left(\frac{t}{u} + \frac{u}{t}\right)$$

So in COM

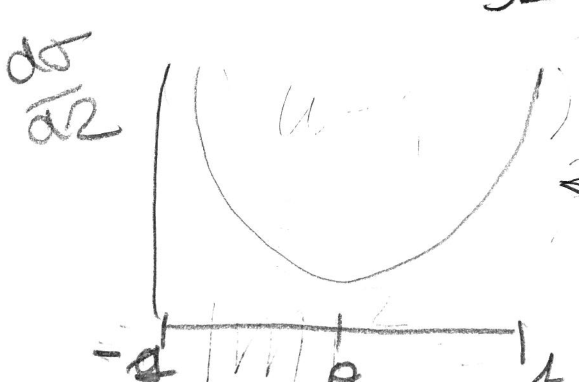


from identical particles

$$\frac{d\sigma}{d\Omega}(e^+e^- \to \gamma\gamma)\Big|_{COM} = \frac{e^4}{32\pi^2 s} \, \frac{1}{2}\left[\frac{\sin^2\theta/2}{\cos^2\theta/2} + \frac{\cos^2\theta/2}{\sin^2\theta/2}\right]$$

backward peak

forward peak



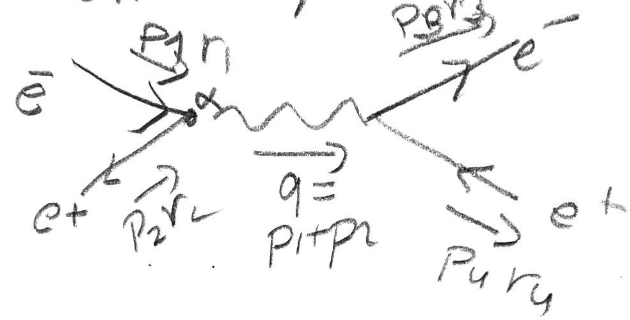
Forward-backward symmetric since $\theta \leftrightarrow \pi - \theta$ exchanged final photons and they are identical

5 Higher orders: renormalization, running coupling constant and magnetic moments

So far we had made predictions for cross sections at lowest order in perturbation theory and good agreement with data

