Chapter 8

Strong interactions : Quantum Chromodynamics (QCD)

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1) Evidence of 3 colours of quarks
2) QCD Lagrangian and Feynman rules
3) Tests of QCD
4) $q\bar{q}$ interaction in QCD : bound states
5) $SU(3)_{FLAVOR}$ and spectrum of light hadrons

① Evidence of 3 colours of quarks

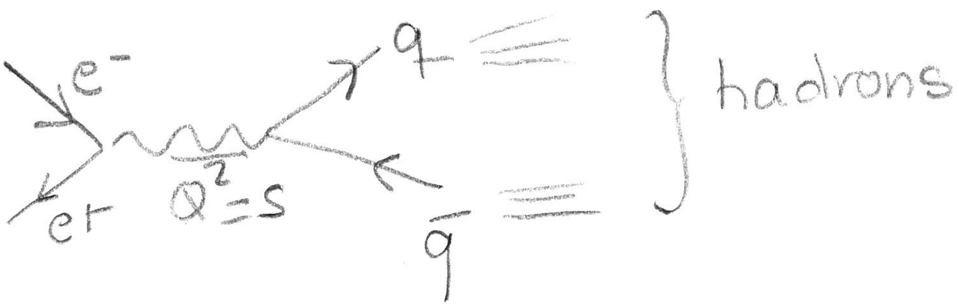
We have seen that at large Q^2

$e^-p \to e^- X$ can be understood as $e^-q \to e^-q$.

X ← hadrons

Equivalently $e^+e^- \longrightarrow$ hadrons at large Q^2

can be understood as



unless; $s = m^2_{meson}$ ($\equiv q\bar{q}$ bound state)

↑ meson in "s" chanel $\Rightarrow$ resonant increase cs.

In QED

$$\sigma(e^+e^- \to \mu^+\mu^-)(s) = \frac{4\pi\alpha^2}{s} \quad (s \gg 4m_\mu^2)$$

So in this picture

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha}{s} \underset{\uparrow}{N_c} \sum_{\text{quarks}} \underset{\uparrow}{e_q^2} \quad \text{for} (s \gg 4m_q^2)$$

N_c: # quark flavours

e_q: quark charge

So $R = \frac{\sigma(e^+e^- \to hadrons)(s)}{\sigma(e^+e^- \to \mu^+\mu^-)} \equiv N_c \sum_{\substack{quark \\ with\, 4m_q^2 < s}} e_q^2$

Using $m_u, m_d, m_s \ll 100\,MeV$, $m_c \sim GeV$, $m_b \sim 5\,GeV$
$m_t \sim 175\,GeV$ $e_u = e_c = 2/3$ $e_d = e_s = e_b = -\frac{1}{3}$

- For $\sqrt{s} \ll 3\,GeV$ $q = u, d, s$ $R = N_c\left(\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = \frac{2}{3} N_c$
- $3 \ll \sqrt{s} \ll 10$ $q = u, d, s, c$ $R = \left(\frac{2}{3} + \frac{4}{9}\right) N_c = \frac{10}{9} N_c$
- $10 \lll \sqrt{s} \ll 300\,GeV$ $q = u, d, s, b$ $R = N_c\left(\frac{10}{9} + \frac{1}{9}\right) = \frac{11}{9} N_c$

Experimentally → Figure $\Rightarrow N_c = 3$

the constituent quark model does not explain
- why quarks are never seen as free states
- why quarks behave as muons (ie free states) when probed at large Q2 collisions

the theory of strong interactions with binds the quarks inside the proton must explain this.

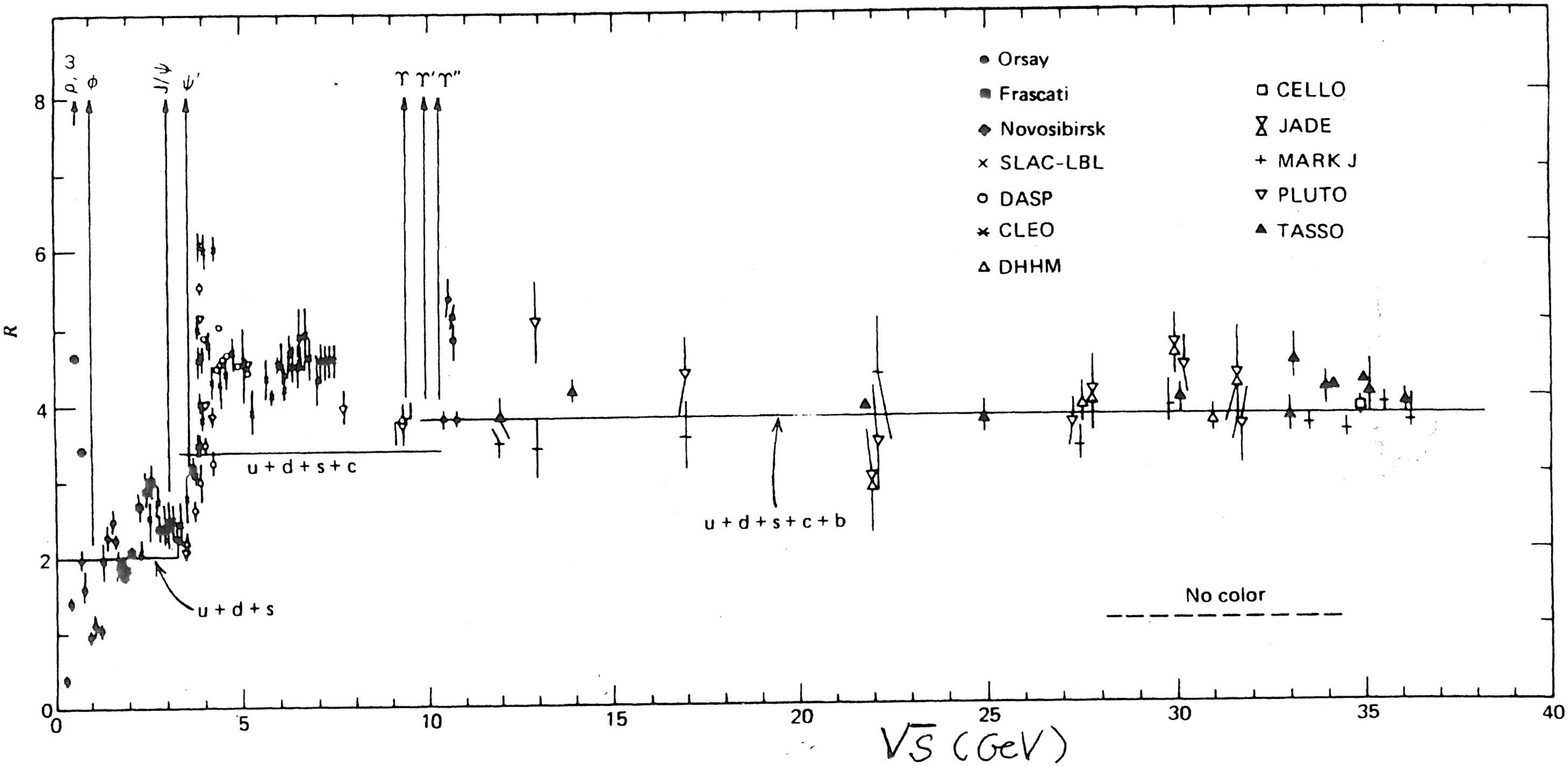


Fig. 11.3 Ratio *R* of (11.6) as a function of the total e^-e^+ center-of-mass energy. (The sharp peaks correspond to the production of narrow 1^- resonances just below or near the flavor thresholds.)

