PAULI PRINCIPLE FOR ONE-DIMENSIONAL BOSONS AND THE ALGEBRAIC BETHE ANSATZ

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ABSTRACT. For the construction of the physical vacuum in exactly solvable one-dimensional models of interacting bosons it is important that the momenta of all the particles be different. We give a formal proof that they are indeed different.

1. The existence of the Pauli principle not only for fermions but also for interacting bosons (identical bosons cannot have equal momenta) is a specific feature of one-dimensional mathematical physics [1]. One can understand the physical reason by considering, for example, the one-dimensional Schrödinger equation for two particles. The wave function in the center of the mass frame for any state with equal momenta has zero energy and generally increases linearly at infinity. Hence, this wave function cannot be normalized as a plane wave; it also cannot be made periodical if the system is put into a box. In other words, this state does not exist in the space of physical states.

A new method for obtaining exact solutions for many one-dimensional models has recently appeared – the quantum inverse scattering method (QISM) [2]. This method allows one to find the spectrum of model Hamiltonians by means of the algebraic Bethe Ansatz. The permitted momenta of the particles have to satisfy a system of 'transcendental equations' which are equivalent to the periodicity conditions for wave functions in configuration space. Up to now, the fact that all these momenta should be different has been postulated. In this paper the Pauli principle for interacting bosons is proved in the framework of QISM for the nonlinear Schrödinger equation (the NS model). Generalizations to other models are also discussed. The existence of the Pauli principle is due to additional transcendental equations which appear if the momenta of the particles become equal.

2. Consider the one-dimensional quantum NS model in a periodic box. The Hamiltonian and the commutation relations of the model read:

$$H = \int_{-L}^{L} \mathrm{d}x(\partial_x \psi^+ \partial_x \psi + \kappa \psi^+ \psi^+ \psi \psi);$$

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$$[\psi(x), \psi^{+}(y)] = \delta(x-y) \quad (-L \leqslant x, y \leqslant L). \tag{1}$$

This model was solved in [3]; it has interesting physical applications [4]. We consider the case $\kappa > 0$ where the model describes the repulsive bose-gas. Due to its relative simplicity, the model allows one to demonstrate the essential structure of QISM in the most explicit way [2, 5, 6]. The construction of eigenfunctions of the Hamiltonian (1) is reduced in QISM to constructing eigenfunctions Φ of the trace of the monodromy matrix $T(\lambda)$ [2, 6]:

$$T(\lambda) = \begin{pmatrix} A(\lambda); & B(\lambda) \\ C(\lambda); & D(\lambda) \end{pmatrix}; \qquad \operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda). \tag{2}$$

The commutation relations between the monodromy matrix elements essential for us are [2]:

$$\begin{aligned} A(\lambda)B(\mu) &= \alpha(\lambda, \mu)B(\mu)A(\lambda) - \beta(\lambda, \mu)B(\lambda)A(\mu); \\ D(\lambda)B(\mu) &= \alpha(\mu, \lambda)B(\mu)D(\lambda) + \beta(\lambda, \mu)B(\lambda)D(\mu); \\ B(\lambda)B(\mu) &= B(\mu)B(\lambda), \;\; \beta(\lambda, \mu) = \alpha(\lambda, \mu) - 1 = i\kappa(\lambda - \mu)^{-1}. \end{aligned} \tag{3}$$

Eigenfunctions Φ are constructed from the pseudovacuum $|0\rangle$ $(\psi(x)|0\rangle \equiv 0)$:

$$C(\lambda)|0\rangle = 0; \qquad A(\lambda)|0\rangle = e^{-i\lambda L}|0\rangle; \qquad D(\lambda)|0\rangle = e^{i\lambda L}|0\rangle \tag{4}$$

by means of the algebraic Bethe Ansatz [2]:

$$\Phi = B(\mu_1)B(\mu_2) \dots B(\mu_n)|0\rangle$$

where μ_j's satisfy the system of transcendental equations [2, 4]. It has always been postulated that all the μ_j's are different in the expression for Φ. We give the proof of this fact below. Let two of the μ_j's in Φ be equal, i.e., consider the following state Φ:

$$\Phi = B^2(\mu_1) \prod_{j=2}^{n} B(\mu_j)|0\rangle. \tag{5}$$

We will prove that an additional equation should be added to the standard system of transcendental equations for $\mu_1, \mu_2, \dots, \mu_n$, the enriched system having no solutions. The following commutation relations, which is easily obtained from (3), will be essential for the proof:

$$\begin{aligned} A(\lambda)B^2(\mu) &= \alpha^2(\lambda, \mu)B^2(\mu)A(\lambda) + i\kappa\beta(\lambda, \mu)B(\lambda)B'(\mu)A(\mu) - \\ &\quad - \beta(\lambda, \mu)B(\lambda)B(\mu)[(1 + \alpha(\lambda, \mu))A(\mu) + i\kappa A'(\mu)] \\ D(\lambda)B^2(\mu) &= \alpha^2(\mu, \lambda)B^2(\mu)D(\lambda) + i\kappa\beta(\lambda, \mu)B(\lambda)B'(\mu)D(\mu) + \\ &\quad + \beta(\lambda, \mu)B(\lambda)B(\mu)[(1 + \alpha(\mu, \lambda))D(\mu) - i\kappa D'(\mu)]. \end{aligned} \tag{6}$$