I.e $e^+e^- \to \mu^+\mu^-$

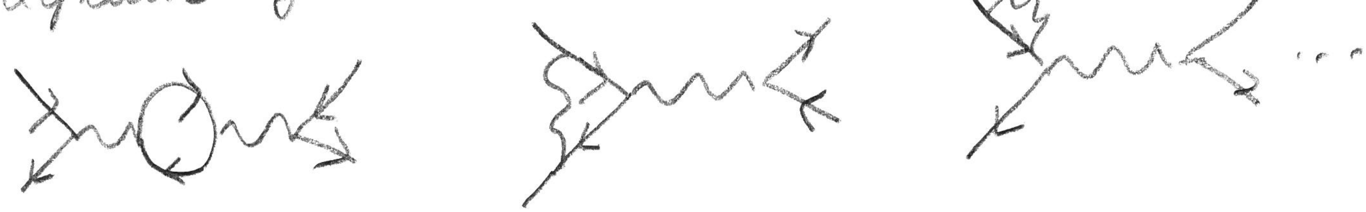
has one diagram at tree level



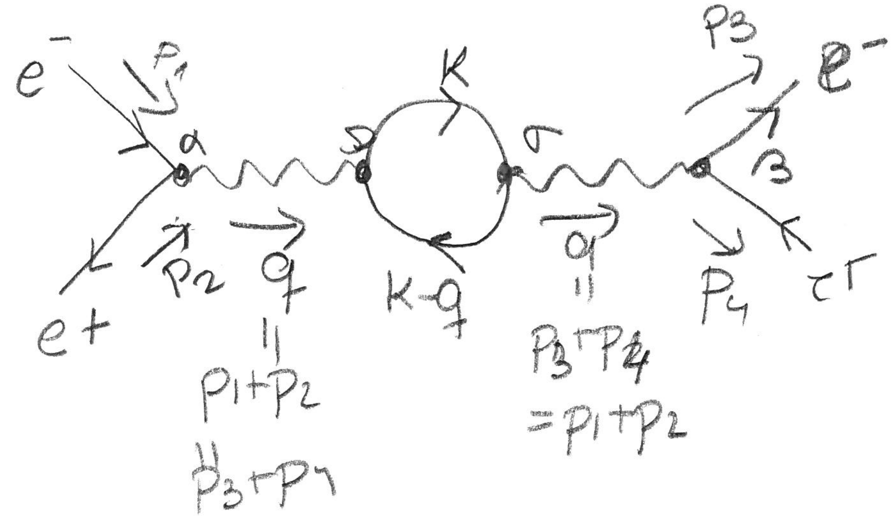
$$\mathcal{M}_{tree}(e^+e^- \to \mu^+\mu^-) = -e^2\left(\frac{-i g_{\alpha\beta}}{q^2 = (p_1+p_2)^2}\right)(\bar{v}_2\gamma^\alpha u_1)(\bar{u}_3\gamma^\beta v_4)$$

$$\equiv e^2 \mathcal{M}_0$$

At next order in perturbation theory there are many diagrams. I.e



Lets have a look at 1st one.



works for any value of k

Using FR

$$\mathcal{M}_{1\,loop} = -e^2 \; \frac{-i g_{\alpha\rho}}{q^2} \; \frac{-i g_{\sigma\beta}}{q^2} \; \bar{v}_2 \gamma^\alpha u_1 \, \bar{u}_3 \gamma^\beta v_4$$

$$\times \underbrace{e^2 \int \frac{d^4k}{(2\pi)^4} \, (-) \, \frac{Tr\left[\gamma^\rho(\not{k}+m)\gamma^\sigma((\not{k}-\not{q})+m)\right]}{(k^2-m^2)((k-q)^2-m^2)}}_{\equiv\; -ie^2 I^{\rho\sigma}(q)}$$

comparing with $\mathcal{M}_{tree}$, $\mathcal{M}_{1\,loop}$ we see that

tree-level

$$\frac{-i g_{\alpha\beta}}{q^2} \longrightarrow \frac{-i g_{\alpha\rho}}{q^2} \left(-ie^2 I^{\rho\sigma}\right) \left(\frac{-i g_{\sigma\beta}}{q^2}\right)$$

It can be show that $I^{\rho\sigma}(q) = -q^{\rho\sigma}\, F(q^2)$

$$F(q^2) = \frac{1}{q^2} \int \frac{d^4k}{(2\pi)^4} \frac{Tr(\gamma^\rho(\not{k}+m)\gamma_\rho(\not{k}-\not{q}+m))}{(k^2-m^2)((k-q)^2-m^2)}$$

So rule $\qquad$ one loop

$$\frac{-ig_{\alpha\beta}}{q^2} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2}\left[-e^2F(q^2)\right]$$

$$\mathcal{M}_{tree}+\mathcal{M}_{1loop} = \mathcal{M}_0\, e^2\left[1-e^2F(q^2)\right]$$

In $F(q^2)$ we are integrating to any value of k^2.

up to $|k| \equiv \Lambda \to \infty$

- $d^4k \to |k|^3 d|k| \to \Lambda^4$
- Numerator $\to |k|^2 \to \Lambda^2$
- Denominator $\to |k^4| \to \Lambda^{-4}$

$\Rightarrow F(q^2)$ diverges when $\Lambda \to \infty$

Doing the calculation

$$F(q^2) = \underbrace{\frac{1}{12\pi^2}\lim_{\Lambda\to\infty}\ln\frac{\Lambda^2}{m^2}}_{\substack{\parallel \\ F(0)\ \xrightarrow[\Lambda\to\infty]{}\ \infty}} + \underset{\text{finite}}{f(q^2)}$$

$$f(q^2) = \begin{cases} \dfrac{1}{60\pi^2}\dfrac{q^2}{m^2} & \text{for } q^2 \ll m^2 \\ -\dfrac{1}{12\pi^2}\ln\dfrac{q^2}{m^2} & q^2 \gg m^2 \end{cases}$$

So $F(q^2)$ is ∞ but the ∞ piece is independent of q^2. It would be the same in any diagram with  or 

What is the meaning of this ∞?