② Quantum Chromodynamics

In principle there is no reason why the answer to these two questions have anything to do with the colour quantum # of the quarks.

But as we are going to see they can be explained if we assume that these facts are due to the colour because colour is "the charge" responsible of strong interactions.

We call quantum chromodynamics (QCD) the gauge theory implementing this idea. Its basic assumption

- each quark has a colour quantum # in 3 possible states which we call r, g, b ≡ red, green, blue
- to specify the quark "in" "i" color state besids the spinor we need a vector $C^{(i)}$ which will be a linear combination of

$$r \equiv c_1 \equiv \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad b \equiv c_2 \equiv \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad g \equiv c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

So we represent a quark wave function

notation ↳ $q_i(x) \equiv \Psi_{q_i}(x) = \Psi_q(x) C^{(i)}$ ← colour vector

$\uparrow$ spinorial part

correspondingly for the antiquarks there are 3 "anticolours" $\bar{r}, \bar{g}, \bar{b}$

- colour is the "charge" responsible of strong interactions and QCD its gauge theory -

Gauge group $SU(3)_c$

SU(3) = dimension 8 non-abelian group ⇒ 8 generators

Its algebra

1. $$[T^a, T^b] = i f^{abc} T^c$$

f_{abc} are totally antisymmetric ⇒ $f_{abc} = f_{cab} = f_{bca} = -f_{bac} = -f_{cba} = -f_{acb}$

the non-zero values are

$f_{123} = 1$, $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$

$f_{458} = f_{678} = \frac{\sqrt{3}}{2}$

r, g, b are a basis for the triplet representation of $SU(3)_c$

In this representation $T_a = \frac{\lambda_a}{2} \leftarrow$ Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & \end{pmatrix}$$

Under $SU(3)_c$ transformation

$$q_i(x) \longrightarrow q'_i(x) = \sum_{j} \underbrace{\left[e^{i \sum_{a=1}^{8} \alpha_a(x) \frac{\lambda^a}{2}} \right]_{ij}}_{3 \times 3 \text{ matrix}} q_j(x)$$

- the coupling constant is g_s.

We can build a Lag for q which is invariant under $SU(3)_c$ in analogy of what we did for e^- in QED with group $U(1)$

$$\mathcal{L} = \sum_{ij} \bar{q}_i [i\gamma^\mu D_\mu]_{ij} q_j - \sum_i m \bar{q}_i q_i$$

ij ← colour indexes

Where to define the covariant derivative $D_{\mu\,ij}$ we need to introduce gauge boson $G_\mu^{(a)}$ $a = 1 \ldots 8$ which we will call gluons.

So the covariant derivative is

$$\sum_j (D^\mu)_{ij}\, q_j = \sum_j \left[\partial^\mu \delta_{ij} + i g_s \sum_{a=1}^{8} \frac{(\lambda^a)_{ij}}{2} G^{\mu(a)} \right] q_j$$

Impossing gauge invariance

$$\sum_j (D^\mu_{ij}\, q)' = \sum_{kj} \left(e^{-i \sum_a \frac{\lambda^a}{2}} \right)_{ij} (D^\mu)_{kj}\, q_j$$

after some algebra

$$\Rightarrow \quad G'^a_\mu = \underbrace{G^a_\mu - \frac{1}{g_s} \partial_\mu \alpha_a}_{\text{same as QED}} - \underbrace{\sum_{b,c} f^{abc} \alpha_b \cdot G^c_\mu}_{\substack{\text{new term } \neq 0 \\ \text{because } f^{abc} \neq 0 \\ \text{(non abelian group)}}}$$

We factorize the colour index of the gluon as

$$G^a_\mu = \underset{\uparrow \text{ vector wavefunction like the photon}}{G_\mu(x)}\; a^{(a)} \leftarrow \text{colour vector in dimension 8}$$

So in QCD there is an additional piece in the gauge transf. of the gluon wrt photon in QED because SU(3) is non-abelian.

Next we need to add the Lag for the gluons

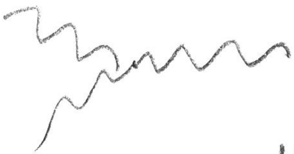
In QED $\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

In QCD $\mathcal{L}_G = -\frac{1}{4} \sum_a G^a_{\mu\nu} G^{\mu\nu,a}$ with

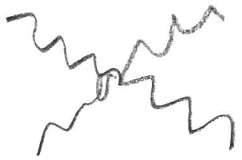
$$G^a_{\mu\nu} = \underbrace{\partial_\mu G^a_\nu - \partial_\nu G^a_\mu}_{\text{like QED}} - i g_s \underbrace{\sum_{bc} f_{abc} G^b_\mu G^c_\nu}_{\text{new wrt QED}}$$

Because of the new part ⇒ $\mathcal{L}_G$ there are terms like

$\sum_{abc} f^{abc} \partial_\nu G^a_\mu G^{\mu,b} G^{\nu,c} \to$ 3 gluon vertices



$\sum_{abcd} f^{abc} f^{ade} G^b_\mu G^c_\nu G^{\mu,d} G^{\nu,e} \to$ 4 gluon vertex



So unlike in QED where the photons do not interact with themselves the gluons of QCD do.

