Chapter 7

"QED as probe of the structure of hadrons"

0) elastic $e^-\mu^- \to e^-\mu^-$ in LAB (μ^- at rest)

1) concept of form factor

2) $e^- p \to e^- p$ elastic scattering

3) $e^- p \to e^- X$ deep inelastic scattering
"anything"

4) Bjorken scaling

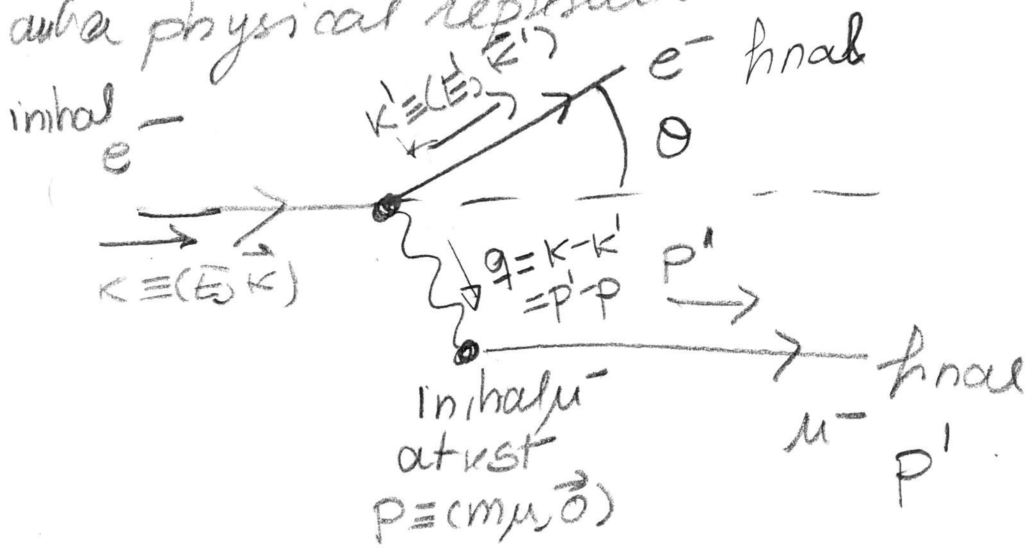
5) the quark-parton model of hadrons

6) Parton model for hadron-hadron collisions.

0) $e^- \mu^- \to e^- \mu^-$ in the LAB

We are going to study scattering of e^- on different targets to study the targets knowing what we know when target is a μ^-

We will represent these processes with a "drawing" which is an admixture of a Feynman diagram and a physical representation



We found the amplitude

$$\mathcal{M} = \frac{-e^2}{q^2} j^\alpha_{ele} \, j_{\alpha\, muon}$$

I am suppressing the helicity indexes

$$j^\alpha_e = \bar{u}_e(k') \gamma^\alpha u_e$$

$$j^\alpha_{muon} = \bar{u}_{muon}(p') \gamma^\alpha u_{muon}(p)$$

We found (for $m_e=0$)

$$|\overline{M}|^2 = \frac{8e^4}{q^4}\left[(k'p')(kp) - (kp')(k'p) - \overset{m_\mu^2}{\overset{\equiv}{M^2}}(kk')\right]$$

using $k^2 = k'^2 = m_e^2 = 0$

$$q^2 = (k-k')^2 \overset{\downarrow}{=} -2kk' = -2EE'(1-\cos\theta) = -4EE'\sin^2\frac{\theta}{2}$$

$$k'p = ME'$$

$$kp = ME$$

$$kp' = k(k+p-k') = kp - kk' = ME + \frac{q^2}{2}$$

$$k'p' = k'(k+p-k') = kk' + ME' = ME' - \frac{q^2}{2} \quad \Rightarrow q^2 = -2(\underbrace{E-E'}_{\nu})M$$

$$|\overline{M}|^2 = \frac{8e^4}{q^4}M^2\left[2EE' - \frac{q^2}{2} + \frac{q^2}{4M^2}\right] = \frac{16e^4}{q^4}M^2EE'\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

so

$$d\sigma = \underbrace{\frac{1}{4ME}}_{\text{initial flux in LAB frame}} \frac{|\overline{M}|^2}{(2\pi)^2}\frac{d^3k'}{2E'}\frac{d^3p'}{2p'_0}\delta^4(p+k-p'-k')$$

$$\delta^4(p+k-p'-k') = \delta^3(\vec{k}-\vec{p}\,'-\vec{k}') \, \delta(E+M-E'-p_0')$$

used to integrate $d^3p' \Rightarrow \vec{p}\,' = \vec{k}-\vec{k}'$

so $\delta(E+M-E'-p_0') = \delta(M+E-E'-\sqrt{M^2+|\vec{k}-\vec{k}'|^2})$

using $|\vec{k}-\vec{k}'|^2 = |\vec{k}|^2 + |\vec{k}'|^2 - 2\vec{k}\cdot\vec{k}' = E^2 + E'^2 - 2EE'\cos\theta$

$$= (\underbrace{E-E'}_{\nu})^2 - q^2$$

$$\delta(M+E-E'-p_0') = \delta(M+\nu-\sqrt{M^2+\nu^2-q^2})$$

$$(M+\nu)^2 - M^2 - \nu^2 + q^2 = 0 = 2M\nu + q^2 =$$

① link between ν and α $\downarrow (\Rightarrow E, \theta)$

We saw that for $e\mu \to e\mu$

$$\left.\frac{d\sigma^{e\mu\to e\mu}}{dE' d\Omega}\right|_{LAB} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \left[\cos^2\frac{\theta}{2} - \frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right] \delta\left(\nu + \frac{q^2}{2m_\mu}\right)$$

with $q^2 = -4EE'\sin^2\theta/2$ $\quad \nu = E - E'$

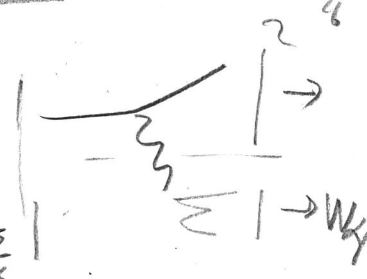
For $ep \to ep$

$$\frac{G_E^2 + \tau G_M^2}{1+\tau} \qquad \tau G_M^2 \qquad \tau = \frac{-q^2}{4M^2}$$

$$\left.\frac{d\sigma^{ep\to ep}}{dE' d\Omega}\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\left[G_1(q^2)\cos^2\frac{\theta}{2} + 2G_2(q^2)\sin^2\frac{\theta}{2}\right]\delta(\nu + \tfrac{q^2}{2M})$$

Functions of (q^2) to be determined from data

For $ep \to eX$ DIS we derived that



$$\left.\frac{d\sigma^{DIS}}{dE' d\Omega}\right|_{LAB} = \frac{\alpha^2}{q^4}\frac{E'}{E} L^{\alpha\beta}_{elec} W_{\alpha\beta}$$

$L^{\alpha\beta}_{elec} = 2(k^\alpha k'^\beta + k^\beta k'^\alpha - 2g^{\alpha\beta}(k k'))$ 2 independent variables)

We need to figure out the most general form of $W_{\alpha\beta}$

- It must be a Lorentz tensor because $L^{\alpha\beta}_{elec}$ is a tensor
⇒ it must be made with q, p
- It must be symmetric under $\alpha \leftrightarrow \beta$ because $L^{\alpha\beta}_{elec}$ is symmetric

the most general form is

$$W^{\alpha\beta} = -W_1(q^2,\nu)\, g^{\alpha\beta} + W_2(q^2,\nu)\frac{p^\alpha p^\beta}{M^2} + W_4(q^2,\nu)\frac{q^\alpha q^\beta}{M^2} + W_5(q^2,\nu)\frac{(p^\alpha q^\beta + q^\alpha p^\beta)}{M^2}$$