To understand it we have to think what is the "e" (coupling constant) that we are using in our calculations.

So far we have been identifying $e \equiv \sqrt{4\pi\alpha}$ with $\alpha \simeq \frac{1}{137}$

But how did we get that numerical value?

It was obtained by comparing some data with a theoretical prediction for that observable made at some order in perturbation theory.

So if we want to use the extracted value of "e" in another prediction we must to the two predictions ~~at~~ the same order

When this is done that apparent ∞ disappears from the prediction since it is "absorbed" in the ~~definition~~ measured coupling constant

Let us call "e" to the parameter in the Lag

≡ this is the "e" in my calculations.

Let us call $e_{phys} \equiv \sqrt{4\pi\alpha} \equiv \sqrt{\frac{4\pi}{137}}$ the meassured

constant in thompson scattering. $e^- \gamma \to e\gamma$ at $q_0^2 \to 0$

at one-loop the prediction fo the compton scattering

CS will depend on

$$e^2(1 - e^2 F(q_0^2))$$

q_0^2 for compton scattering $q_0^2 \simeq 0$

so what we are extracting from data

$$e^2_{phys} \equiv e^2(1 - e^2 F(0)) \equiv \frac{4\pi}{137}$$

When we make prediction for other opreran at 1 loop

the amplitude will be proportional to

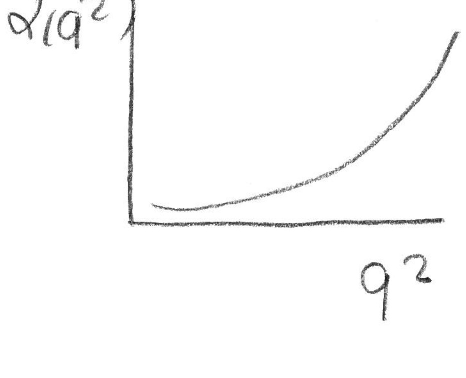
q^2 in what ever process we are predicting

$$e^2(1 - e^2 F(q^2)) = \underbrace{e^2(1 - e^2 F(q_0^2))}_{e^2_{phys} \text{ which is a \#}} \frac{(1 - e^2 F(q^2))}{(1 - e^2 F(q_0^2))}$$

and $\dfrac{1 - e^2 F(q^2)}{1 - e^2 F(q_0^2)}$ is finite because the ∞ in the numerator and in the denomi. cancel this process is called "renormalisation of the coupling constant". One can do similar procedure with all parameters of the theory (masses, wave funct normalisation) and after this all predictions in terms of renormalized parameters is finite to any order in perturbation theory. This is so because QED is a gauge theory and gauge theories have been proven to be renormalisable. After renormalisation of α we are left with finite q^2 dependent corrections to the tree level predictions. I.e the coupling constant for $q^2 \gg m_e^2$

$$\alpha_{phys}(q^2) = \frac{e_{phys}(q^2)}{4\pi} \simeq \alpha(0)\left(1 + e^2 \underbrace{f(q^2)}_{= -\frac{1}{12\pi^2}\ln\frac{q^2}{m_e^2}}\right)$$

$$\simeq \frac{\overbrace{\alpha(0)}^{= 1/137}}{1 - \underbrace{\frac{\alpha(0)}{3\pi}\ln\frac{q^2}{m_e^2}}_{>0}} \Rightarrow \alpha(q^2) > \alpha(0)$$



we say that the effective coupling constant "runs" with energy. In QED ~~this~~ increases with com the energy of the process

For QED this running is very slow but measurable

For example for $e^+e^- \to \mu^+\mu^-$ at $q^2 \simeq M_Z^2 \simeq (91\,GeV)^2$

the observed $\alpha[(100\,GeV)^2] \simeq \frac{1}{128} > \frac{1}{137}$

How sure are we that QED is the quantum theory of em interactions?