So with this the FR for QCD are

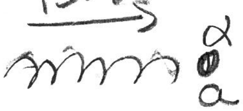
- External lines

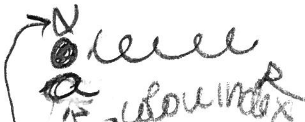
quark incoming  $C^{(i)}\ u^s(p)$

quark outgoing  $\bar{u}^s(p)\, C^{(i)\dagger}$

anti " incoming  $\bar{v}^s(p)\, C^{(i)\dagger}$

anti " outgoing  $C^{(i)}\, v^s(p)$

gluon incoming  $[\epsilon^{\lambda}(p)]_{\alpha}\ a^{(a)}$

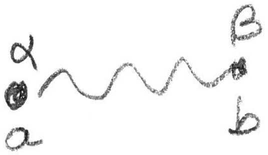
gluon outgoing  $a^{(a)\dagger}\ [\epsilon^{\lambda}(p)]^{*}_{\alpha}$

lorentz index like in QED

gluon represented by a "spring-like" line

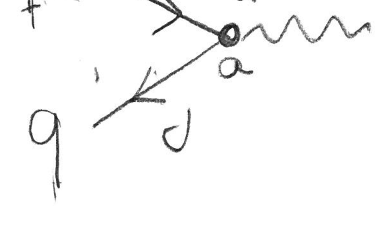
- Propagators

quark  $\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$ (same as e^-)

gluon  $-\dfrac{i g_{\alpha\beta}}{p^2+i\epsilon}\ \delta_{ab}$

- Vertices

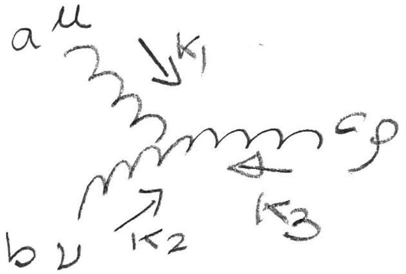
same flavour different colour



$-i g_s \gamma^{\mu}\ \dfrac{\lambda^a_{ij}}{2}$

Equiv to QED with "charge" $\dfrac{\lambda^a_{ij}}{2}$

Additional vertices



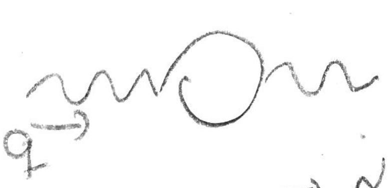
$$g_s f^{abc}\left[(k_1-k_2)_\rho g_{\mu\nu} + (k_2-k_3)_\mu g_{\nu\rho} + (k_3-k_1)_\nu g_{\mu\rho}\right]$$

$$= -i g_s^2 \sum_e \left[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\rho\lambda}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\nu\lambda}) \right]$$

③ Some tests of QCD

Running of αs

In QED we saw that when including higher order we get some apparent divergences which disappeared once we renormalised the parameters of the Lag (coupling constant, mass, ...) . After the renormalisation the predictions were finite but it remained a finite energy dependent corrections. I.e the em coupling constant "e" (α)



+ . . . .

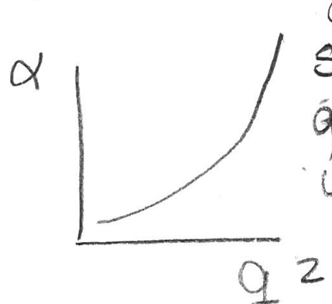
$\alpha_0 \leftarrow 1/137 = \alpha$,

$$\Rightarrow \alpha(q^2) = \frac{\alpha_0}{1-\frac{\alpha_0}{3\pi}\ln\frac{q^2}{m_e^2}}$$

$\alpha_0 = \alpha(\mu^2=0)$ where μ^2 is the energy at which we measured ~~the~~

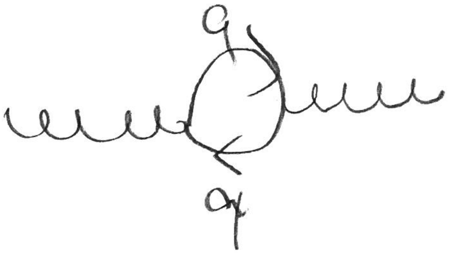
If we had measured α at some $\mu^2 \gg m_e^2$

$$\alpha(q^2) = \frac{\alpha(\mu^2)}{1-\underset{\substack{\uparrow \\ \text{negative}}}{\frac{\alpha(\mu^2)}{3\pi}}\ln\frac{q^2}{\mu^2}} \Rightarrow$$



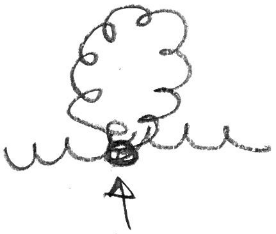
QED strength grows with q²

For QCD we can do the same but now we also have



like QED

but also



these additional loop can be computed using the FR of QCD and they lead to a totally different running of $\alpha_S \equiv \frac{g_s^2}{4\pi}$

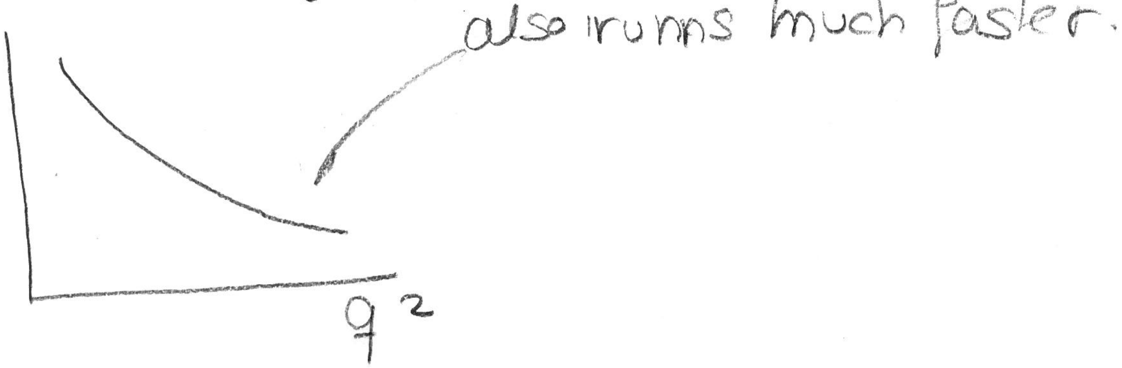
$$\alpha_S(q^2) = \frac{\alpha_S(\mu^2)}{1+\frac{\alpha_S(\mu^2)}{12\pi}(33-2N_f)\ln\frac{q^2}{\mu^2}}$$

flavour of quarks with $4m_q^2 \ll q^2$

For any q^2 $N_f < 6$ (u, d, s, c, b, t)

$$\Rightarrow 33-2N_f \geq 11 > 0$$

So α_S



- So at $q^2 \to 0 \Rightarrow \alpha \to$ very large

  $\Rightarrow$ confinement of non-relativistic quarks inside the hadrons

  $\Rightarrow$ non pertubative regime $\Rightarrow$ phemenological hadron models

- at $q^2 \to$ large $\Rightarrow \alpha \to$ small

  $\Rightarrow$ strong interactions become weak enough $\equiv$ asymptotic freedom

  $\Rightarrow$ even if quarks are confined inside the hadrons when they are probed with high Q^2 (like DIS) we can treat them as free. This is the basics of why the parton model works

