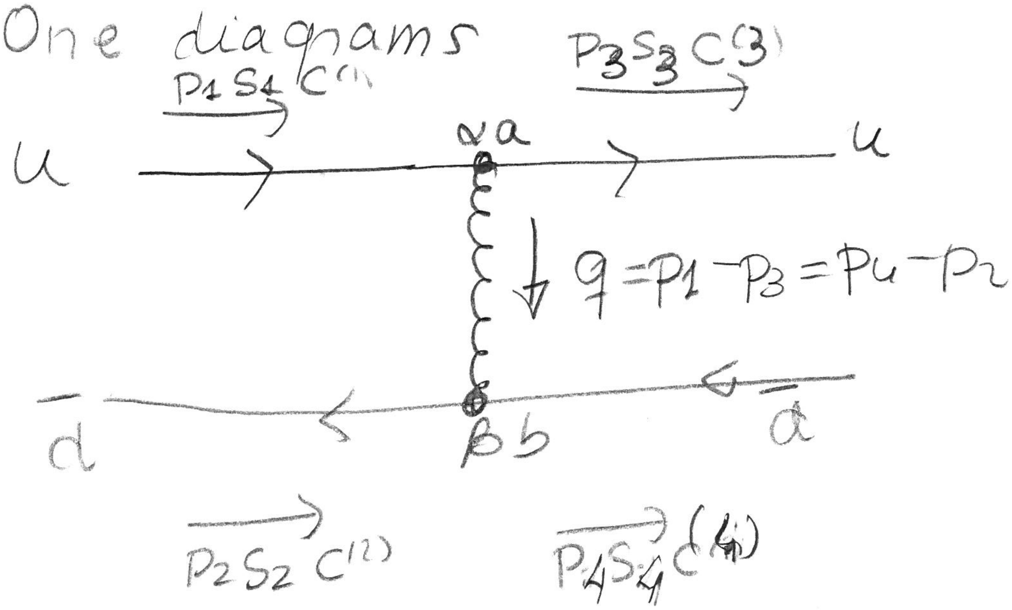
④ $q\bar{q}$ interaction in QCD

Let us compute the amplitude $u\bar{d} \to u\bar{d}$ in QCD

$u\bar{d} \to u\bar{d}$
$p_1 p_2 \quad p_3 p_4$
$s_1 s_2 \quad s_3 s_4$
color $c^{(1)} c^{(2)} \quad c^{(3)} c^{(4)}$

One diagrams



$$i\mathcal{M} = \sum_{ab} \left[\bar{u}_3 c^{(3)\dagger} \left[-ig_s \frac{\lambda^a}{2}\gamma^\alpha\right] u_1\right] c^{(1)} \left(\frac{-ig_{\alpha\beta}}{q^2}\right)\delta_{ab}$$

$$\bar{v}_2 c^{(2)\dagger} \left[-i\frac{g_s}{2}\frac{\lambda^a}{2}\gamma^\beta\right) v_4 c^{(4)}$$

$$= \frac{ig_s^2}{q^2}(\bar{u}_3\gamma^\alpha u_1)(\bar{v}_2\gamma_\alpha v_4) \underbrace{\sum_a c^{(3)\dagger}\frac{\lambda^a}{2}c^{(1)} c^{(2)\dagger}\frac{\lambda^a}{2}c^{(4)}}_{\overset{|||}{f_{\text{color}}}}$$

For $\overset{1}{e^-}\overset{2}{\mu^+} \to \overset{3}{e^-}\overset{4}{\mu^-}$ in QED

$$i\mathcal{M}^{QED} = +\frac{ie^2}{q^2}(\bar{u}_3\gamma^\alpha u_1)(\bar{v}_2\gamma_\alpha v_4)$$

Same with $g_s \to e$

$f_{color} \to 1$

In QED the coloumb potenhal betwen and e^- and a μ^+ is

$$V^{em}_{(r)} = \overset{\text{negative} \Rightarrow \text{attractive potential}}{-}\frac{\alpha}{r}$$

