Nuclear Physics B340 (1990) 752–758
North-Holland

TEMPERATURE AUTOCORRELATIONS OF THE TRANSVERSE ISING CHAIN AT THE CRITICAL MAGNETIC FIELD

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Received 26 February 1990

A partial differential equation for a temperature autocorrelator of the transverse Ising chain at the critical magnetic field is obtained. It is essentially the same equation as for the equal-time temperature correlator of the impenetrable bosons. The formula for the asymptotics of the autocorrelator as time tends to infinity is obtained.

1. Introduction

The important particular case of the spin-$\frac{1}{2}$ XY model in a magnetic field [1] is the transverse Ising model at the critical magnetic field specified by the hamiltonian

$$H = -\tfrac{1}{2} \sum_{l} \{ \sigma_l^x \sigma_{l+1}^x + \sigma_l^z \} , \tag{1}$$

where σ_l are Pauli matrices at the lth site of a one-dimensional lattice. Correlation functions in this model were investigated by many authors. We use in particular the results and notations of ref. [2] (other references in this connection can be found in ref. [2]).

0550-3213/90/$03.50

The correlation function $X_n(t)$ of the first spin components

$$X_n(t) \equiv \langle \sigma_0^x(t)\sigma_n^x(0)\rangle_T = \frac{\mathrm{Tr}(\exp\{-\beta H\}\sigma_0^x(t)\sigma_n^x(0))}{\mathrm{Tr}(\exp\{-\beta H\})} \tag{2}$$

($\beta = 1/T$ is the inverse temperature) generates all the other correlators of the model. For example,

$$\langle \sigma_0^y(t)\sigma_n^y(0)\rangle_T = -\partial_t^2 X_n(t).$$

It should also be mentioned that correlators of the isotropic XY model,

$$H_{XY} = -\tfrac{1}{4}\sum_l \{\sigma_l^x\sigma_{l+1}^x + \sigma_l^y\sigma_{l+1}^y\}$$

can also be expressed in terms of $X_n(t)$, e.g.

$$\langle \sigma_0^x(t)\sigma_n^x(0)\rangle_T^{(XY)} = \begin{cases} X_{n/2}^2(t/2), & n \text{ even} \\ X_{(n-1)/2}(t/2)X_{(n+1)/2}(t/2), & n \text{ odd.} \end{cases}$$

2. Autocorrelator as a Fredholm determinant

In what follows, the autocorrelator $X_0(t)$ in the transverse Ising model

$$X_0(t) = \langle \sigma_n^x(t)\sigma_n^x(0)\rangle_T \tag{3}$$

is considered. In ref. [2] the representation of this correlator as the Fredholm determinant of linear integral operator $\mathscr{K}$ was obtained:

$$X_0(t) = \exp\{-t^2/2\}\det(1-\mathscr{K}). \tag{4}$$

The kernel $\mathscr{K}(z_1, z_2)$ of operator $\mathscr{K}$ is

$$\mathscr{K}(z_1,z_2) = \tanh\left(\beta\sqrt{1-z_1^2}\right)\frac{\sin(it(z_1-z_2))}{\pi(z_1-z_2)}, \qquad -1 \leqslant z_1, z_2 \leqslant 1. \tag{5}$$

Our aim is to use the approach of refs. [3–5] and to consider the Fredholm determinant as a τ-function for some integrable equation. To this end it is convenient to introduce variables x, τ

$$x \equiv it/\beta\,; \qquad \tau \equiv \beta^2\,, \tag{6}$$

spectral parameters $\lambda \equiv \beta z_1$, $\mu \equiv \beta z_2$ and to symmetrize the kernel. One obtains

$$X_0(t) = \exp\{-t^2/2\}\Delta(x,\tau,\gamma)|_{\gamma=1/\pi} \tag{7}$$

with

$$\Delta(x,t,\gamma) = \det(1-\gamma K)\,. \tag{8}$$

The kernel $K(\lambda,\mu)$ of operator K is

$$K(\lambda,\mu) = \sqrt{\vartheta(\lambda)}\,\frac{\sin x(\lambda-\mu)}{(\lambda-\mu)}\sqrt{\vartheta(\mu)}\,, \qquad -\sqrt{\tau} \leqslant \lambda,\mu \leqslant \sqrt{\tau}\,. \tag{9}$$

The "weight" $\vartheta(\lambda)$ is given here as

$$\vartheta(\lambda) \equiv \vartheta(\lambda^2-\tau) = \tanh\sqrt{\tau-\lambda^2} \tag{10}$$

satisfying the obvious relation

$$(2\lambda\partial_\tau + \partial_\lambda)\vartheta(\lambda) = 0\,. \tag{11}$$

Comparing expression (8) with the corresponding Fredholm determinant for the impenetrable bosons [4] one notices the difference in only three aspects:

(i) variable x is now pure imaginary;

(ii) the explicit form of weight $\vartheta(\lambda)$ is different but relation (11) is the same;

(iii) operator K now acts on the interval $[-\sqrt{\tau}, \sqrt{\tau}]$ and not on the whole axis.