Experimentally the most precisely measured em property is the magnetic moment of the e^-

g_e which gives the coupling of e^- to $\vec{B}$ as

$$\frac{-e}{2m} \frac{g_e}{2} \vec{\sigma} \cdot \vec{B}$$

and it has been measured to be

$$g_e^{exp} = 2(1.001159652180\,76 \pm 0.000000000000\,28) \quad \text{precision } 3 \text{ in } 10^{12}$$

$$\Rightarrow \sigma_{exp} \sim 3 \times 10^{-12}$$

In QED this coupling is part of the vertex

$$-e\bar{\Psi}\gamma^\alpha\Psi A_\alpha$$

(which use Dirac Eq's)

using Gordon identities one can write

$$-e\bar{\Psi}\gamma^\alpha\Psi A_\alpha = \frac{-e}{2m}\left[i\bar{\Psi}\partial^\alpha\psi A_\alpha - i\partial^\alpha\bar{\Psi}\psi A_\alpha - \bar{\Psi}\sigma^{\alpha\beta}\psi\partial_\beta A_\alpha\right]$$

where $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha,\gamma^\beta] \Rightarrow \sigma^{ij} = \epsilon^{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}$

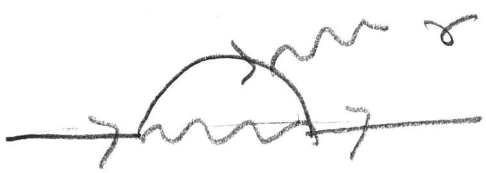
So the last term contains

$$\sigma^{ij}\partial_j A_i = \underbrace{\epsilon^{ijk}\sigma_k\,\partial_j A_i}_{-(\vec{\nabla}\times\vec{A})_k \equiv -\vec{B}_k} = -\vec{\sigma}\cdot\vec{B}$$

So in QED the vertex coupling to $\vec{B}$ is

$$\frac{-e}{2m_e}\vec{\sigma}\,\vec{B} \Rightarrow g_e^{QED,\ \text{tree level}} = 2$$

at next order we have a correction



$$\Rightarrow g_e^{QED\ 1\ loop} = 2(1.00116)$$

better

the prediction including all diagrams up to 9 loop

$$g_e^{\text{9-loops}} = 2(1.001159652181643 \pm 0.000000000000764) \Rightarrow \sigma_{theo} \simeq 8 \times 10^{-12}$$

$$\Rightarrow g_e^{exp} - g_e^{\text{9 loops prediction}} = 2(-883 \pm \underbrace{\sqrt{764^2+280^2}}_{813}) \times 10^{-15}$$

agree in 1.1σ !!

For the muon (PDG)

$$g_\mu^{exp} = 2(1.00116592059^{+} \pm 0.00000000022)$$

and the prediction

$$g_\mu^{\text{5 loops}} = 2(1.00116591954 \pm 0.00000000057) \quad \text{or} \quad 2(1.00116591810 \pm 0.\dots\dots.43)$$

$$g_\mu^{exp} - g_\mu^{\text{5 loop}} = \begin{cases} 2\times(105 \pm \overbrace{\sqrt{22^2+57^2}}^{61}) \to \text{agreement at} \sim 1.5\sigma \\ 2\times(249 \pm \underbrace{\sqrt{22^2+43^2}}_{48}) \to \simeq 5\sigma \text{ disagreement} \end{cases}$$

New physics?