From above comparison we expect the strong potential betwen u and $\bar{d}$

$$V^{strong}_{(r)} = -f_{color}\frac{\overset{\text{dep}}{\alpha(r)}}{r}$$

Let us calculate f_{colour} for some colour configs

$(u\bar{d})_{init}$ and $(u\bar{a})_{final}$ can be

composing $3_c \times \bar{3}_c = \underset{}{\overset{\text{colour octet}}{8_c}} \oplus \underset{\text{colour singlet}}{1_c}$

the colour singlet state is

$$\left(C^{(1)}C^{(2)+}\right)_{singlet} = \frac{1}{\sqrt{3}}\left(r\bar{r}+b\bar{b}+c\bar{c}\right) = \frac{1}{\sqrt{3}}\sum_i c_i c_i^+$$

$$r = C_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad b = C_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad g = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

and in the same way

$$\left(C^{(3)+}C^{(4)}\right)_{singlet} = \frac{1}{\sqrt{3}}\sum_j c_j^+ c_j$$

$$\Rightarrow f^{colour}_{singlet} = \frac{1}{4}\,\frac{1}{3}\sum_a \underbrace{\sum_j \underbrace{\sum_i \underbrace{c_j^+\lambda^a c_i}_{\lambda^a_{ji}}\ \underbrace{c_i^+\lambda^a c_j}_{\lambda^a_{ij}} =}_{(\lambda^a)^2_{ii}}}_{=\ \mathrm{Tr}(\lambda^a)^2}$$

$$= \frac{1}{12}\sum_a \mathrm{Tr}(\lambda^a)^2 = \frac{4}{3} > 0$$

So in colour singlet state the potential is attractive.

For any non-singlet combination fe $\overset{1}{r}\overset{2}{g}\ \overset{3}{r}\overset{4}{\bar{g}}$

$$C^{(1)}C^{(3)+} = c_1 c_3^+$$

$$C^{(3)+}C^{(2)} = c_3 c_1^+$$

$$\Rightarrow f_{colour} = \frac{1}{4}\sum_a \lambda^a_{11}\lambda^a_{33} = -\frac{1}{6} < 0$$

⇒ repulsive potential. - Same for any non-singlet configuration

So mesons ($\equiv q\bar{q}'$ bound states of strong interaction) can only be formed if constituent $q\bar{q}'$ are in color singlet state.

For baryons $qq'q''$ the color state is obtained composing

repulsive among the 3 quarks

$$3_c \times 3_c \times 3_c \Rightarrow 1_c \times 8_{cs} \times 8_{ca} + 10_c$$

attractive potential among the 3q

attractive between 2 and repulsive with the 3rd

the colour wave function for 1_s is totally anti-symmetric in colour

$$\frac{1}{\sqrt{6}}\left(q_r q'_g q''_b + q_b q'_r q''_g + q_g q'_b q''_r - q_g q'_r q''_b - q_b q'_g q''_r - q_r q'_b q''_g \right)$$

⑥ Bound states of strong interactions and SU(3) FLAVOR

Starting with the Lagrangian of QCD in terms of quarks one cannot describe perturbatively the physics of bound states (hadrons) because this is a regime in which α_s in very large and perturbative expansion does not work.

But using the global symmetries of $\mathcal{L}_{QCD}$ as guide one can infer some of the features of the bound states, in particular those made up of light quarks u, d, s

If we neglect the mass differences for these 3 quarks the QCD Lag

$$\mathcal{L}_{QCD} = \sum_{\substack{ij \\ \text{color}}} \Big\{ \bar{u}_i \left(i\gamma^\mu D_{\mu ij} - m\delta_{ij} \right) u_j + \bar{d}_i \left(i\gamma^\mu D_{\mu ij} - m\delta_{ij} \right) d_j + \bar{s}_i \left(i\gamma^\mu D_{\mu ij} - m\delta_{ij} \right) s_j \Big\}$$

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s \sum_{a=1}^{8} \frac{\lambda^a_{ij}}{2} G^a_\mu$$

can be written in matrix form

$$\mathcal{L}_{QCD} = \sum_{j} (\bar{u}_i \; \bar{d}_i \; \bar{s}_i) \left[i\gamma^{\mu} D_{ij} \, I_{3\times3} - M_q \delta_{ij} \right] \begin{pmatrix} u_j \\ d_j \\ s_j \end{pmatrix}$$

$$M_q = \begin{pmatrix} m & & \\ & m & \\ & & m \end{pmatrix}$$

this Lag is invariant under a global rotation in flavour space

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \underset{\text{3×3 unitary matrix} \in SU(3)_{Flav.}}{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

one can think of $\begin{pmatrix} u \\ d \\ s \end{pmatrix} \equiv 3_F$ as a triplet of $SU(3)_F$

and correspondingly $(\bar{u}, \bar{d}, \bar{s})$ is a $\bar{3}$ of $SU(3)_F$

consequently the wavefunction of hadrons made of these quarks

$$\Psi_{had} = \Psi_{spatial} \times \Psi_{spin} \times \underset{\text{singlet}}{\Psi_{colour}} \times \Psi_{flavour}$$

where Ψ_{FLAV} must be in some representation of $SU(3)_F$ obtained from the composition of the 3_F and $\bar{3}_F$ of its constituents

