Conformal dimensions in Bethe ansatz solvable models

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Abstract. Models solvable by the hierarchy of Bethe ansätze (i.e. by the multicomponent Bethe ansatz) are considered. The spectrum of conformal dimensions which determines the long-distance asymptotics of correlations is calculated. This asymptotics in a general case is described by the direct sum of conformal theories, each possessing a central charge equal to one.

Phase transition in quantum models with one space dimension takes place at zero temperature. If there is no gap in the energy spectrum then correlators at zero temperature decay as some power of the distance, this power being called the critical exponent. Conformal quantum field theory (see, for example, [1]) is very useful for a description of the critical behaviour. The spectrum of conformal dimensions in the conformal quantum field theory describes the spectrum of critical exponents. It was shown in [2-4] that conformal dimensions can be expressed in terms of finite-size corrections to the energy spectrum of a model.

Much attention has been paid lately to completely integrable models. Great progress in their investigation and classification is due to the quantum inverse scattering method [5]. Completely integrable models associated with the simplest (4×4)-dimensional rational or trigonometric R matrices describe one type of wave and can be solved by the ordinary (one-component) Bethe ansatz [5, 6]. The non-relativistic Bose gas and the Heisenberg magnetic chain of spins $\frac{1}{2}$ are the examples of such models. Critical exponents for such models within the frame of the quantum inverse scattering method were calculated (see [7] and references therein).

The finite-size correction approach to calculating the spectrum of conformal dimensions in such models was given in [8, 9], permitting us also to obtain oscillating terms in the long-distance asymptotics.

The central charge Virasoro algebra in these models is equal to one. So the critical exponents can depend on a continuous parameter. For integrable models the Bethe ansatz permits us to calculate this dependence explicitly, e.g. the dependence on the external magnetic field in the case of the Heisenberg magnet [7-9].

The aim of this paper is to generalise the results mentioned above for the models solvable by the multicomponent Bethe ansatz which are connected with R matrices (solutions of the Yang-Baxter equation) of higher dimensions. The main result given below is the formula for conformal dimensions (and hence for critical exponents) in such models. This formula expresses critical exponents in terms of the values of the

'dressed charge' matrix $Z_{\alpha\beta}$. It is essential that the equation defining the matrix $Z_{\alpha\beta}$ depends only on the R matrix of the model. The formula for critical exponents is, in this sense, universal. It should be mentioned that, in a particular case of some magnets with higher symmetries in zero magnetic fields, the spectrum of conformal dimensions has already been obtained [10-12].

R matrices of the models solvable by the multicomponent Bethe ansatz can be classified with respect to representations of Lie algebras [13-19]. We will consider corresponding integrable models in 'external magnetic fields' h_α, taking the Hamiltonians in the form

$$H = H_0 + \sum_{\alpha=1}^{M} h_\alpha Q_\alpha \tag{1}$$

where Q_α are conserved charges: $[Q_\alpha, H_0] = 0$. Let us give some examples of such integrable systems. First of all, there are magnets with higher symmetries corresponding to algebras su(n), so(n) and sp($2n$). The Hamiltonian H_0 and charges Q_α for them are

$$H_0 = \sum_{l=1}^{L} h_{l,l+1} \qquad Q_\alpha = \sum_{l=1}^{L} q_l^{(\alpha)} \tag{2}$$

where L is the number of sites of a one-dimensional space lattice.

For the su(n) magnet one has [16, 18]

$$h_{l,l+1} = \sum_{i,j=1}^{n} e_{ij}^l e_{ji}^{l+1} \tag{3}$$

$$q_l^{(\alpha)} = e_{\alpha\alpha}^l - e_{\alpha+1,\alpha+1}^l \qquad (\alpha = 1, \ldots, n-1; \quad M = n-1).$$