If we define Λ_{QCD} as the scale at which α becomes very large

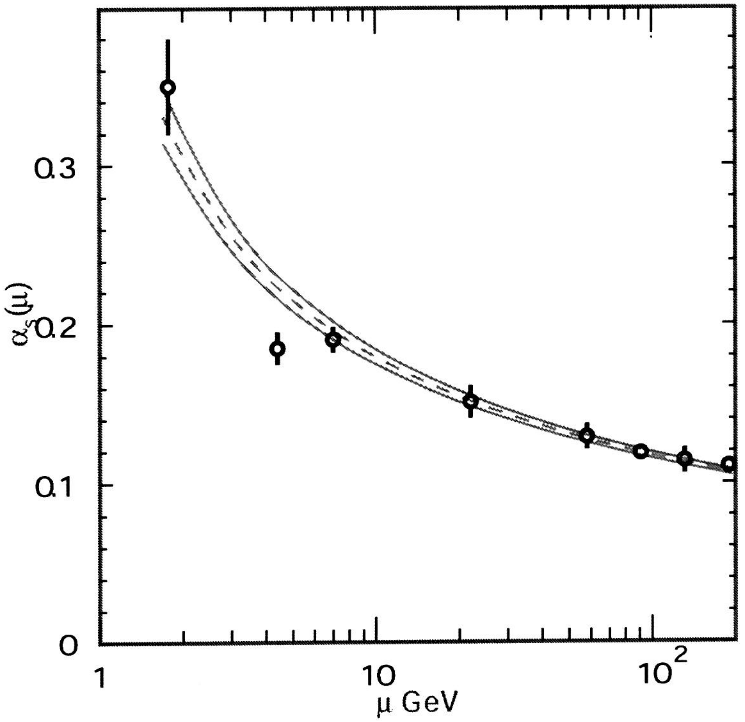
$$0 \simeq \frac{1}{\alpha_s(\Lambda^2)} = \frac{1}{\alpha(\mu^2)} + \frac{33-2N_f}{12\pi} \ln\frac{\Lambda^2}{\mu^2}$$

$$\Lambda^2 = \mu^2 e^{\frac{-12\pi}{(33-2N_f)\alpha_s}} \qquad \Rightarrow \alpha_s(q^2) = \frac{12\pi}{(33-2N_f)\ln\frac{q^2}{\Lambda^2}}$$

QCD prediction

$$\alpha_s(q^2) = \frac{\alpha_s(q_0^2)}{1 + \frac{\alpha_s(q_0^2)}{12\pi}(33 - 2N_f)\ln\left(\frac{q^2}{q_0^2}\right)}$$

Comparing to data



Consequences:

- At short distances the strong potential between two quarks separated by r is Coulomb-like

  $$\frac{\alpha_S(q^2 \sim r^{-2})}{r}$$

  If we separate them apart r grows, and so does α

  $\Rightarrow$ effective strong potential grows (predicted to grow linearly) with the separation

  $\Rightarrow$ quarks cannot escape the potential

  $\Rightarrow$ quarks are *confined* inside the hadrons

- If we insist in breaking the hadron, once the quarks are at sufficient distance from each other the potential energy is huge

  $\Rightarrow$ Energetically favorable to create a $q\bar{q}$ and bind them to the original quarks to form hadrons

  $\Rightarrow$ quarks cannot be observed in isolation

- Conversely going to smaller and smaller distances, or equivalently, to larger and larger energies, the strong coupling constant becomes weak.

  For example at $q^2 = (91\,\mathrm{GeV})^2$, $\alpha_s \simeq 0.12$.

  $\Rightarrow$ perturbation theory can be trusted.

  $\Rightarrow$ quarks behave as free asymptotic states when probed at very high energies.

  We say that QCD is an *asymptotically-free* theory.

  Asymptotic freedom:

  $\Rightarrow$ We can treat quarks as free outgoing fermions when produced in high energy collisions

  $\Rightarrow$ We can treat quarks as free incoming fermions inside the hadrons when we collide the hadrons at very high energies

  $\Rightarrow$ We can use Feynman calculus to compute perturbatively the expectation from QCD

  Ultimately it was the reason why QCD was accepted as the theory of strong interactions which unified the hadronic low energy physics with the very high-energy strong interaction effects.

The strong coupling constant becomes small at high energies

$\Rightarrow$ one can calculate perturbatively the expectations in QCD for quarks and gluons at high energies.

But in the real world the quarks and gluons have to *hadronize* ($\equiv$ become real white hadrons)

$\Rightarrow$ comparison is not straight-forward

The cleanest data to do this comparison is to use data on

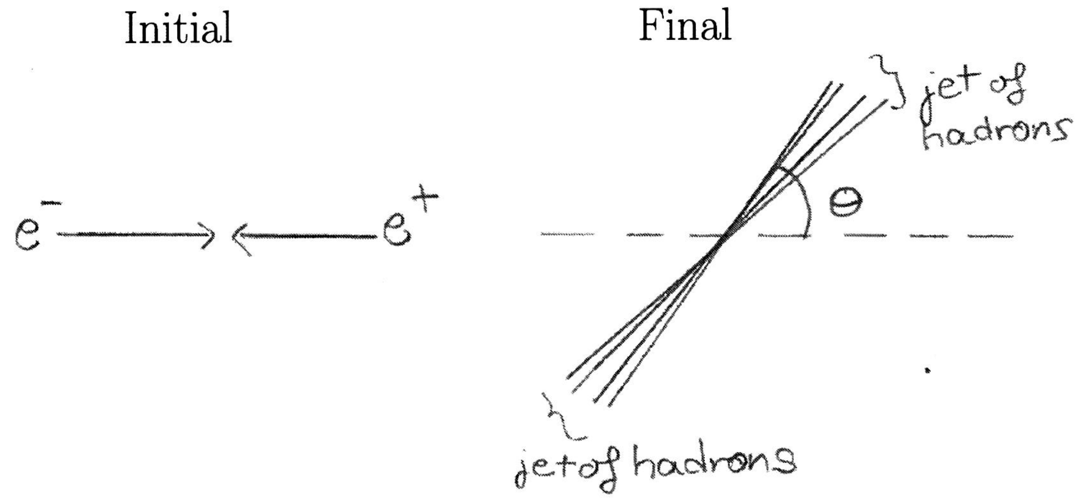
$$e^+e^- \rightarrow \text{hadrons} \qquad \text{at large} \qquad s = (p_e^+ + p_e^-)^2$$

and see if it can be understood in terms of

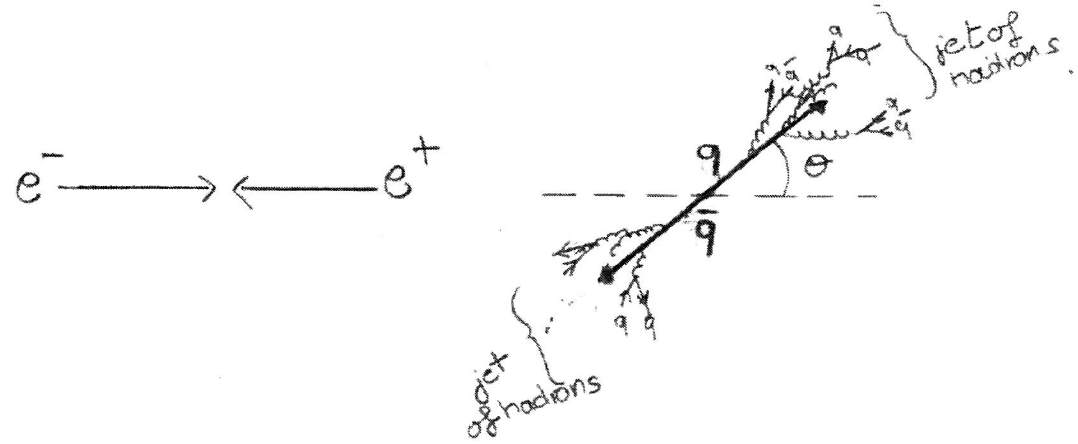
$$e^+e^- \rightarrow \text{quarks and gluons}$$