It appears that one can nevertheless use the approach of the references mentioned above to obtain a partial differential equation for the determinant. The calculations are almost literally the same, so we give mainly the answers.

The main result obtained below is the following partial differential equation for the autocorrelator X_0 [eqs. (3) and (7)]. The function

$$\begin{aligned}\sigma(x,\tau) &\equiv \ln\left(\exp\{t^2/2\}X_0(t)\right)\\ &= \ln\det(1-\pi^{-1}K)\end{aligned} \tag{12}$$

satisfies the equation

$$\left(\partial_\tau\partial_x^2\sigma\right)^2 = -4\partial_x^2\sigma\left[2x\partial_\tau\partial_x\sigma + (\partial_\tau\partial_x\sigma)^2 - 2\partial_\tau\sigma\right] \tag{13}$$

with the initial conditions

$$\sigma(x,\tau) = -\mathrm{d}(\tau)x - \mathrm{d}^2(\tau)x^2/2, \qquad (x \to 0)$$

$$\sigma(x,\tau=0) = 0, \qquad \mathrm{d}(\tau) = \frac{1}{\pi}\int_{-\sqrt{\tau}}^{\sqrt{\tau}} \vartheta(\mu)\,\mathrm{d}\mu. \tag{14}$$

3. Partial differential equation for a temperature autocorrelator

Introduce functions $f_\pm(\lambda)$ as solutions of the integral equations

$$f_\pm(\lambda) - \gamma\int_{-\sqrt{\tau}}^{\sqrt{\tau}} K(\lambda,\mu) f_\pm(\mu)\,\mathrm{d}\mu = e_\pm(\lambda), \tag{15}$$

where

$$e_\pm(\lambda) = \sqrt{\vartheta(\lambda)}\,\exp\{\pm i\lambda x\}.$$

Defining also the "potential B,"

$$B_{\alpha\beta}(x,\tau,\gamma) = \gamma\int_{-\sqrt{\tau}}^{\sqrt{\tau}} e_\alpha(\lambda) f_\beta(\lambda)\,\mathrm{d}\lambda; \qquad \alpha,\beta = +,-,$$

$$B_{++} = B_{--}, \qquad B_{+-} = B_{-+}, \qquad B_{\alpha\beta}(x,0,\gamma) = 0 \tag{16}$$

one obtains, as in ref. [4], the zero curvature representation for the vector function $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$

$$\partial_x f = (i\lambda\sigma_3 + B_{++}\sigma_1) f,$$

$$(2\lambda\partial_\tau + \partial_\lambda) f = (ix\sigma_3 - i\partial_\tau B_{+-}\sigma_3 - \partial_\tau B_{++}\sigma_2) f. \tag{17}$$

The compatibility condition for these equations (at arbitrary values of the spectral parameter λ) gives partial differential equations for the potentials valid for arbitrary γ which are just the same as in the case of the nonrelativistic Bose gas

$$\partial_x B_{+-} = B_{++}^2,$$

$$\partial_\tau B_{+-} = x + \frac{\partial_\tau\partial_x B_{++}}{2B_{++}}. \tag{18}$$

This results in the same equation for B_{++} [4]:

$$\partial_\tau\left(B_{++}^2\right) = 1 + \partial_x\left(\frac{\partial_\tau\partial_x B_{++}}{2B_{++}}\right). \tag{19}$$

Initial conditions for this equation are obtained directly from integral equations (15):

$$B_{++}(x,\tau,\gamma) = \sum_{k=0}^{\infty} b_k(\tau,\gamma)x^k = \gamma\beta_0(\tau) + \left(\gamma\beta_0(\tau)\right)^2 x + \dots,$$

$$\left(x \to 0,\ \beta_0 = \int_{-\sqrt{\tau}}^{\sqrt{\tau}} \vartheta(\lambda)\,\mathrm{d}\lambda\right), \tag{20}$$

$$B_{++}(x,\tau,\gamma) \to \tilde{b}_1(x,\gamma)\tau \to 0, \qquad (\tau \to 0). \tag{21}$$