For the so(n) magnet one has to distinguish between $n = 2k+1$ and $n = 2k$. For the so($2k+1$) magnet [19]:

$$h_{l,l+1} = \sum_{i,j=1}^{2k+1} (e_{ij}^l e_{ji}^{l+1} - 2/(2k-1)e_{ij}^l e_{i'j'}^{l+1}) \qquad (i' = 2k+2-i, j' = 2k+2-j)$$

$$q_l^{(\alpha)} = e_{\alpha\alpha}^l - e_{\alpha+1,\alpha+1}^l - e_{\alpha',\alpha'}^l + e_{\alpha'-1,\alpha'-1}^l \qquad (\alpha = 1, \ldots, k-1; \alpha' = 2k+2-\alpha) \tag{4}$$

$$q_l^{(k)} = 2(e_{kk}^l - e_{k+2,k+2}^l) \qquad M = k.$$

For the so($2k$) magnet [19]:

$$h_{l,l+1} = \sum_{i,j=1}^{2k} (e_{ij}^l e_{ji}^{l+1} - 1/(k-1)e_{ij}^l e_{i'j'}^{l+1}) \qquad (i' = 2k+1-i; j' = 2k+1-j)$$

$$q_l^{(\alpha)} = e_{\alpha\alpha}^l - e_{\alpha+1,\alpha+1}^l - e_{\alpha',\alpha'}^l + e_{\alpha'-1,\alpha'-1}^l \qquad (\alpha = 1, \ldots, k-1; \alpha' = 2k+1-\alpha) \tag{5}$$

$$q_l^{(k)} = e_{k-1,k-1}^l - e_{k+2,k+2}^l + e_{k,k}^l - e_{k+1,k+1}^l \qquad M = k.$$

For the sp($2n$) magnet [19]:

$$h_{l,l+1} = \sum_{i,j=1}^{2n} (e_{ij}^l e_{ij}^{l+1} + 1/(n+1)\varepsilon_i\varepsilon_j e_{ij}^l e_{i'j'}^{l+1}) \qquad (i' = 2n+1-i; j' = 2n+1-j)$$

$$q_l^{(\alpha)} = e_{\alpha\alpha}^l - e_{\alpha+1,\alpha+1}^l - e_{\alpha',\alpha'}^l + e_{\alpha'-1,\alpha'-1}^l \qquad (\alpha = 1, \ldots, n-1; \alpha' = 2n+1-\alpha) \tag{6}$$

$$q_l^{(n)} = e_{nn}^l - e_{n+1,n+1}^l \qquad M = n$$

$$\varepsilon_j = 1 \qquad (j = 1, \ldots, n)$$

$$\varepsilon_j = -1 \qquad (j = n+1, \ldots, 2n).$$

Here e_{jk}^l are matrices $(e_{jk})_{ab} \equiv \delta_{ja}\delta_{kb}$ acting in the lth site of the lattice (local quantum operators).

The model of a different kind which can be also solved by the multicomponent Bethe ansatz is the multicomponent non-linear Schrödinger equation. The Hamiltonian H_0 in (1) and the charges Q_α in the simplest case are (see, e.g., [19])

$$H_0 = \int_0^L dx \left(-\sum_{\alpha=1}^M \partial_x \psi_\alpha^+ \partial_x \psi_\alpha - \varkappa \sum_{\alpha,\beta=1}^M \psi_\alpha^+ \psi_\beta^+ \psi_\alpha \psi_\beta \right)$$

$$Q_\alpha = \int_0^L dx\, \psi_\alpha^+ \psi_\alpha \tag{7}$$

where L is the length of a box and ψ, ψ^+ are canonical Bose fields $([\psi_\alpha(x), \psi_\beta^+(y)] = \delta_{\alpha\beta}\delta(x-y))$. The symmetry group of the R matrix for this model is $SU(M+1)$.

For the given model of this kind there are M (rank of the group) bare momenta $p_\alpha^{(0)}(\lambda)$ and M bare energies $\varepsilon_\alpha^{(0)}(\lambda)$ ($\alpha = 1, \ldots, M$; λ is a complex spectral parameter).