6.4 Some tests of QCD: $e^+e^- \rightarrow jets$

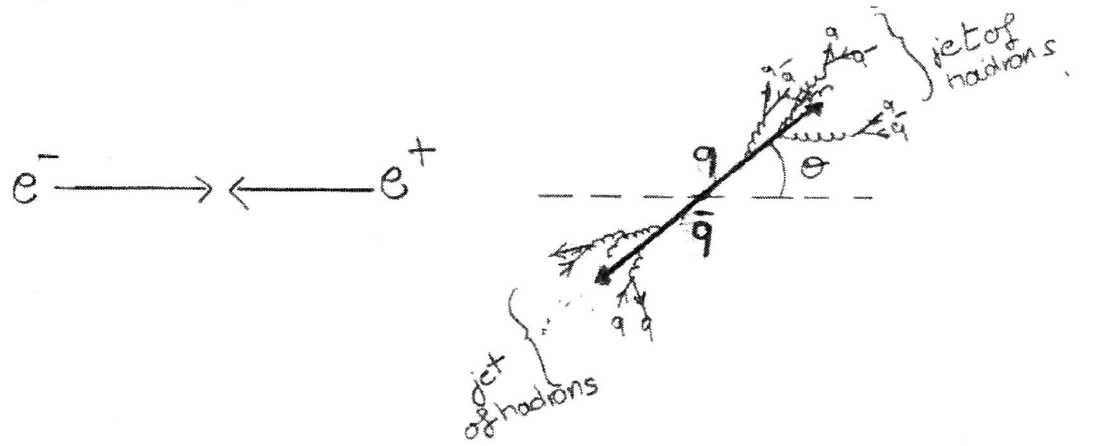
For example we can look at the process with two hadron jets in the final state. In COM



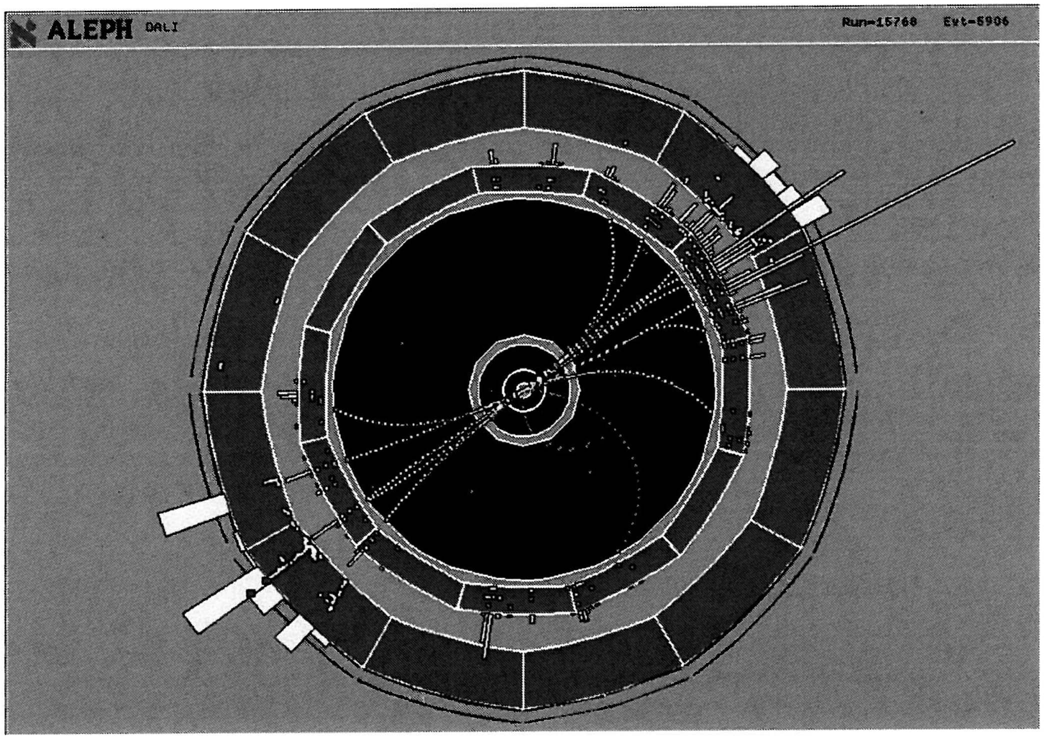
In QED+QCD the asymptotic free picture for this process is



In QED+QCD the asymptotic free picture for $e^+e^- \to 2$ hadron; jets is



In the experiment



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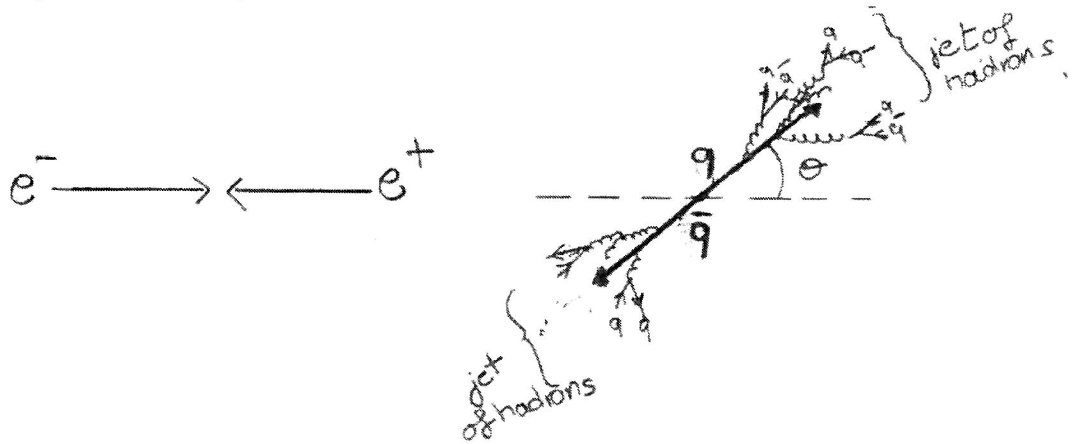