Demand now that the function Φ (5) is an eigenfunction of $A(\lambda) + D(\lambda)$ at any value of λ. The result of the action of $A(\lambda)$ on Φ can be put into the following form:

$$
\begin{aligned}
&A(\lambda)B^2(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle \\
&= \Lambda B^2(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle + \\
&\quad + B(\lambda)B^2(\mu_1)\prod_{l=2}^{n}\Lambda_l^{(1)}\prod_{j\neq l, j=2}^{n}B(\mu_j)|0\rangle + \\
&\quad + \Lambda^{(2)}B(\lambda)B(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle + \\
&\quad + \Lambda^{(3)}B(\lambda)B'(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle.
\end{aligned}
\tag{7}
$$

Here Λ, $\Lambda^{(i)}$ are c-number functions. To obtain this result it is convenient to rewrite Φ (5) in the form $\Phi = (\Pi_{j=2}^{n}B(\mu_j))B^2(\mu_1)|0\rangle$ and to then use Equations (3) and (6). As the structure of the right-hand side of (7) is known, the coefficients Λ, $\Lambda^{(i)}$ are easily calculated in the way which is standard for the algebraic Bethe Ansatz [2]:

$$
\begin{aligned}
&\Lambda = e^{-i\lambda L}\,\alpha^2(\lambda,\mu_1)\prod_{j=2}^{n}\alpha(\lambda,\mu_j), \\
&\Lambda_l^{(1)} = -e^{-i\mu_l L}\beta(\lambda,\mu_l)\alpha^2(\mu_l,\mu_1)\prod_{j\neq l, j=2}^{n}\alpha(\mu_l,\mu_j), \\
&\Lambda^{(2)} = -e^{-i\mu_1 L}\beta(\lambda,\mu_1)[1+\alpha(\lambda,\mu_1)]\prod_{j=2}^{n}\alpha(\mu_1,\mu_j) - \\
&\qquad -i\kappa\beta(\lambda,\mu_1)\frac{\partial}{\partial\mu_1}\left[e^{-i\mu_1 L}\prod_{j=2}^{n}\alpha(\mu_1,\mu_j)\right], \\
&\Lambda^{(3)} = i\kappa\, e^{-i\mu_1 L}\,\beta(\lambda,\mu_1)\prod_{j=2}^{n}\alpha(\mu_1,\mu_j).
\end{aligned}
\tag{8}
$$

One obtains in the same way:

$$
\begin{aligned}
&D(\lambda)B^2(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle \\
&= \widetilde{\Lambda}B^2(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle +
\end{aligned}
$$

$$
+ B(\lambda)B^2(\mu_1)\sum_{l=2}^{n}\widetilde{\Lambda}_l^{(1)}\prod_{j=2,j\neq l}^{n}B(\mu_j)|0\rangle + \tag{9}
$$

$$
+ \widetilde{\Lambda}^{(2)}B(\lambda)B(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle + \widetilde{\Lambda}^{(3)}B(\lambda)B'(\mu_1)\prod_{j=2}^{n}B(\mu_j)|0\rangle;
$$

where

$$
\widetilde{\Lambda} = e^{i\lambda L}\,\alpha^2(\mu_1,\lambda)\prod_{j=2}^{n}\alpha(\mu_j,\lambda),
$$

$$
\widetilde{\Lambda}_l^{(2)} = e^{i\mu_l L}\,\beta(\lambda,\mu_l)\alpha^2(\mu_1,\mu_l)\prod_{j=2,j\neq l}^{n}\alpha(\mu_j,\mu_l),
$$

$$
\widetilde{\Lambda}_l^{(2)} = e^{i\mu_1 L}\,\beta(\lambda,\mu_1)[1+\alpha(\mu_1,\lambda)]\prod_{j=2}^{n}\alpha(\mu_j,\mu_1) - \tag{10}
$$

$$
- i\kappa\beta(\lambda,\mu_1)\frac{\partial}{\partial\mu_1}\left[e^{i\mu_1 L}\prod_{j=2}^{n}\alpha(\mu_j,\mu_1)\right],
$$

$$
\widetilde{\Lambda}^{(3)} = i\kappa\; e^{i\mu_1 L}\,\beta(\lambda,\mu_1)\prod_{j=2}^{n}\alpha(\mu_j,\mu_1).
$$

One can see that a new 'unwanted term' containing $B'(\mu_1)$ appears in (7) and (9), but is not present in the case where all the μ_j's are different.