$W_{1,2,4,5}(q^2,\nu)$ parametrize our ignorance of the form of the em hadronic current. they need to be extracted from data. they are related

because

$$W^{\alpha\beta} \propto \sum_X \underbrace{J^\alpha_X \; J^\beta_X}_{\text{em hadronic current which is conserved}} \Rightarrow \partial_\alpha J^\alpha_X = 0$$

$$\Rightarrow q_\alpha W^{\alpha\beta} = 0$$

$$0 = -W_1 q^\beta + \frac{W_2}{M^2}(pq)p^\beta + \frac{W_4 q^2}{M^2} q^\beta + \frac{W_5}{M^2}\left((pq)q^\beta + q^2 p^\beta\right)$$

$$= q^\beta\left[-W_1 + W_4\frac{q^2}{M^2} + \frac{W_5}{M^2}(pq)\right] \Rightarrow = 0$$

$$+ p^\beta\left[\frac{W_2}{M^2}(pq) + \frac{W_5}{M^2}q^2\right] \Rightarrow = 0$$

$$\Rightarrow W_5 = \frac{-(pq)}{q^2} W_2 \qquad W_4 = \frac{M^2}{q^2} W_1 + \frac{(pq)^2}{q^4} W_2$$

$$\Rightarrow W^{\alpha\beta}(q^2,\nu) = W_1(q^2,\nu)\left[-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{M^2}\right] + W_2(q^2,\nu)\left(p^\alpha - \frac{pq}{M^2}q^\alpha\right)\left(p^\beta - \frac{pq}{M^2}q^\beta\right)$$

$$\text{so } L^{\alpha\beta} W_{\alpha\beta} = 4 W_1(q^2 \nu)\,(k k') + 2\,\frac{W_2(q^2 \nu)}{M^2}\left(2(pk)(pk') - M^2(kk')\right)$$

$$\overset{\text{LAB}}{=} 4EE'\left(2 W_1(q^2 \nu) \sin^2\frac{\theta}{2} + W_2(q^2 \nu) \cos^2\frac{\theta}{2}\right)$$

$$\left.\frac{d\sigma^{DIS}}{d\Omega}\right|_{LAB} = \frac{\alpha^2}{4EE' \sin^4\frac{\theta}{2}} \left\{ W_2(q^2 \nu) \cos^2\frac{\theta}{2} + 2 W_1(q^2 \nu) \sin^2\frac{\theta}{2} \right\}$$

$$= \delta\left(\nu + \frac{q^2}{2M}\right) \frac{2\sqrt{M^2+\nu^2-q^2}}{2M} = \frac{1}{2MA}\,\delta\left(E' - \frac{E}{A}\right)$$

using $q^2 = -2EE'\sin^2\theta/2$

with $A = 1 + \frac{2E}{M}\sin^2\theta/2$

$$d^3k' = |\vec{k}'|^2\, d|\vec{k}'|\, d\Omega \overset{m_e=0}{=} E'^2 dE'\, d\Omega$$

so we get

$$\left.\frac{d\sigma(e^-\mu^- \to e^-\mu^-)}{d\Omega\, dE'}\right|_{LAB} = \underbrace{\frac{(2\alpha E')^2}{q^4}}_{\frac{\alpha^2}{4E^2\sin^4\theta/2}} \left\{\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right\} \delta\left(\nu + \frac{q^2}{2M}\right)$$

link between E' and θ. so only 1 independent kinematic variable

integrating E' with δ

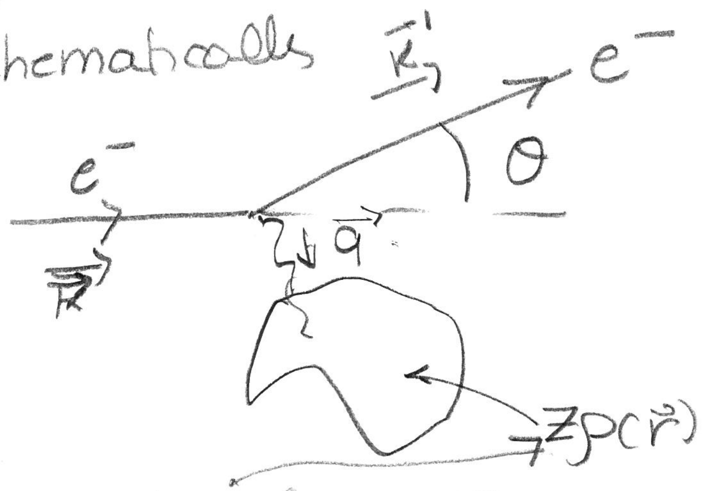
$$\left.\frac{d\sigma}{d\Omega}(e^-\mu^- \to e^-\mu^-)\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left\{\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right\}$$

with $E' = \frac{E}{1 + \frac{2E}{M}\sin^2\theta/2}$ and $\frac{q^2}{2M} = E' - E = \frac{2E^2\sin^2\frac{\theta}{2}}{M + 2E\sin^2\frac{\theta}{2}}$

① Form factor of a charge distribution

In classical em the scattering of e^- onto a static charge distribution $Z\rho(\vec{r})$ is used to obtain information about its structure

Schematically



Static $\Rightarrow |\vec{k}| = |\vec{k}'| \equiv E$

$\Rightarrow |\vec{q}|^2 = |\vec{k} - \vec{k}'|^2$

$= 4E^2 \sin^2 \theta/2$

$\Rightarrow$ if we know E and measure $\theta \Rightarrow |\vec{q}|$

we normalise $\int \rho(\vec{r})\, d^3\vec{r} = 1$

To obtain information on $\rho(\vec{r})$ we compare

$$\frac{d\sigma}{d\Omega}(eZ\,\text{point charge}) \text{ with } \frac{d\sigma}{d\Omega}(eZ\rho)$$

and the ratio is the form factor

$$\frac{d\sigma}{d\Omega}(eZ\rho) \equiv F(\vec{q})\, \frac{d\sigma}{d\Omega}(eZ\,\text{point})$$

$F(\vec{q})$ is the fourier transform of ρ

$$F(\vec{q}) = \int e^{-i\vec{q}\vec{r}} \rho(\vec{r})\, d^3\vec{r}$$

since we have normalised $\int \rho(\vec{r})\, d\vec{r} = 1$

$$\Rightarrow F(0) = 1$$

For a spherically symmetric distubution

only depends on modulus

$$F(|\vec{q}|) = \int \rho(r)\, r^2 dr \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\, e^{i|\vec{q}|r\cos\theta}$$

expanding in $|\vec{q}|$

$$F(|\vec{q}|) = 1 - i \int |\vec{q}|\, r^3 \rho(r) \underbrace{\int_{-1}^{1} d\cos\theta \cos\theta}_{0} \int_0^{2\pi} d\varphi$$