So we predict

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2 \text{ jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}} \times \text{Prob(final } q \to \text{jet)} \times \text{Prob(final } \bar{q} \to \text{jet)}$$

where the sum extend over all possible quark flavors light enough to be produced

But quarks and gluons always hadronize $\Rightarrow$ Prob(final $q \to$ jet) $=$ Prob(final $\bar{q} \to$ jet) $= 1$

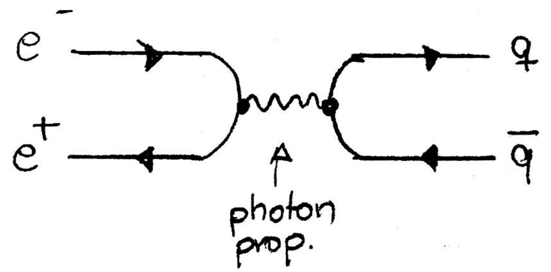
So we predict

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2 \text{ jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}}$$

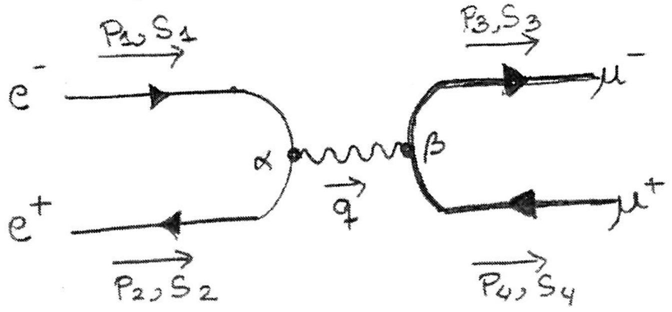
We predict

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2\,\text{jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}}$$

We compute the cross section for $e^+ e^- \to q\,\bar{q}$ from lowest order diagram in QED



If quarks are fermions that amplitude is totally analogous to the one for



$$\frac{d\sigma}{d\Omega}\Big|_{\text{COM}} = \frac{\alpha^2}{4\,s}\left[1 + \cos^2\theta\right]$$

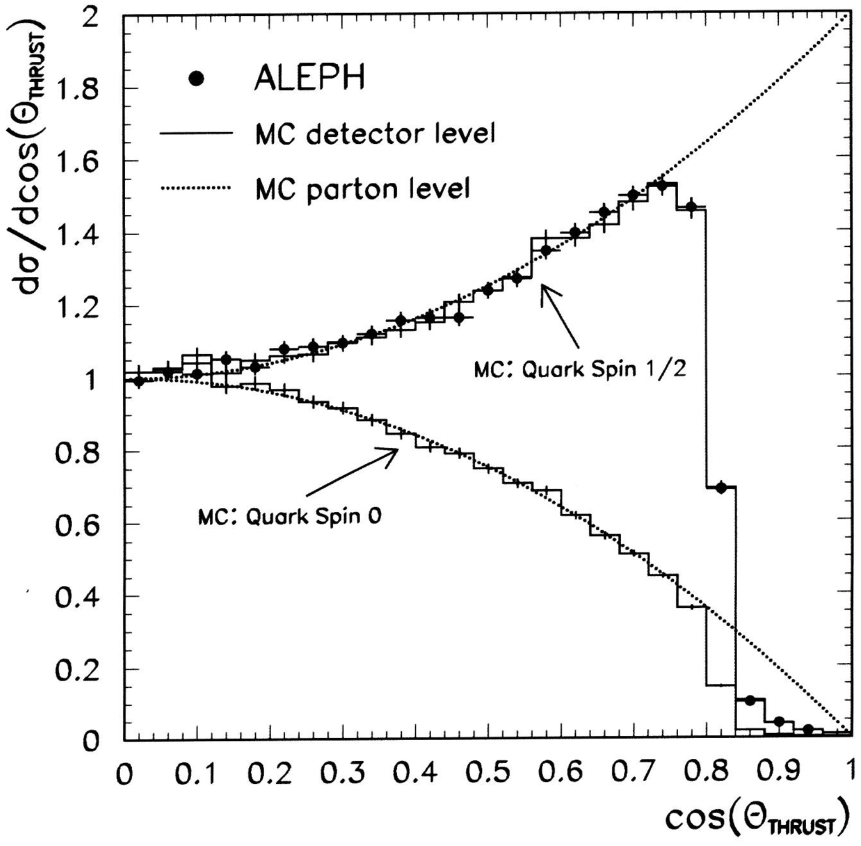
up to the number of colors and the charge of the quarks.

So

$$\frac{d\sigma}{d\Omega}(e^+ e^- \to 2\text{ jets})\Big|_{\text{COM}} = \sum_q \frac{d\sigma}{d\Omega}(e^+ e^- \to q\,\bar{q})\Big|_{\text{COM}} = 3 \times \sum_q Q_q^2 \frac{d\sigma}{d\Omega}(e^+ e^- \to \mu^+ \mu^-)\Big|_{\text{COM}}$$

$$= 3 \times \sum_q Q_q^2 \frac{\alpha^2}{4\,s}\left[1 + \cos^2\theta\right]$$

Comparing with data from ALEPH experiment



$\Theta_{\text{Thrust}} \equiv \theta_{\text{jet axis}}$

This confirms

- the picture of asymptotic free quarks

- that quarks are spin 1/2 particles like the muons

if quarks had spin 0

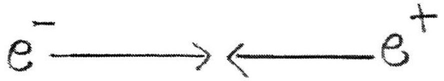
$\Rightarrow$ the amplitude would be proportional to $\sin\theta$

$$\Rightarrow \frac{d\sigma}{d\Omega}(e^+ e^- \to 2\text{ jets})\Big|_{\text{COM}} \propto \sin^2\theta = 1 - \cos^2\theta$$

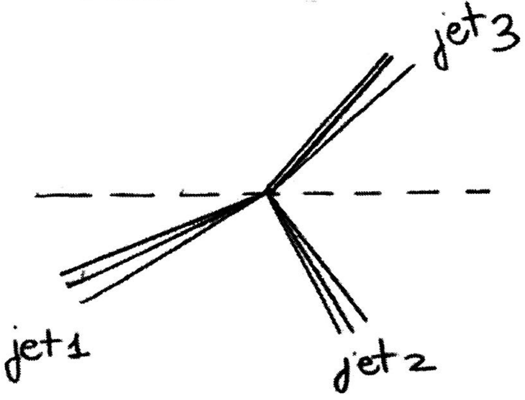
(dash-line)