The requirement that Φ (5) is an eigenfunction of operator $A(\lambda) + D(\lambda)$ leads to the following equations:

$$
[A(\lambda) + D(\lambda)]\,\Phi = (\Lambda + \widetilde{\Lambda})\Phi, \tag{11}
$$

$$
\Lambda_l^{(1)} + \widetilde{\Lambda}_l^{(1)} = 0, \quad l = 2, \ldots, n, \tag{12}
$$

$$
\Lambda^{(2)} + \widetilde{\Lambda}^{(2)} = 0, \tag{13}
$$

$$
\Lambda^{(3)} + \widetilde{\Lambda}^{(3)} = 0. \tag{14}
$$

The system of Equations (12)–(14) is the system of transcendental equations for the case considered; Equation (13) being an additional one.

Let us show that the system (12)–(14) has no solutions. Indeed, it follows from (13) and (14) that $\Lambda^{(2)}(\Lambda^{(3)})^{-1} = \widetilde{\Lambda}^{(2)}(\widetilde{\Lambda}^{(3)})^{-1}$, which is easily reduced to the following form:

$$
2 + \kappa L + \kappa^2\sum_{k=2}^{n}[(\mu_1 - \mu_k)^2 + \kappa^2]^{-1} = 0. \tag{15}
$$

This last equation has no solutions for $\kappa > 0$. So we have proved the statement.

Notice that generalization of the case of more than two equal μ's is trivial. Indeed, when $A(\lambda)$ is acting on the state $B^m(\mu_1)\Pi_{j=2}^n B(\mu_j)|0\rangle$ then states containing

$$[\mathrm{d}^k B(\mu)/\mathrm{d}\mu^k]_{\mu=\mu_1} \prod_{j=2}^{n} B(\mu_j)|0\rangle \quad (k = 1, 2, ..., m)$$

arise. This results in $(m - 1)$ additional equations. It is not difficult to show that the system thus obtained also has no solution.

The proof given above can be literally applied to the lattice NS model [7]. One has only to change

$$\mathrm{e}^{\mp i\mu L} \to \left(1 \mp \frac{i\mu\Delta}{2}\right)^N; \qquad \mathrm{e}^{\mp i\lambda L} \to \left(1 \mp \frac{i\lambda\Delta}{2}\right)^N$$

in (4), (8) and (10), Equation (15) being transformed into

$$2 + 2\kappa N\Delta(4 + \mu^2\Delta^2)^{-1} + \kappa^2 \sum_{j=2}^{n} [(\mu - \mu_j)^2 + \kappa^2]^{-1} = 0 \tag{16}$$

which also has no solutions at $\kappa > 0$. N is the number of sites and Δ is the lattice constant.

3. Let us now examine the result obtained in the previous section from another point of view. Eigenvalues of $A(\lambda) + D(\lambda)$ in the lattice NS model are the following [7]:

$$\left(1 - \frac{i\lambda\Delta}{2}\right)^N \prod_{j=1}^{n} \left(\frac{\lambda - \mu_i + i\kappa}{\lambda - \mu_j}\right) + \left(1 + \frac{i\lambda\Delta}{2}\right)^N \prod_{j=1}^{n} \left(\frac{\lambda - \mu_j - i\kappa}{\lambda - \mu_j}\right). \tag{17}$$

The trace $(A(\lambda) + D(\lambda))$ of the monodromy matrix should be a polynomial in λ. There are, however, poles in λ in (17). One obtains a complete system of transcendental equations requiring that all the residues at these poles vanish (this is usually called Manakov's principle). We may obtain the Pauli principle in this way. Suppose that the value $\mu = \mu_1$ is present twice in (17). Then at $\lambda = \mu_1$ there is a pole of second order as well as a pole of first order. So the requirement that there is no singularity at $\lambda = \mu_1$ in (17) gives now two equations which can be easily seen to have exactly the form of (13) and (14). This leads to the Pauli principle. If the value $\mu = \mu_1$ is present m times at (17), then the order of the corresponding pole is equal to m. The requirement that the singularity at $\lambda = \mu_1$ should be absent leads again to $(m - 1)$ complementary equations.

So we have proved the Pauli principle for the repulsive NS-model. Note that the Pauli principle is necessary to construct the physical vacuum. Were the Pauli principle absent, all the vacuum pseudoparticles would have the same momenta corresponding to the lowest energy of the pseudoparticle. This degenerate vacuum would be physically unacceptable. Let us discuss the Pauli principle for other exactly solvable models (e.g., for the sine-Gordon model). It is evident from the proof that additional equations, analogous to (15), will arise in all these models. If the momenta of m particles are equal $(m - 1)$ additional equations will arise. So the number of complementary equations increases with the number of particles having identical momenta. This leads to the impossibility of the degenerate vacuum. Notice finally that equations like (15) proved to be useful in computing the norms of the Bethe wave function [8].

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