The Fredholm determinant Δ (8) itself satisfies a partial differential equation in x, τ valid at arbitrary γ. To obtain this one first calculates for $\sigma \equiv \ln \Delta$:

$$\partial_x \sigma = -B_{+-},$$

$$\partial_\tau \sigma = -x\partial_\tau B_{+-} + \tfrac{1}{2}(\partial_\tau B_{+-})^2 - \tfrac{1}{2}(\partial_\tau B_{++})^2 \tag{22}$$

which by virtue of eq. (18) leads to

$$\left(\partial_\tau \partial_x^2 \sigma\right)^2 = -4\partial_x^2\sigma\left[2x\partial_\tau\partial_x\sigma + (\partial_\tau\partial_x\sigma)^2 - 2\partial_\tau\sigma\right], \tag{23}$$

with the initial conditions

$$\sigma(x,\tau,\gamma) = \sum_{k=1}^{\infty} \sigma_k(\tau,\gamma)x^k = -\gamma\beta_0 x - \gamma^2\beta_0^2 x^2/2 - \dots, \qquad (x \to 0) \tag{24}$$

and

$$\sigma(x,\tau,\gamma) \to \tilde{\sigma}_1(x,\gamma)\tau \to 0 \qquad (\tau \to 0). \tag{25}$$

For $\gamma = 1/\pi$, using eq. (7) one obtains the equations for the correlator $X_0(t)$ given at the end of sect. 2 [eqs. (12)–(14)].

It should be mentioned that the correlator $X_0(t)$ at zero temperature was completely described in ref. [2]. It satisfies the ordinary differential equation (the fifth Painlevé transcendent) in this case. At $T = 0$ ($\beta = \infty$) the correlator depends on variable $\tau_0 \equiv x\sqrt{\tau} = it$ only, and for the function $\sigma_0 \equiv \tau_0(\partial \ln \Delta(\tau_0, \gamma)/\partial\tau_0)$ eq.

(23) gives

$$(\tau_0\sigma_0'')^2 = -4(\tau_0\sigma_0' - \sigma_0)(4\tau_0\sigma_0' + \sigma_0'^2 - 4\sigma_0) \tag{26}$$

which is just the corresponding ordinary differential equation mentioned. It is to be noted that at $T=0$ this equation also describes the equal-time correlator for impenetrable bosons [6].

4. Conclusion

We have shown that the autocorrelator in the transverse Ising model at the critical magnetic field satisfies the same partial differential equation as the equal-time field correlator of the one-dimensional impenetrable bosons.

The most interesting question is the asymptotics of the autocorrelator at $t \to \infty$. In ref. [2] it was stated that $\det(1 - \mathscr{K})$ in eq. (4) grows for $t \to \infty$ more slowly than $\exp\{t^2/2\}$. Our calculations agree with this statement. Our hypothetical answer for the main term of the asymptotics obtained from the analysis of eqs. (15) and (23) is ($t \to \infty$, β fixed, $\epsilon < \beta < \epsilon^{-1}$, $\epsilon \to 0$)

$$X_0(t) \to \exp\left\{ -\frac{t^2}{2} + 4t - \frac{i}{\pi} t \int_{-1}^{1} \mathrm{d}\mu \ln\left(\frac{2 + \tanh \beta\sqrt{1-\mu^2}}{2 - \tanh \beta\sqrt{t-\mu^2}} \right) + \text{minor terms} \right\}. \tag{27}$$

We hope to give the proof of this formula as well as the complete asymptotical expansion of the autocorrelator in our next paper.

We thank L.D. Faddeev for discussions. This work was supported in part by NSF grant NSF-PHY-89-08495.

Appendix A

For very large temperature $\beta \to 0$ (and fixed time t, which is large) one obtains

$$X_0(t) = \exp\left\{ -\frac{t^2}{2} - \frac{i\beta t}{2} \right\} \left[1 + \text{const} \frac{\beta^2}{t^3} e^{4t} \right]. \tag{A.1}$$

This formula describes the crossover region $\beta \to 0$, $t \to \infty$. It is valid under the condition $\beta^2 e^{4t} \to 0$. Eq. (A.1) shows how the asymptotic formula (27) turns into the infinite temperature formula $X_0(t) = \exp\{-t^2/2\}$ at $\beta = 0$, as obtained in ref. [7].

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