Next we can look at the process with thee hadron jets in the final state. In COM

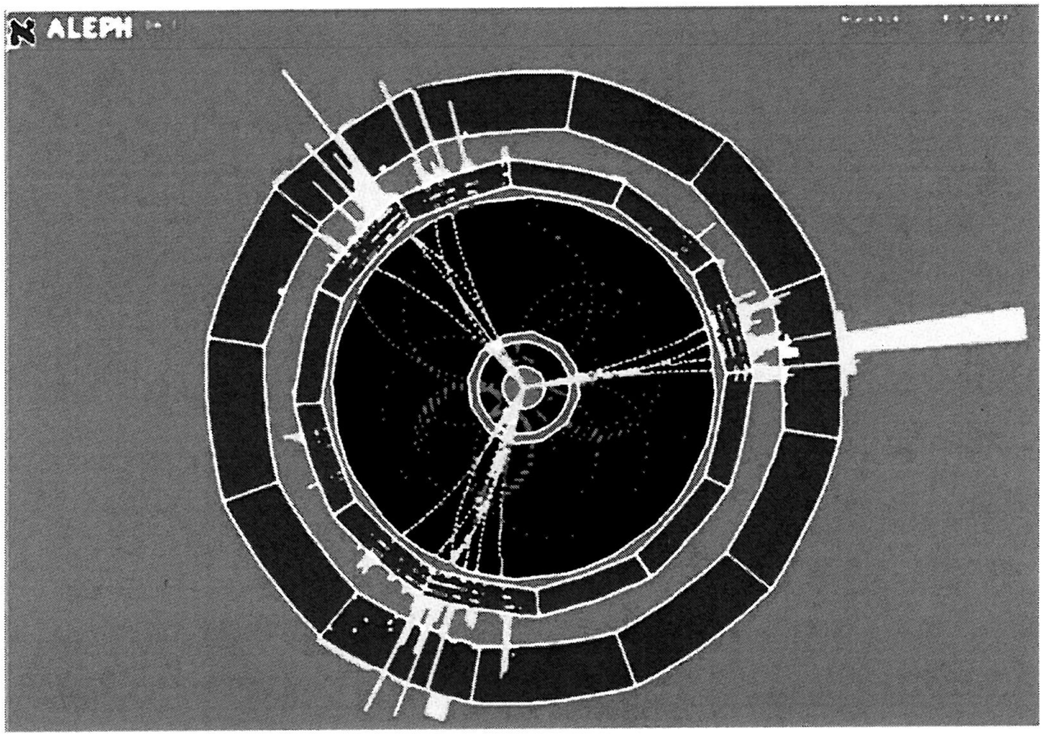
Initial



Final



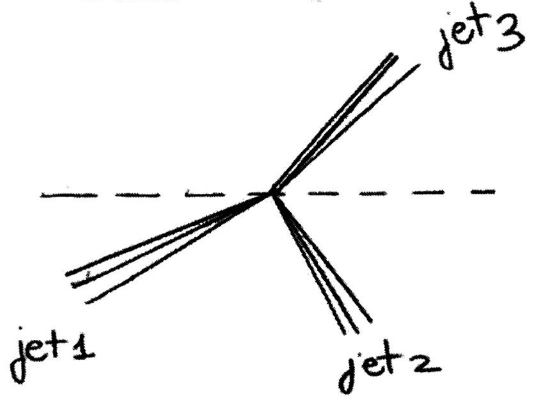
In the experiment



Next we can look at the process with thee hadron jets in the final state. In COM

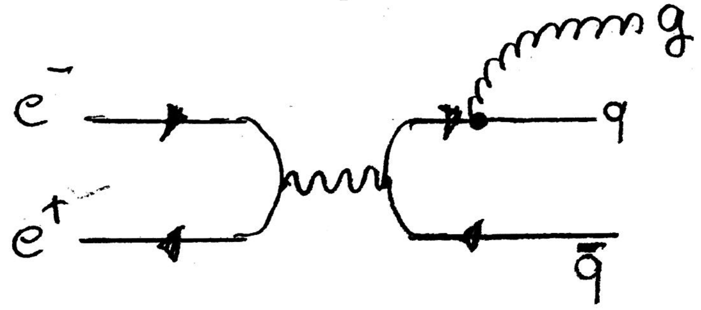
Initial

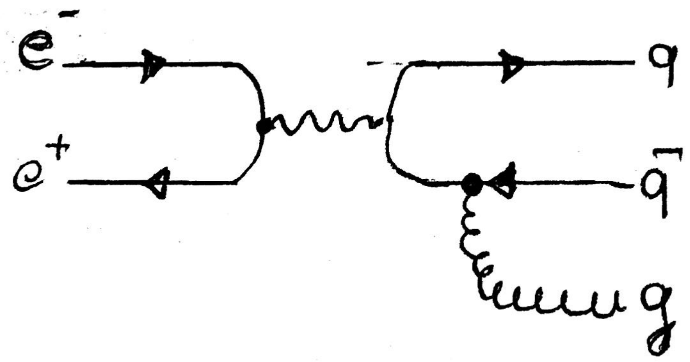
Final



In QED+QCD the asymptotic free picture for this process is that the third jet comes from a gluon

So the cross section can be predicted at lowest order from the diagrams





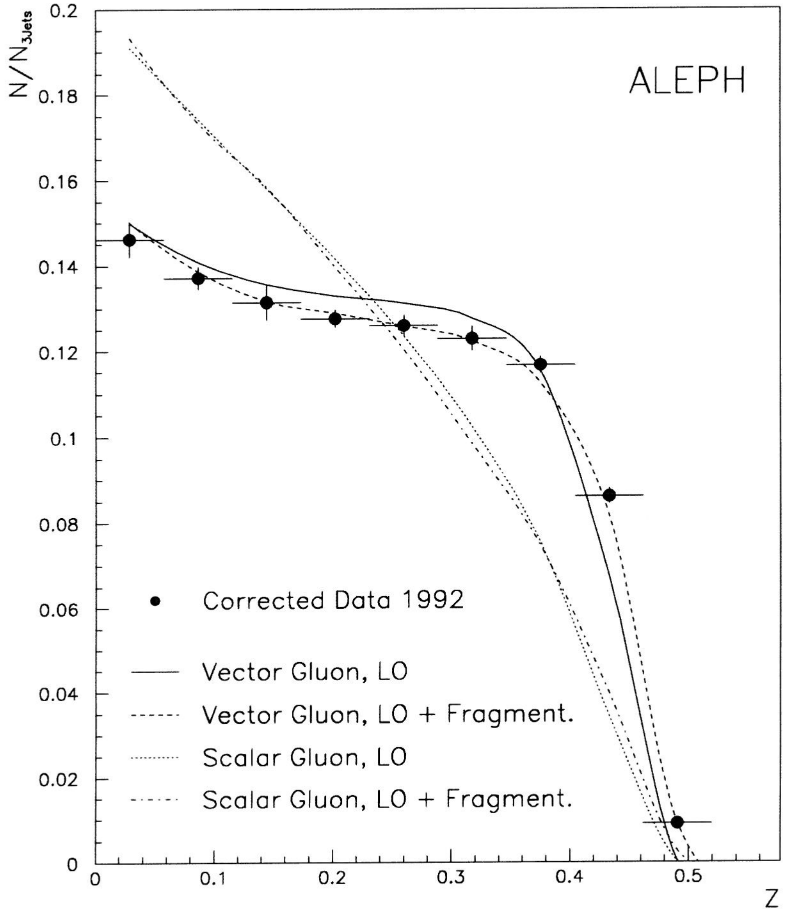
which involve the QCD vertex of quarks to gluons

⇒ comparing to data we can verify that gluons are vector particles and test gluon-q-$\bar{q}$ QCD vertex

$e^+ e^- \to 3\ jets$ first observed at the PETRA collider in DESY in Hamburg, Germany in the late 70's.