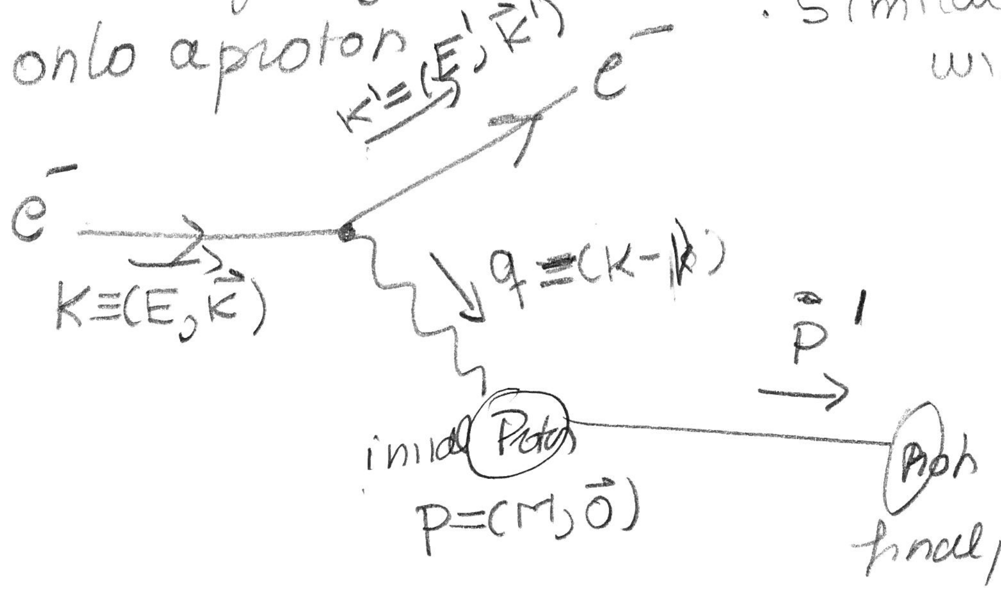
$$- \frac{1}{2} \int |\vec{q}|^2 r^4 \rho(r) \underbrace{\int_{-1}^{1} d\cos\theta \cos^2\theta}_{2/3 = \frac{1}{3}\int_{-1}^{1} d\cos\theta} \int_0^{2\pi} d\varphi$$

$$= 1 - \frac{|\vec{q}|^2}{6} \underbrace{\int r^2 d^3\vec{r}}_{\langle r^2 \rangle}$$

So meassuring the differencial scattering cs at low scattering angles ($\equiv$ low $|\vec{q}|$) we infer the charge radius of the distubution

② $e^- p \to e^- p$

We are going to study elastic e^- scattering onto a proton. Similar to the $Zp \to eZp$ with $Z=1$ and include spin:



As for $e^-\mu^- \to e^-\mu^-$ we can write

$$\mathcal{M} = -\frac{e^2}{q^2} J^\alpha_{elec} J_{\alpha\,proton}$$

with $J^\alpha_{elec} = \bar{u}_e(k')\gamma^\alpha u_e(k)$

But for $J_{\alpha,proton} = \bar{u}_p(p')[\,?\,]^\alpha u_p(p)$

We know that

- $[\,?\,]^\alpha$ is Lorentz 4-vector
- $\Rightarrow$ must be formed with p^α, p'^α, q^α or γ^α
- must be conserved $\Rightarrow \partial_\alpha J^\alpha_{proton} = 0 \Rightarrow q_\alpha J^\alpha_{proton} = 0$

the most general form verifying these conditions is

$$[2]^{\alpha} = F_1(q^2) + \frac{\kappa}{2M} F_2(q^2)\,\sigma^{\alpha\beta} q_\beta$$

with

$$\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^{\alpha}, \gamma^{\beta}]$$

$\kappa \equiv g_{proton} - 2 \equiv$ anomalous magnetic moment of the proton

$F_1(q^2), F_2(q^2) \equiv$ electromagnetic form factors of the proton.

they parametrize our ignorance of the structure of the proton. they must be determined experimentally (also κ) by measuring the angular distribution of the scattered e^-.

For $q^2 \to 0$ ($\lambda_\gamma \sim \frac{1}{\sqrt{q^2}}$ very large)

the photon "sees" the proton as a point charge with anomalous magnetic moment κ.

$\Rightarrow F_1(0) = F_2(0) = 1$

Integrating the phase space as for $e^-\mu^- \to e^-\mu^-$

$$\left.\frac{d\sigma_{\bar{e}p\to\bar{e}p}}{dE'd\Omega}\right|_{LAB} = \frac{(2\alpha E')^2}{q^4}\left\{\left(F_1^2 - \frac{\kappa^2 q^2}{4M^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}(F_1+\kappa F_2)^2\sin^2\frac{\theta}{2}\right\} \times \delta\left(\nu + \frac{q^2}{2M}\right)$$

In practice is better to use combinations of F_1 and F_2 which do not interphere

$$G_E(q^2) = F_1(q^2) + \frac{\kappa q^2}{2M^2}F_2(q^2) \equiv \text{electric form factor}$$

$$G_M(q^2) = F_1(q^2) + \kappa F_2(q^2) \equiv \text{magnet " "}$$

integrating dE' with the δ we get

$$\left.\frac{d\sigma(\bar{e}p\to\bar{e}p)}{d\Omega}\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right)$$

with $\tau = -\dfrac{q^2}{4M^2}$

Data on $e^-p^- \to e^-p^-$ is used to fit these form factors

We get

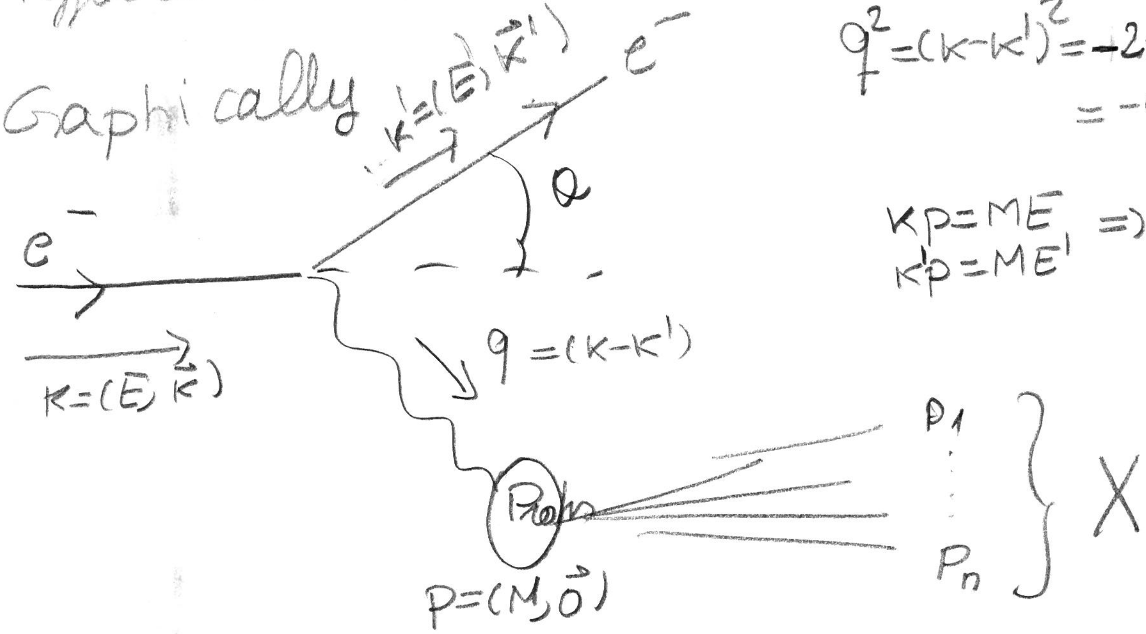
$$G_E(q^2) \simeq \left(1 - \frac{q^2}{0.7\,\text{GeV}}\right)^{-2} \xrightarrow{q^2\to 0} 1 + \frac{2q^2}{0.71\,\text{GeV}^2}$$

$$\Rightarrow \langle r^2\rangle_{\text{proton}} = 6\left.\frac{dG_E}{dq^2}\right|_{q^2=0} = \frac{12}{0.71\,\text{GeV}^2} = (0.81\,\text{fm})^2$$

3 Deep inelastic scattering $e^- p \to e^- X$

If in the $e^- p$ collision the e^- transfers a large $|q^2|$ to the proton it can "break it" and then the final state will be a collection of hadrons (protons, neutrons, pions, kaons.....). Generically refer to this process as deep inelastic scattering. (DIS)

Graphically



$$q^2 = (k-k')^2 = -2EE'(1-\cos\theta)$$
$$= -4EE'\sin^2\theta/2$$

$$kp = ME \quad k'p = ME' \Rightarrow kq = M(E-E') = M\nu$$

Now if we write

$$\mathcal{M} = -\frac{e^2}{q^2} J^{\alpha}_{elec} J_{\alpha, X} \qquad J^{\alpha}_{elec} = \bar{u}_e(k')\gamma^{\alpha} u_e(k)$$

but we do not know what to write for $J_{\alpha X}$ since X can be many different states.