The spectrum of the models solved by the multicomponent Bethe ansatz is defined from the system of Bethe equations [16, 19, 20]:

$$Lp_\alpha^{(0)}(\lambda_j^\alpha) = 2\pi n_j^\alpha - \sum_{\beta=1}^M \sum_{k=1}^{N_\beta} \Phi_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) \qquad (\alpha = 1, \ldots, M; j = 1, \ldots, N_\alpha). \tag{8}$$

Here L is the length of the box. Integer numbers N_β $(\beta = 1, \ldots, M)$ are eigenvalues of conserved quantities and can be considered as numbers of interacting particles of different kinds or 'spins'. Scattering phases $\Phi_{\alpha\beta}$ are defined only by the R matrix of the model and possess the antisymmetry property, i.e.

$$\Phi_{\alpha\beta}(\lambda) = -\Phi_{\beta\alpha}(-\lambda). \tag{9}$$

Their explicit form is not essential in what follows and can be found, e.g., in [21]. The numbers n_j^α are integers if N_α is odd and half-integers if N_α is even. The momentum P of the state with given numbers n_j^α is equal to

$$P = \sum_{\alpha=1}^M \sum_{j=1}^{N_\alpha} p_\alpha^{(0)}(\lambda_j^\alpha) = \frac{2\pi}{L} \sum_{\alpha=1}^M \sum_{j=1}^{N_\alpha} n_j^\alpha \tag{10}$$

and the energy E is

$$E = \sum_{\alpha=1}^M \sum_{j=1}^{N_\alpha} \varepsilon_\alpha^{(0)}(\lambda_j^\alpha). \tag{11}$$

Considerations similar to those in the one-component case [22] show that for a given R matrix the functions $p_\alpha^{(0)}(\lambda)$ and $\varepsilon_\alpha^{(0)}(\lambda)$ can be arbitrary (except that $\varepsilon_\alpha^{(0)}(\lambda) = \varepsilon_\alpha^{(0)}(-\lambda)$; $p_\alpha^{(0)}(\lambda) = p_\alpha^{(0)}(-\lambda)$). In other words, there exist completely integrable models with any given bare momenta $p_\alpha^{(0)}(\lambda)$ and bare energies $\varepsilon_\alpha^{(0)}(\lambda)$.

The explicit form of functions $\varepsilon_\alpha^{(0)}(\lambda)$ and $p_\alpha^{(0)}(\lambda)$ for the models with Hamiltonians (1)-(7) can be found in [19].

In the thermodynamical limit ($L \to \infty$, densities of different kinds of particles $D_\alpha = N_\alpha/L$ remain fixed) values λ_j^α in the ground state are distributed on the segment $[-q_\alpha, q_\alpha]$ with spectral distribution functions $\rho_\alpha(\lambda)$ satisfying

$$\rho_\alpha(\lambda_\alpha) - \frac{1}{2\pi} \sum_{\beta=1}^M \int_{-q_\beta}^{q_\beta} K_{\alpha\beta}(\lambda_\alpha - \lambda_\beta)\rho_\beta(\lambda_\beta)\, d\lambda_\beta = (1/2\pi)p_\alpha^{(0)'}(\lambda_\alpha) \equiv \rho_\alpha^{(0)}(\lambda_\alpha). \tag{12}$$

Here the kernels K are defined as

$$K_{\alpha\beta}(\lambda_\alpha - \lambda_\beta) = \frac{\partial}{\partial\lambda_\alpha} \Phi(\lambda_\alpha - \lambda_\beta) = K_{\beta\alpha}(\lambda_\beta - \lambda_\alpha). \tag{13}$$

The densities D_α in the ground state are

$$D_\alpha = \int_{-q_\alpha}^{q_\alpha} \rho_\alpha(\lambda)\,d\lambda_\alpha. \tag{14}$$

The physical (dressed) energies $\varepsilon_\alpha(\lambda)$ of one-particle excitations over the ground state are to be calculated from

$$\varepsilon_\alpha(\lambda_\alpha) - \frac{1}{2\pi} \sum_{\beta=1}^{M} \int_{-q_\beta}^{q_\beta} K_{\alpha\beta}(\lambda_\alpha - \lambda_\beta)\varepsilon_\beta(\lambda_\beta)\,d\lambda_\beta$$

$$= \varepsilon_\alpha^{(0)}(\lambda_\alpha) \qquad (\alpha = 1, \ldots, M) \tag{15}$$

$$\varepsilon_\alpha(\pm q_\alpha) = 0. \tag{16}$$

Defining Fermi velocities v_α as derivatives of physical energies with respect to physical momenta at the boundary of the Fermi zone, one has M different Fermi velocities:

$$v_\alpha = \varepsilon_\alpha'(\lambda)/(2\pi\rho_\alpha(\lambda))|_{\lambda=q_\alpha}. \tag{17}$$