More precise data from ALEPH: Ordering $E_{jet,1} \leq E_{jet_2} \leq E_{jet_3}$

Study the $Z = \frac{1}{3}\left(\frac{2\,E_{jet_2}}{\sqrt{s}} - \frac{2\,E_{jet_3}}{\sqrt{s}}\right)$ distribution of events compared to QCD prediction



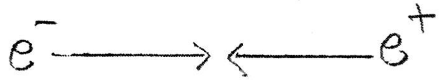
Z distribution would be different if gluon had spin=0

Data agrees with the QCD prediction

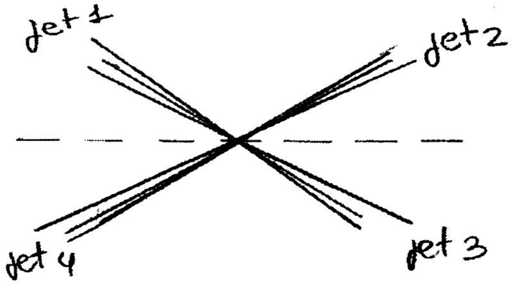
of a vector gluon-q-$\bar{q}$ coupling.

Next we can look at the process with four hadron jets in the final state. In COM

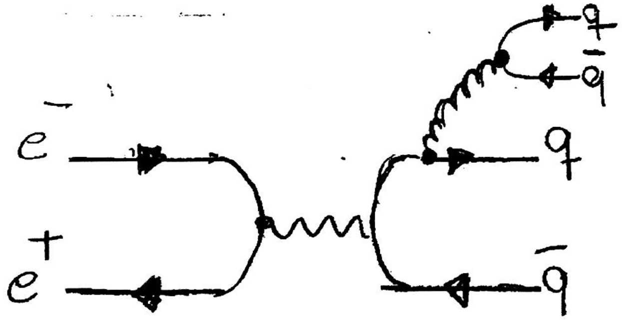
Initial

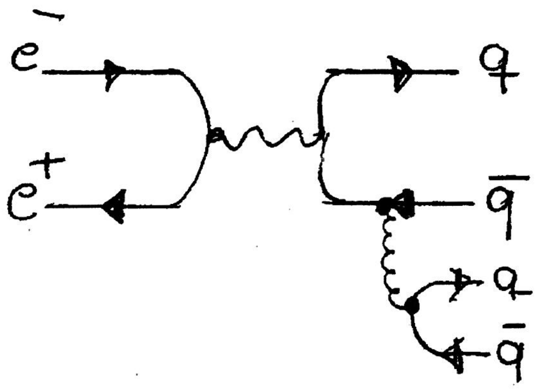


Final

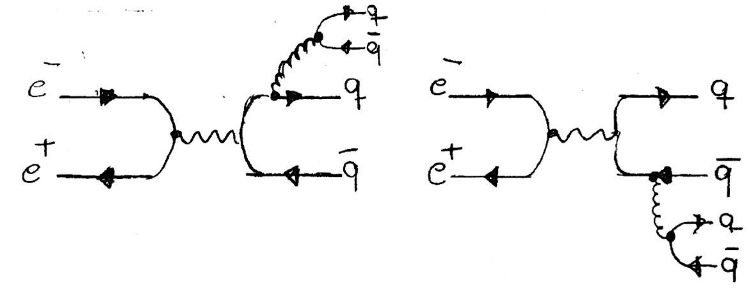


In QED+QCD the asymptotic free picture for this process the four jets can be either four quarks

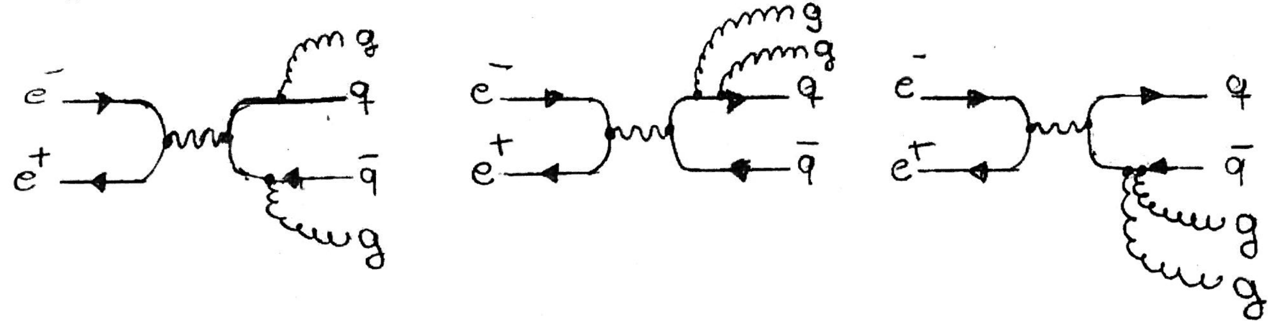


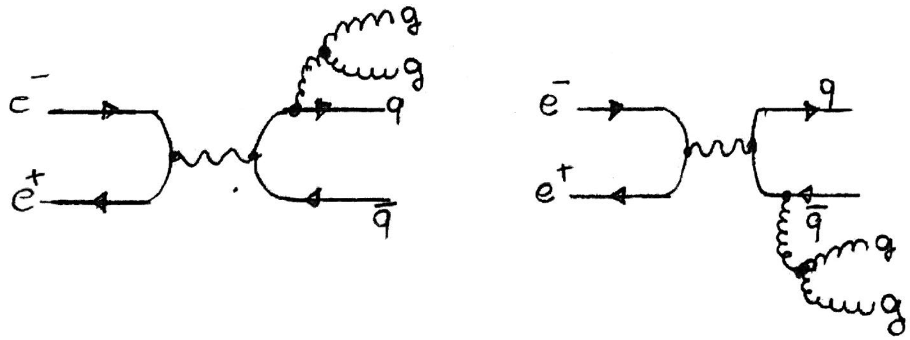


In QED+QCD the asymptotic free picture for $e^+ e^- \to 4$ jets the four jets can be either four quarks



or 2 quarks and 2 gluons and there are very different diagrams contributing





⇒ test of the gluon self-coupling

⇒ test of symmetry group $SU(N)_{\text{color}}$

Fit to data:

$C_A \equiv N = 2.9 \pm 0.6$ and $C_F \equiv \frac{N^2-1}{2N} = 1.35 \pm 0.27$

In agreement with QCD prediction :

$$C_A = 3,\ C_F = \tfrac{4}{3} = 1.33$$