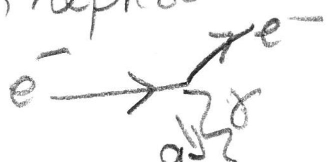
But whatever $J_{\alpha X}$ is we can always write

$$|\overline{M}|^2 = \frac{e^4}{q^4} L^{\alpha\beta}_{elec} K_{\alpha\beta}(X)$$

where $L^{\alpha\beta}_{ele} = 2(k^\alpha k'^\alpha + k^\beta k'^\alpha - 2g^{\alpha\beta}(kk')]$

which represents the part of the square amplitude

$$\Bigg| \qquad\qquad\qquad \Bigg|^2$$



while $K_{\alpha\beta}(X)$ represents the pia

$$\Bigg| \qquad\qquad\qquad \Bigg|^2$$

the cross section in the LAB and suming over any X

$$d\sigma^{DIS}_{LAB} = \frac{1}{4ME}\sum_X |\overline{M}|^2 \frac{d^3k'}{(2\pi)^3 2E'} \underbrace{\prod_{i\in X}\frac{dP_i^3}{(2\pi)^3 2E_i}(2\pi)^4\delta^4(p+k-k'-\sum p_i)}$$

$$= \frac{1}{4ME}\frac{e^4}{q^4}L^{\alpha\beta}_{ele}\frac{d^3k'}{(2\pi)^3 2E'}\sum_X K_{\alpha\beta}(X)(\qquad)$$

Since $d^3k' = E'^2\, dE'\, d\Omega$

$$\left.\frac{d\sigma^{DIS}}{d\Omega\, dE'}\right|_{LAB} = \underbrace{\frac{e^4}{16\pi^2 q^4}\frac{E'}{E}}_{\frac{\alpha^2}{q^4}\frac{E'}{E}} L^{\alpha\beta}_{elec}\left[\underbrace{\frac{1}{4\pi M}\sum_X K_{\alpha\beta}(X)\prod_{i\in X}\frac{d^3p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^4(k+p-k'-\sum p_i)}_{W_{\alpha\beta}}\right]$$

Now we have to figure out the most general form of $W_{\alpha\beta}$

But before that let us notice that we have lost the link betwen E' and θ because for each X the δ gives a different relation betwen E' an θ.

For each X we can define the "X invariant mass"

$$m_X^2 \equiv \sum_i P_i^2 \overset{\text{from } \delta(p+k-k'-\sum p_i)}{\overset{\downarrow}{=}} (p+q)^2 = M^2 + 2pq + q^2$$
$$= M^2 + 2M\nu + q \Rightarrow q^2 = M_X^2 - M^2 - 2M\nu$$

So for each M_X there is a link betwen q^2 and ν ($\equiv$ link betwen E' and θ). But as m_X is not fixed because we are counting events with any X now the two variables (E', θ) or (q^2, ν) are independent

What is the most general form of $W^{\alpha\beta}$

What is the most general form of $W^{\alpha\beta}$ (since $L^{\alpha\beta}_{elec}$ is)

- It must be a Lorentz tensor (since $L^{\alpha\beta}_{elec}$ is)

$\Rightarrow$ it must be made with, q, p and g

the most general form is

$$W^{\alpha\beta} = -W_1(q^2,\nu)\, g^{\alpha\beta} + \frac{W_2(q^2,\nu)}{M^2} p^\alpha p^\beta$$
$$+ \frac{W_4(q^2,\nu)}{M^2} q^\alpha q^\beta + \frac{W_5(q^2,\nu)}{M^2} (p^\alpha q^\beta + q^\alpha p^\beta)$$

(symmetric under $\alpha \leftrightarrow \beta$ because $L^{\alpha\beta}_{elec}$ is symmetric).

$W_{1,2,4,5}(q^2,\nu)$ parametrize our ignorance of the form of the hadronic current. they need to be extracted from data. But not the 4 form factors are independent because still

$$W^{\alpha\beta} = \sum_X J^\alpha_X J^\beta_X \quad \text{with } \partial^\alpha J^\alpha_X = 0 \text{ (em current conservation)}$$

$$\Rightarrow q_\alpha W^{\alpha\beta} = q_\beta W^{\alpha\beta} = 0$$

$$0 = -q^\alpha W_1 + \frac{W_2}{M^2}(pq)p^\alpha + \frac{W_4}{M^2} q^2 q^\alpha + \frac{W_5}{M^2}\left(q^2 p^\alpha + (pq) q^\alpha\right)$$

$$= q^\alpha \underbrace{\left[-W_1 + \frac{W_4}{M^2} q^2 + \frac{W_5}{M^2}(pq)\right]}_{(1)}$$

$$+ p^\alpha \underbrace{\left[(pq)\frac{W_2}{M^2} + \frac{W_5}{M^2} q^2\right]}_{(2)}$$

$$\Rightarrow \quad (1) = 0 \quad W_4 = \frac{M^2}{q^2} W_1 + W_2 \frac{(pq)^2}{q^4}$$

$$(2) = 0 \Rightarrow W_5 = -\frac{(pq)}{q^2} W_2$$

So altogether

$$W^{\alpha\beta}_{(q^2\nu)} = W_1(q^2\nu)\left[-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{M^2}\right]$$

$$+ \frac{W_2(q^2\nu)}{M^2}\left(p^\alpha - \frac{(pq)}{M^2} q^\alpha\right)\left(p^\beta - \frac{(pq)}{M^2} q^\beta\right)$$

$$\text{So } L^{\alpha\beta} W_{\alpha\beta} = 4\, W_1(q^2\nu)\,(kk') + \frac{2\, W_2(q^2\nu)}{M^2}\left(2(pk)(pk') - M^2 kk'\right)$$

$$\overset{\text{LAB}}{=} 4EE'\left[2\, W_1(q^2\nu)\,\sin^2\frac{\theta}{2} + W_2(q^2\nu)\,\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \boxed{\left.\frac{d\sigma^{DIS}}{dE'\,d\Omega}\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left\{W_2(q^2\nu)\cos^2\frac{\theta}{2} + 2W_1(q^2\nu)\sin^2\frac{\theta}{2}\right\}}$$