Another important quantity is the 'dressed charge' matrix Z which is defined as the solution of the following equation:

$$Z_{\alpha\beta}(\lambda_\beta) - \frac{1}{2\pi} \sum_{\gamma=1}^{M} \int_{-q_\gamma}^{q_\gamma} Z_{\alpha\gamma}(\lambda_\gamma)K_{\gamma\beta}(\lambda_\gamma - \lambda_\beta)\,d\lambda_\gamma = \delta_{\alpha\beta} \qquad (\alpha, \beta = 1, \ldots, M). \tag{18}$$

Let us now make the following important remark. In integrable models there exists an infinite number of conservation laws. Hence one can construct a family of commuting Hamiltonians with the same ground state but with different Fermi velocities v_α. That is the reason why one can consider v_α as independent variables. However, the matrix $Z_{\alpha\beta}(\lambda)$ will be the same for all these Hamiltonians.

To calculate the central charge of the conformal algebra and the spectrum of conformal dimensions one calculates finite-size corrections. The results are as follows. The ground state energy E_g is

$$E_g = (L/2\pi) \sum_{\alpha=1}^{M} \int_{-q_\alpha}^{q_\alpha} \varepsilon_\alpha(\lambda)p_\alpha^{(0)'}(\lambda)\,d\lambda - (\pi/6L) \sum_{\alpha=1}^{M} v_\alpha + o(1/L). \tag{19}$$

The energy and the momentum of the excited state, are

$$\delta E = E - E_g = \frac{2\pi}{L} \sum_{\alpha=1}^{M} v_\alpha\{\tfrac{1}{4}(\hat{Z}^{-1}d)_\alpha^2 + (\hat{Z}^{T}l)_\alpha + I_\alpha^+ + I_\alpha^-\} \tag{20}$$

$$\delta P = P - P_g = 2\pi \sum_{\alpha=1}^{M} l_\alpha D_\alpha + \frac{2\pi}{L} \sum_{\alpha=1}^{M} (I_\alpha^+ - I_\alpha^- + d_\alpha l_\alpha). \tag{21}$$

Here $d = \{d_\alpha\} = \{d_1, \ldots, d_M\}$ and $l = \{l_\alpha\} = \{l_1, \ldots, l_M\}$ are M-dimensional vectors with integer components. Numbers $I_\alpha^\pm$ are also non-negative integers. The number d_α gives the change of the 'number of particles', N_α, in the excited state with respect to the ground state: $d_\alpha = N_\alpha - D_\alpha L$. The number l_α is the total number of transitions of bare

particles of the kind α from the vicinity of momentum $(-k_F^\alpha)$ to the vicinity of momentum $+k_F^\alpha$ (each transition of this kind gives to l_α a contribution equal to $+1$) and transitions of the kind $+k_F^\alpha \to (-k_F^\alpha)$ give a contribution equal to -1). Non-negative integers $I_\alpha^\pm$ describe excitations with the momenta in the vicinity of $\pm k_F^\alpha$, respectively. (Fermi momenta k_F^α are defined as $k_F^\alpha = \pi D_\alpha$.)

The matrix $\hat{Z}$ in (20) is defined as follows:

$$(\hat{Z}) \equiv Z_{\alpha\beta}(q_\beta) = Z_{\alpha\beta}(-q_\beta). \tag{22}$$

Due to the arbitrariness of Fermi velocities v_α it follows from equation (19)-(21) that one has the sum of M conformal algebras, each of them possessing the central charge equal to 1. It means that the effective infrared Hamiltonian and momentum are given as a direct um:

$$H_{\text{eff}} = \bigoplus_{\alpha=1}^{M} v_\alpha \mathcal{H}_\alpha \qquad P_{\text{eff}} = \bigoplus_{\alpha=1}^{M} \mathcal{P}_\alpha$$