For example for a meson ($q\bar{q}'$) we need to compose

$$3_F \times \bar{3}_F = 8_F \times 1_F \rightarrow 1 \text{ octet and } 1 \text{ singlet}$$

$\equiv 9$ mesons for each

$\underbrace{\text{given } \psi_{spatial} \text{ and } \psi_{spin}}$

(for a given S_{spin}, P) :

In this limit in which $m_u = m_d = m_s$ all hadrons in a given $SU(3)_F$ rep have the same mass

Since in reality $m_u \simeq m_d < m_s$ the hadrons with only u and d quarks and antiquarks have more similar mass ($SU(2)_F \equiv$ quark isospin)

If we look at PDG we find 9 pseudoscalar ($S=0$, $P=-1$) mesons with similar mass

($\pi^\pm$ π^0, $K^\pm$, $K^0 \bar{K}^0$, η, η')

and 9 vector mesons ($S=1$, $P=-1$)

($\rho^\pm \rho^0$ $K^{*\pm}$ $K^{*0} \bar{K}^{*0}$, ω, ϕ)

Slides

In this model a meson $q_1\bar{q}_2$ has a mass

$$M_{meson} = \underbrace{m_1 + m_2}_{\substack{\uparrow \\ \text{quark} \\ \text{constituent mass}}} + \underset{\substack{\uparrow \\ \text{constant to} \\ \text{be fitted to data}}}{A} \frac{\overset{\text{spins of quarks}}{\langle \vec{S}_1 \cdot \vec{S}_2 \rangle}}{m_1 m_2}$$

$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle$ can be computed from the spin wave function of the meson: $S_1 = \frac{1}{2}$ $S_2 = \frac{1}{2} \Rightarrow \vec{S}_{12} \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right.$

$$\langle \vec{S}_1 \vec{S}_2 \rangle = \frac{S(S+1)}{2} - \frac{3}{4} \begin{cases} -3/4 & \text{for } S=0 \text{ pseudoscal} \\ \frac{1}{4} & \text{for } S=1 \text{ vector} \end{cases}$$

(slides)

For baryons ($\equiv qq'q''$) $\Rightarrow$ wave function must be totally antisymmetric under exchange of 2 of them

For light baryons ($l=0$) $\Rightarrow \Psi_{spat}$ is totally symmet

We have seen that Ψ_{color} is totally antisymmetric

$\Rightarrow \Psi_{spin} \times \Psi_{FLAV}$ must be totally symmetric

Composing

$$3_F \times 3_F \times 3_F = \underset{\uparrow}{10_F} \oplus \underbrace{8_{FS} \oplus 8_{FA}} \oplus \underset{\uparrow}{1_F}$$

$\psi_{FLAV} \Rightarrow$ totally symmetric

partially symmetric and partially anti symmetric

totally antisymmetric

For spin

$$2_S \times 2_S \times 2_S = \underset{\uparrow}{4_S} \oplus 2_{SS} \oplus 2_{SA}$$

ψ_{SPIN} totally symmetric

partly symmetric and " antisymmetric

So possible combination are

$10_F \times 4_S \Rightarrow$ decuplet of $S = 3/2$ baryons

$$\left.\begin{array}{l} 8_{FS} \times 2_{SS} \\ 8_{FA} \times 2_{FA} \end{array}\right\}$$ octet of $S = 1/2$ baryons

- For the pseudoscalar mesons

particles	$S_{\text{strangeness}}$	Mass (MeV)	Q	I	I_3	$\psi_{flavour}$
K^0, K^+	1	496	0,+1	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	$d\bar{s}$, $u\bar{s}$
π^-, π^0, π^+	0	138	-1,0,+1	1	-1,0,+1	$d\bar{u}$, $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $u\bar{d}$
η	0	776	0	0	0	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$
η'	0	957	0	0	0	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$
K^-, $\overline{K}^0$	-1	496	-1,0	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	$s\bar{u}$, $s\bar{d}$