4 Bjorken scaling

Let's collect the expessious we got so far ($Q^2 = -q^2 > 0$)

$$\left.\frac{d\sigma^{DIS}}{dE'd\Omega}\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left[W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}\right]$$

with $Q^2 = -q^2 = 4EE'\sin^2\frac{\theta}{2}$ and $\nu = E - E'$ being 2 independent variables

$$\left.\frac{d\sigma_{e^-p\to e^-p}}{dE'd\Omega}\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^2\frac{\theta}{2}}\left[\underbrace{G_1(Q^2)}_{\left(F_1^2 - \frac{K^2q^2}{4M^2}F_2^2\right)}\cos^2\frac{\theta}{2} + 2\underbrace{G_2(Q^2)}_{\frac{Q^2}{2M^2}(F_1+KF_2)^2}\sin^2\frac{\theta}{2}\right]\delta\left(\nu - \frac{Q^2}{2M}\right)$$

so we can say

$$W_{1,2}^{ep\to ep} = G_{1,2}(Q^2)\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$ which depends on both Q^2, ν

$$\frac{d\sigma}{dE'd\Omega}{}_{e^-\mu\to e^-\mu} = \frac{\alpha^2}{4E^2\sin^2\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2M_\mu^2}\sin^2\frac{\theta}{2}\right]\delta\left(\nu - \frac{Q^2}{2M_\mu^2}\right)$$

$$\text{So } W_2^{e\mu\to e\mu}(Q^2\nu) = \delta\left(\nu - \frac{Q^2}{2M_\mu}\right) = \frac{1}{\nu}\,\delta\left(1 - \frac{Q^2}{2M_\mu\nu}\right)$$

$$2W_1^{e\mu\to e\mu}(Q^2\nu) = \frac{Q^2}{2M_\mu^2}\,\delta\left(\nu - \frac{Q^2}{2M_\mu}\right) = \frac{Q^2}{2M_\mu^2\nu}\,\delta\left(1 - \frac{Q^2}{2M_\mu\nu}\right)$$

So when e^- scatters on a point like particle the coefficients of the $\sin^2\theta/2$ and $\cos^2\theta/2$ terms verify

$$M W_1^{e^- \text{point}}(Q^2 \nu) = \frac{1}{2}\underbrace{\frac{Q^2}{2M\nu}}_{\equiv 1/\omega}\delta\Big(1-\underbrace{\frac{Q^2}{2M\nu}}_{\equiv \frac{1}{\omega}}\Big) \equiv F_1(\omega)$$

$$\nu W_2^{e^- \text{point}}(Q^2, \nu) = \delta\left(1-\frac{Q^2}{2M\nu}\right) \equiv F_2(\omega)$$

with $F_1(\omega) = \frac{1}{2\omega} F_2(\omega)$

So these two coefficient depend on a single combined of Q^2 and ν (or E', θ)

$$\omega = \frac{2M\nu}{Q^2} = \frac{2M\overset{\nu}{(E-E')}}{2EE'\sin^2\theta/2}$$

Now if we look at the data for $e^-p \to e^-p$ DIS at large Q^2 we find that

$$\left.\begin{aligned} M W_1^{DIS}(Q^2, \nu) &\xrightarrow{Q^2 \text{ large}} F_1(\omega) \\ \nu W_2^{DIS}(Q^2, \nu) &\xrightarrow{\qquad} F_2(\omega) \end{aligned}\right\} \equiv \text{Bjorken scaling}$$

So when Q^2 is large $\Rightarrow$ the γ has very short wavelength the DIS behaves as if the e^- was scattering over a point-like particle. Why? and what is the meaning of the variable "ω"?

mass scale is explicitly present; it is set by the empirical value 0.71 GeV in the dipole formula for $G(Q^2)$ which reflects the inverse size of the proton, see (8.20). As Q^2 increases above $(0.71\ \mathrm{GeV})^2$, the form factor depresses the chance of elastic scattering; the proton is more likely to break up. The point structure functions, on the other hand, depend only on a dimensionless variable $Q^2/2m\nu$, and no scale of mass is present. The mass m merely serves as a scale for the momenta Q^2, ν.

The discussion can be summarized as follows: if large Q^2 virtual photons resolve "point" constituents inside the proton, then

$$MW_1(\nu, Q^2) \xrightarrow[\text{large } Q^2]{} F_1(\omega),$$

$$\nu W_2(\nu, Q^2) \xrightarrow[\text{large } Q^2]{} F_2(\omega), \tag{9.5}$$

where

$$\omega = \frac{2q \cdot p}{Q^2} = \frac{2M\nu}{Q^2}. \tag{9.6}$$

Note that in (9.5) we have changed the scale from what it was in (9.3). We have introduced the proton mass instead of the quark mass to define the dimensionless variable ω. The presence of free quarks is signaled by the fact that the inelastic structure functions are independent of Q^2 at a given value of ω [see (9.5)]. This is equivalent to the onset of $\sin^{-4}(\theta/2)$ behavior for large momentum transfers in the Rutherford experiment, which reveals the "point" charge of the nucleus in the atom. A sample of data is shown in Fig. 9.2. νW_2 at $\omega = 4$ is independent of Q^2; the photon is indeed interacting with point-like particles. No form factors, leading to additional Q^2 dependence as in (9.4), are present. Are these particles (called partons by Bjorken) the same as the quarks discovered in the spectroscopy of hadrons (Chapter 2)?

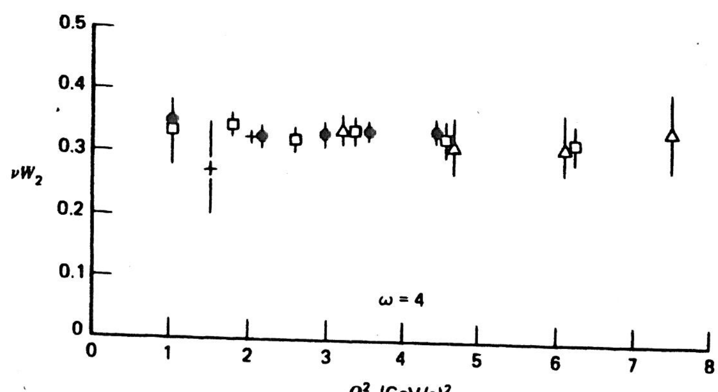
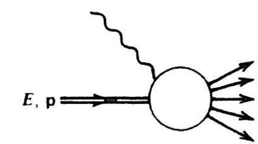


Fig. 9.2 The structure function νW_2 determined by electron–proton scattering as a function of Q^2 for $\omega = 4$. Data are from the Stanford Linear Accelerator.

9.2 Partons and Bjorken

Now that scaling is an a
identification of (9.2) expl



Equation (9.7) recognizes
the proton (i = u, d, . . . ,
latter do not interact with
fraction x of the parent p
momentum distribution