Equations (20) and (21) permit us to calculate the spectrum of conformal dimensions $\Delta_\alpha^\pm$. Let us consider the local operator $\mathcal{O}_{(d)}(x, t)$ (x is a coordinate, $0 \le x \le L$, and t is the Euclidean time) which changes the number of particles of the kind α by d_α. Such an operator can be introduced in the model with arbitrary functions $p_\alpha^{(0)}$, $\varepsilon_\alpha^{(0)}$ (see, for example, [23]). From the general considerations of conformal field theory [1-4, 8, 9] the asymptotics of the correlator

$$\langle \mathcal{O}_{(-d)}(x, t) \mathcal{O}_{(d)}(0, 0) \rangle \tag{23}$$

contains a sum of the terms of the form

$$\exp\left(-2\pi i x \sum_{\alpha=1}^{M} m_\alpha D_\alpha\right) \prod_{\pm} \prod_{\alpha=1}^{M} (\pm i x + v_\alpha t)^{-2\Delta_\alpha^\pm} \tag{24}$$

where m_α are integers. In the box of finite length L it should be replaced by

$$\exp\left(-2\pi i x \sum_{\alpha=1}^{M} m_\alpha D_\alpha\right) \prod_{\pm} \prod_{\alpha=1}^{M} \left(\frac{\pi/L}{\sinh(\pi(v_\alpha t \pm ix)/L)}\right)^{2\Delta_\alpha^\pm}. \tag{25}$$

The contribution of the state (20) and (21) to the correlator (23) is proportional to

$$\exp\{-t\delta E - ix\delta P\}. \tag{26}$$

Comparing (26) and (25) one has $l_\alpha = m_\alpha$, and using equations (20) and (21) one obtains the spectrum of conformal dimensions:

$$2\Delta_\alpha^\pm(d, l, I^\pm) = 2I_\alpha^\pm + \tfrac{1}{4}(\hat{Z}^{-1}d)_\alpha^2 + (\hat{Z}^T l)_\alpha^2 \pm d_\alpha l_\alpha. \tag{27}$$

Here the arbitrariness of Fermi velocities v_α was again essentially used, as well as the fact that for a local operator quantities $(\Delta_\alpha^+ - \Delta_\alpha^-)$ are integers independent of coupling constant. Equation (27) shows that the sector with fixed l and d corresponds to a conformal modulus [1].

The leading term in the asymptotics for the correlator (23) can be now written down as

$$\langle \mathcal{O}_{(-d)}(x, t) \mathcal{O}_{(d)}(0, 0) \rangle \sim \prod_{\pm} \prod_{\alpha=1}^{M} (\pm i x + v_\alpha t)^{-2\Delta_\alpha^\pm(d,0,0)}. \tag{28}$$

Non-zero l and $I^\pm$ give the corrections to the asymptotics, non-zero l resulting in the appearance of oscillating terms [7, 9]. It is to be noted that, for concrete models, in

general, there can also exist logarithmic corrections to the power terms (16) [24, 25]. However, in a general case the first several terms in the asymptotics have no such corrections.

From our point of view, one of the important results of this paper is that critical exponents are defined completely by the R matrix of the model and by the structure of the ground state. Hence, models possessing the same R matrix and the same ground state but different dispersion laws $\varepsilon^{(0)}(\lambda)$ and $p^{(0)}(\lambda)$ are unified in the same universality class. This interesting fact is confirmed by independent calculations based on the quantum inverse scattering method [7].

We have considered the case of non-zero magnetic fields in a general position. Models of magnets corresponding to vector representation with zero magnetic fields for simply laced algebras were considered in [10-12]. It is not difficult to see that our results agree with the results obtained in those papers. Indeed, at $h_\alpha \to 0$ ($\alpha = 1, \ldots, M$) all the Fermi velocities v_α become equal and M non-interacting conformal theories with central charges $c = 1$ are unified with the theory with $c = M$. The limiting spectrum of dimensions is obtained from equation (27) at $h_\alpha \to 0$ ($q_\alpha \to \infty$) after summing up over α from $\alpha = 1$ to $\alpha = M$. It is interesting to note that the limit $h_\alpha \to 0$ for the higher-spin magnets is not regular (the ground state for these magnets is filled with n-strings with $n > 1$). For example, taking the limit $h \to 0$ in the integrable XXX spin-s magnet one obtains the theory with $c = 1$.

On the other hand, calculating the central charge from finite-size corrections at $h = 0$ one obtains $c = 2s/(s+1) > 1$ and the spectrum of dimensions of the Wess-Zumino model. It means that, at the limit $h \to 0$, only a part of the conformal field theory describing the long-distance asymptotics of the correlators at $h = 0$ is restored.

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