- $K^+, K^0, \overline{K}^0, K^-, \pi^+, \pi^0, \pi^-\eta$ are the octet and η' is the singlet
- For the vector mesons

Particles	S	Mass(MeV)	Q	I	I_3	$\psi_{flavour}$
K^{*0}, K^{*+}	1	896	0, +1	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	$d\bar{s}$, $u\bar{s}$
ρ^-, ρ^0, ρ^+	0	776	-1,0,+1	1	-1,0,+1	$d\bar{u}$, $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $u\bar{d}$
ω	0	783	0	0	0	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$
ϕ	0	1020	0	0	0	$s\bar{s}$
K^{*-}, $\overline{K}^{*0}$	-1	896	-1,0	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	$s\bar{u}$, $s\bar{d}$

- $K^{*+}, K^{*0}, \overline{K}^{*0}, K^{*-}, \rho^+, \rho^0, \rho^-$ are 7 members of the octet
- ω and ϕ are an admixture of the singlet and the neutral component of the octet.

Mass formula for the mass of a meson compose of a quark 1 and and antiquark 2 as

$$\mathrm{M(meson)} = m_1 + m_2 + A\,\frac{\langle \vec{S}_1.\vec{S}_2 \rangle}{m_1\,m_2}$$

with

$$\langle \vec{S}_1.\vec{S}_2 \rangle = \frac{s(s+1)}{2} - \frac{3}{4} = \begin{cases} -\frac{3}{4} \text{ for pseudoscalar mesons (s = 0)} \\ \frac{1}{4} \text{ for vector mesons (s = 1)} \end{cases}$$

For the values $m_u = m_d = 310$ MeV $m_s = 483$ MeV and $A = (2m_u)^2 \times 160$ MeV

Meson	**Calculated**	**Observed**
π	140	138
K	484	496
η	559	549
ρ	780	776
ω	780	783
K^*	896	892
ϕ	1032	1020

So this model could explain the masses of all the light meson (except the η') to about 1% accuracy.

- Baryons with Spin $s = \frac{1}{2}$: 8 particles

particles	S	Mass (MeV)	Q	I	I_3	quark content
n, p	0	939	0,+1	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	*udd*, *uud*
$\Sigma^-, \Sigma^0, \Sigma^+$	-1	1193	-1,0,+1	1	-1,0,+1	*dds*, *uds*, *uus*
Λ^0	-1	1114	0	0	0	*uds*
Ξ^-, Ξ^0	-2	1318	-1,0	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	*dss*, *uss*

- Baryons with Spin $s = \frac{3}{2}$: 10 particles

particles	S	Mass (MeV)	Q	I	I_3	quark content
Δ^-,Δ^0,Δ^+,Δ^{++}	0	1232	-1,0,+1,+2	$\frac{3}{2}$	$-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$	*ddd udd*, *uud*, *uuu*
$\Sigma^{*-}, \Sigma^{*0}, \Sigma^{*+}$	-1	1384	-1,0,+1	1	-1,0,+1	*dds*,*uds*, *uus*
Ξ^{*-}, Ξ^{*0}	-2	1533	-1,0	$\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$	*dss*, *uss*
Ω^-	-3	1672	-1	0	0	*sss*

Again, the masses of all baryons in a given multiplet are not all the same because the $SU(3)_{\text{flavour}}$ symmetry is broken by the different quark masses.

But they are more similar within each multiplet so the potential between the quarks must be spin dependent.

$$\text{M(baryon)} = m_1 + m_2 + m_2 + A' \left(\frac{\langle \vec{S}_1 . \vec{S}_2 \rangle}{m_1 \, m_2} + \frac{\langle \vec{S}_1 . \vec{S}_3 \rangle}{m_1 \, m_3} + \frac{\langle \vec{S}_2 . \vec{S}_3 \rangle}{m_2 \, m_3} \right)$$

For the baryon decuplet in all spin configurations one finds

$$\langle \vec{S}_i . \vec{S}_j \rangle = \frac{1}{4}$$

The mass of all baryons in the decouplet can well explained with $A' = (2m_u)^2 \, 50$ MeV.