$f_i(x) =$

which describes the proba
proton's momentum p. A

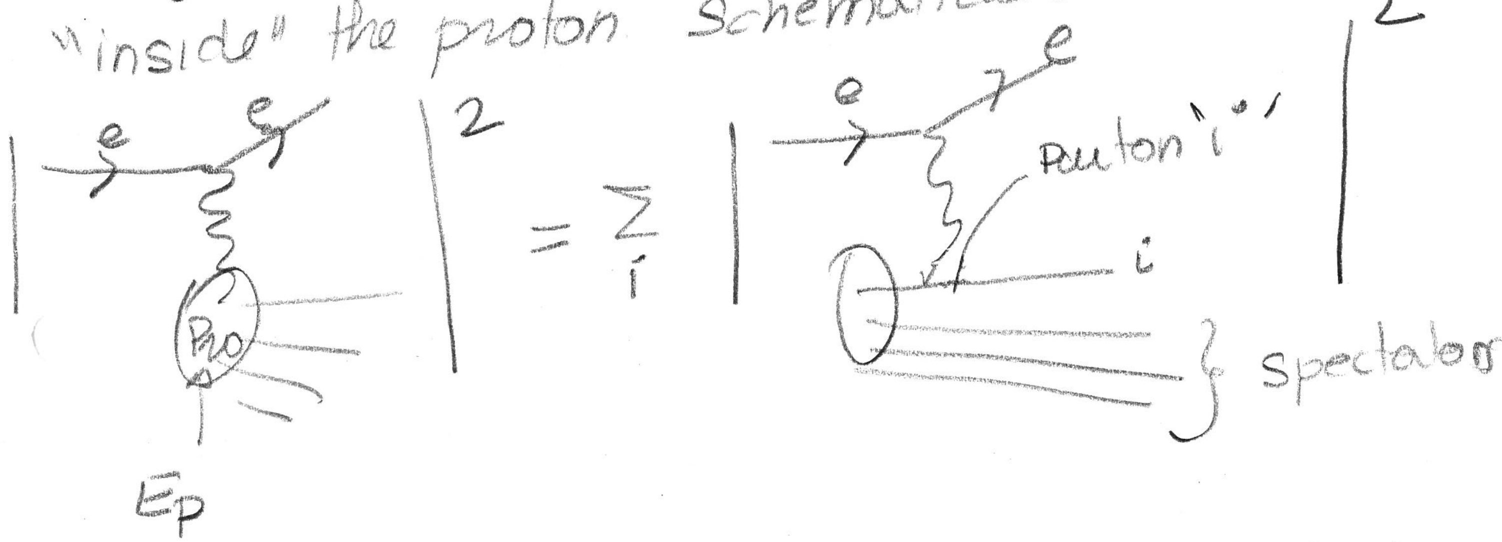
Here, i' sums over all the
the photon. The kinemati

Prot
↓

Energy	E
Momentum	p_L
	$p_T =$
Mass	M

(5) the parton (≡ quark) model of hadrons

In the quark model of the proton the observed Bjorken scaling in $e^-p \to e^- X$ DIS is understood as due to the interaction of the exchanged γ with one of the quarks (≡ point like state) which is "inside" the proton. Schematically



So γ interacts with only one of the partons which are inside proton. that parton carries a fraction x of the energy of the proton E_p. the other partons do not participate in the interaction (≡ spectators)

Let us define the "parton distribution function"

$f_p^i(x)$ ≡ probability for parton "i" in the proton to carry a fraction x of the E and $\vec{p}$ of the proton (we assume "i" goes parallel to proton)

	Proton	Parton
Energy	E_P	$E_i = x\,E_p$
3-momentum	$\vec{P}$	$\vec{P}_i = \vec{P}\,x$
Mass	M	$m_i = \sqrt{E_i^2 - \lvert\vec{p}_i\rvert^2} = xM$

conservation of momentum $\Rightarrow$

$$\sum_i \int_0^1 dx\, \vec{P}_i\, f_i^P(x) = \vec{P} \Rightarrow \sum \int_0^1 dx\, x\, f_i^P(x) = 1$$

the cs for the interaction $e\,i \to e\,i$ is like $e^-\mu^- \to e^-\mu^-$ just introducing "e_i $\equiv$ charge of parton "i" (assuming "i" is a fermion)

So

proton

$$M\,W_1^{ei\to ei}(Q^2,\nu) = \frac{M}{2}\,\frac{Q^2}{\underset{x^2M^2}{2m_i^2}\nu}\, e_i^2\,\delta\Big(1 - \frac{Q^2}{2\nu \underset{xM}{m_i}}\Big)$$

$$= \frac{e^{i2}}{2\omega x}\,\delta\Big(1 - \frac{1}{x\omega}\Big) \equiv F_1^i(\omega)$$

$$\nu W_2^{ei\to ei}(Q^2,\nu) = e_i^2\,\delta\Big(1 - \frac{1}{x\omega}\Big) \equiv F_2^i(\omega) = \frac{2}{\omega}F_1^i(\omega)$$

the δ's $\Rightarrow$ the variable $\omega \equiv \frac{2M\nu}{Q^2}$ which we found to be the only relevant in DIS data is the inverse of the "x" $\equiv$ momentum fraction carried by the interacting parton

In this model

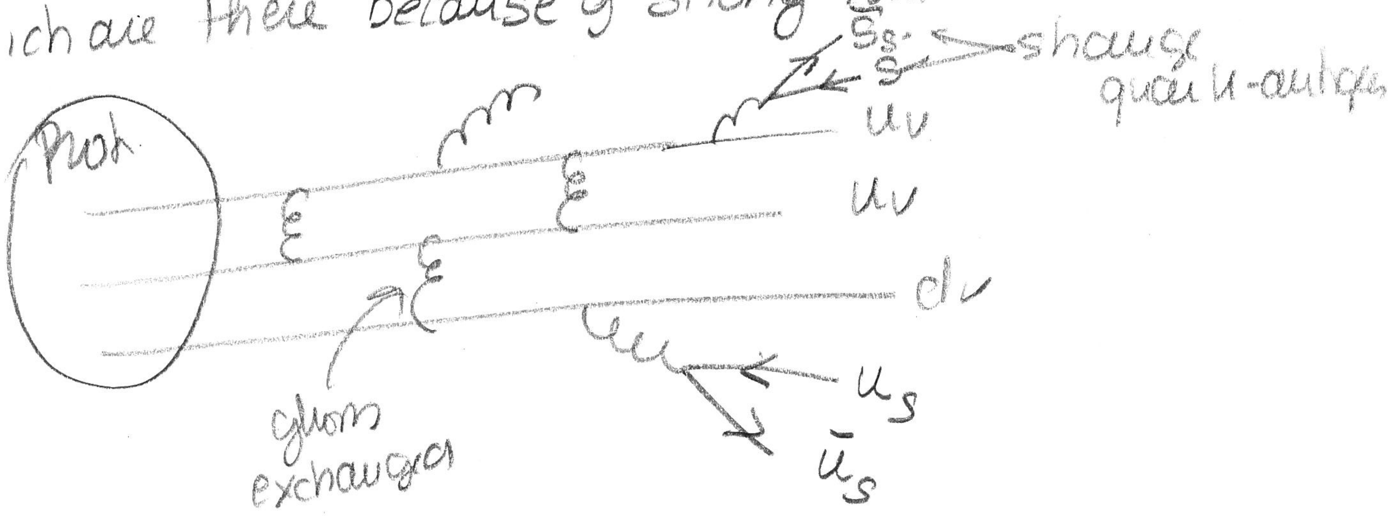
$$F_2^{DIS}(\omega) = \int \sum_i f_i(x)\, F_2^i(\omega)\, dx = \sum_i e_i^2 f_i^p\left(x=\tfrac{1}{\omega}\right)\frac{1}{\omega}$$

$$F_1^{DIS}(\omega) = \frac{\omega}{2}\, F_2^{DIS}(\omega) \quad \text{or} ,$$

Or equivalently

$$\left.\begin{aligned} F_2^{DIS}(x) &= \sum_i e_i^2\, x\, f_i^p(x) \\ F_1^{DIS}(x) &= \frac{1}{2x}\, F_2^{DIS} \end{aligned}\right\} = \text{Callen-Gross relation}$$

In this simple model the proton is understood as made of 3 valence quarks $u_v u_v d_v$ (valence = responsible for quantum # of the proton) and a "sea" of quark-antiquark pairs and gluons which are there because of strong interactions



Strong interactions $\Rightarrow$ probability of emission of a gluon, or of a $q_s\bar{q}_s$ pair with momentum fraction $x \sim \frac{1}{x}$

Notation $f_q^P(x) \equiv q^P(x)$

In this model

$$\frac{1}{x}F_2^{DIS}(x) = \overbrace{\left(\frac{2}{3}\right)}^{e_u}{}^2\left[u_v^P(x) + u_s^P(x) + \bar{u}_s^P(x)\right]$$
$$+\left(\frac{1}{3}\right)^2\left[d_v^P(x) + d_s^P(x) + \bar{d}_s^P(x)\right]$$
$$+\left(\frac{1}{3}\right)^2\left[s^P(x) + \bar{s}^P(x)\right]$$

As sea quarks are produced in pair one expects

$$q_s^P = \bar{q}_s^P$$

Furthermore neglecting quark mass effects

$$u_s^P = d_s^P = s^P \equiv S^P \sim \frac{1}{x}$$

$$\Rightarrow \frac{1}{x}F_2^{ep}(x) = \left(\frac{2}{3}\right)^2(u_v^P + 2S^P) + \left(\frac{1}{3}\right)^2(d_v^P + 2S^P) + \left(\frac{1}{3}\right)^2 2S^P$$

$$= \frac{4}{9}u_v^P + \frac{1}{9}d_v^P + \frac{4}{3}S^P$$

For $en \to eX$ $\overset{\text{neutron}}{n} = udd$ using that for strong interaction $u \Leftrightarrow$ makes no difference

$$\Rightarrow u_v^n = d_v^p \equiv d_v$$
$$d_v^n = u_v^p \equiv u_v$$
$$s^n = s^p = s$$

In this simplified model

$$\frac{1}{x} F_2^{ep}(x) = \frac{1}{9}\left[4 u_v(x) + d_v(x)\right] + \frac{4}{3} s(x)$$
$$\frac{1}{x} F_2^{en}(x) = \frac{1}{9}\left[4 d_v(x) + u_v(x)\right] + \frac{4}{3} s(x)$$

Furthermore

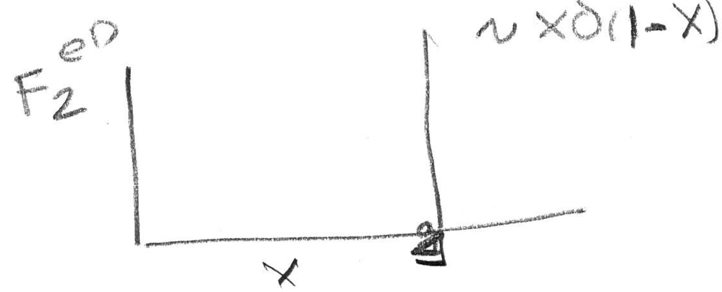
$$\int u_v(x)\,dx = 2 \qquad \int d_v(x)\,dx = 1 \Rightarrow u_v > d_v$$

How do we expect $F_2^{ep}(x)$ depend on x?

If proton was made

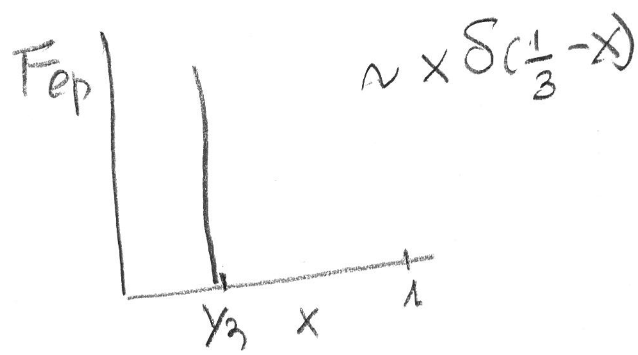
- only 1 quark u

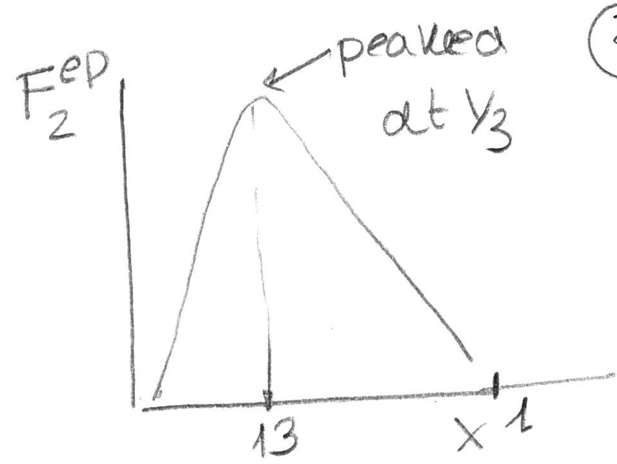
  $\Rightarrow x = 1$



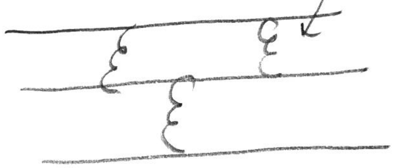
- 3 valence quark which do not ~~interact~~ among them

  $\Rightarrow$ each one carries $x = 1/3$



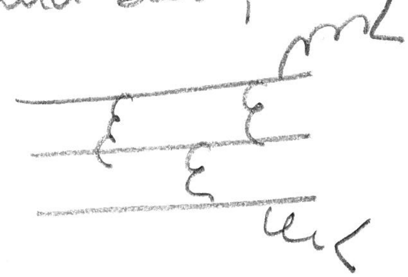


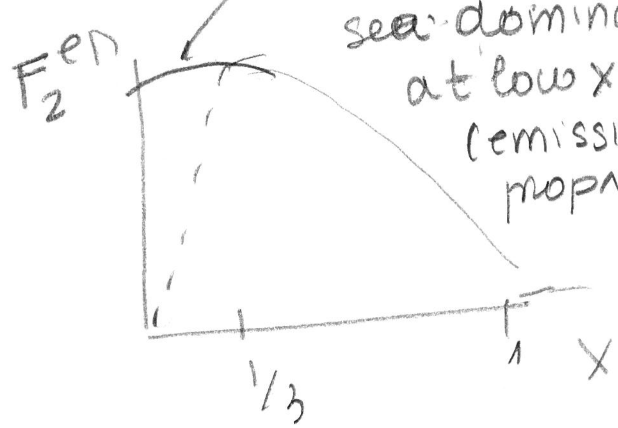
- 3 valence quarks interacting



some exchange of momentum so $x \neq 1/3$

- 3 valence quarks interacting and sea quark





contribution for sea dominant at low x (emission prop $\sim \frac{1}{x}$)

In this simplified model

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow{x \to 0} \frac{S(x)}{S(x)} = 1$$

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow[(u_v > d_v)]{x \to 1} \frac{u_v}{4u_v} = \frac{1}{4}$$

also $F_2^{ep}(x)\ F_2^{en}(x) = \frac{1}{3}(u_v(x) - d_v(x))$ must peak at $x = 1/3$

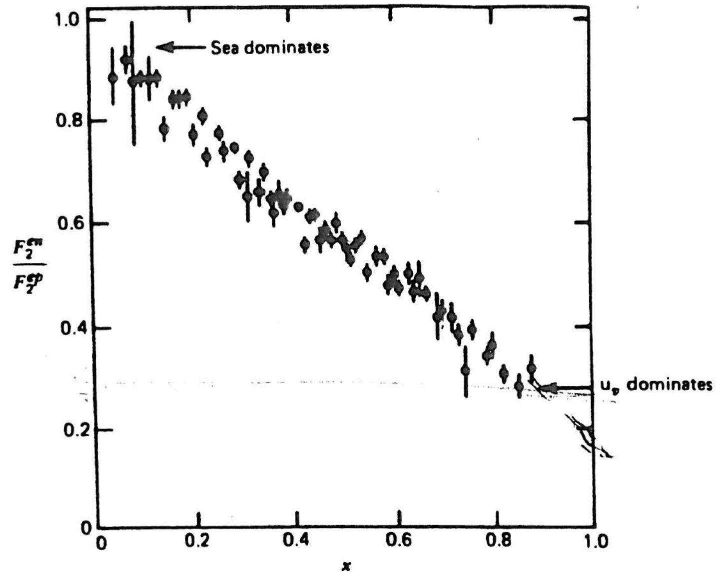


Fig. 9.6 The ratio F_2^{en}/F_2^{ep} as a function of x, measured in deep inelastic scattering. Data are from the Stanford Linear Accelerator.

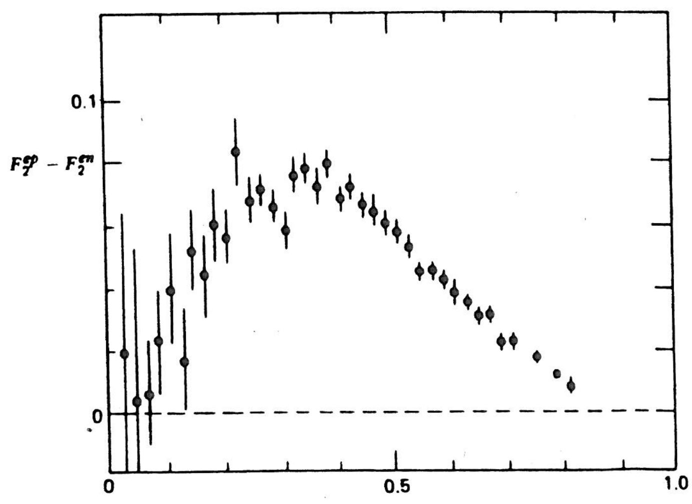


Fig. 9.8 The difference $F_2^{ep} - F_2^{en}$ as a function of x, as measured in deep inelastic scattering. Data are from the Stanford Linear Accelerator.

An alternative phenomenological approach ...

From the experimental results of $F_2^{ep}(x)$ and $F_2^{en}(x)$ we find

$$\int_0^1 dx\, F_2^{ep}(x) \simeq 0.18 \simeq \frac{4}{9}\int dx(\bar{u}+u)x + \frac{1}{9}\int x(\bar{d}+d)x$$

$$= \frac{4}{9}\langle x\rangle_u + \frac{1}{9}\langle x_d\rangle$$

$$\int_0^1 dx\, F_2^{en}(x) \simeq 0.12$$

$$= \frac{4}{9}\langle x\rangle_d + \frac{1}{9}\langle x\rangle_u$$

Solving we find $\langle x\rangle_u \simeq 0.36 \simeq 2\langle x\rangle_d$

$\Rightarrow \langle x\rangle_u + \langle x_d\rangle \simeq 0.54 \Rightarrow$ only about 54% of the momentum of the proton is carried by the the quarks. The 46% "missing" is carried by gluons. they do not contribute to DIS because they have no electric charge. But for strong interactions we must also consider a gluon distribution function $g^p(x) = g^n(x) \equiv g(x)$

6 Parton model for hadron hadron collisions

the quark and gluon distribution functions which we extract from DIS and other data are "universal" ≡ they characterise the parton in the proton in any proton collision.

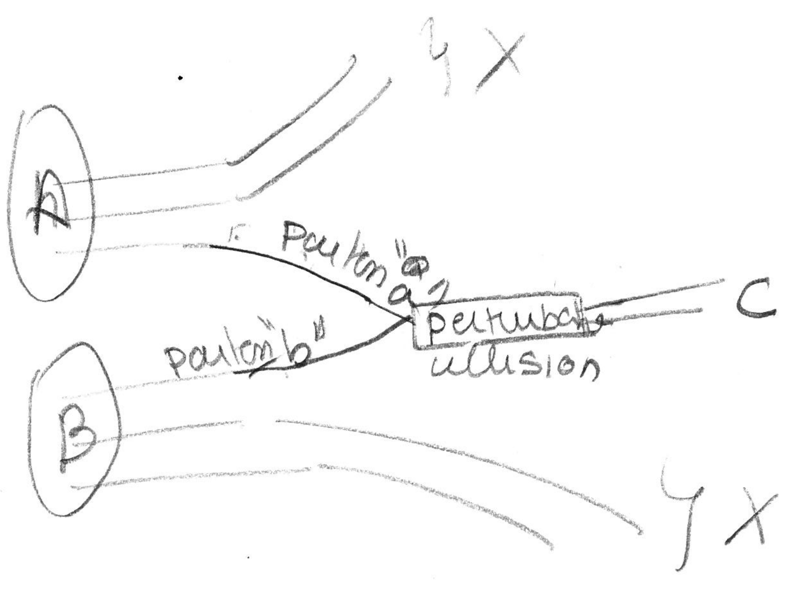
Furthermore for antiprotons it is expected that

$$u_v^p = \bar{u}_v^{\bar{p}} \quad d_v^p = \bar{d}_v^{\bar{p}} \quad S^p = S^{\bar{p}}$$

So we can use these distributions functions (also called "structure functions") to predict cs in pp or $p\bar{p}$ collisions

Schematically

$$\underset{\text{hadron}}{A} + \underset{\text{hadron}}{B} \longrightarrow \underset{\text{set of high energy particles}}{C} + X$$



So we can predict

$$\sigma(A+B \longrightarrow C+X)(S) =$$

$$= \sum_{a,b} \underset{\substack{\uparrow \\ \text{color factor}}}{C_{ab}} \int_0^1 dx_a \int_0^1 dx_b \left[f_a^A(x_a)\, f_b^B(x_b) + \underset{(\text{if } a \neq b)}{A \Leftrightarrow B} \right] \hat{\sigma}(a+b \to c)(\hat{S})$$

Neglecting the mass of A, B, a, b

$$p_a = x_a P_A \qquad \Rightarrow \quad \hat{S} \equiv (p_a + p_b)^2 = 2 x_a x_b P_A P_B = x_a x_b (P_A + P_B)^2$$

$$p_b = x_b P_B \qquad \quad = x_a x_b S$$

Since $x_a, x_b < 1 \Rightarrow \hat{S} < S$.

In fact $\hat{S} \ll S$.

To produce a state of mass M_c we need

$$\hat{S} > M_c^2 \Rightarrow S > \frac{M_c^2}{x_a x_b} \gg M_c^2$$

For example at LHC $\sqrt{S} = 13\,\text{TeV}$ but we can hardly see a particle with $M \gtrsim 2\,\text{TeV}$

this is more explicit if we make a change of variables from x_a, x_b to $x_a, \tau = x_a x_b$

So $\int_0^1 dx_a \int_0^1 dx_b \, \hat{\sigma} \Rightarrow \int_{\tau_{min}}^1 d\tau \int_\tau^1 d\frac{x_a}{x_a} \hat{\sigma}$

where τ_{min} is $\frac{\hat{s}_{min}}{s}$ the minimum COM energy in the partonic collision required to kinematically produced the final state.

Finally the colour factors account for the fact that as we will see we have 3 possible colours of quarks and 8 for gluons, but we define the distribution functions $q(x)$, $g(x)$ as summing over all colours. So if the partonic process requires for example that the colour of "a" and "b" are the same $\Rightarrow$

$C_{ab} = \frac{1}{3}$ for a, b quarks or antiquarks