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LATTICE VERSIONS OF QUANTUM FIELD THEORY MODELS IN TWO DIMENSIONS

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The quantum inverse scattering method allows one to put quantum field theory models on a lattice in a way which preserves the dynamical structure. The trace identities are discussed for these models.

1. Introduction

Lattice versions (both classical and quantum) of completely integrable field models are constructed in this paper. We consider the non-linear Schrödinger equation (the NS model) and the sine-Gordon model (the SG model) in the formalism of the quantum inverse scattering method (QISM) [1]. The NS model can be quantized directly in the continuous case [2] but the situation is different for the SG model. The reason is that one has to solve the problem of ultraviolet divergences for relativistic quantum models. Due to these divergences the classical expression for the hamiltonian is not valid in the quantum case and the hamiltonian requires a more precise definition. Normally such a definition is given by renormalization by means of momentum cutoff Λ of the perturbation series. The hamiltonian is then defined by adding counter terms which diverge at $\Lambda \to \infty$.

Perturbation series are asymptotic and make sense for small values of the coupling constant. To define a theory at large coupling constant is a non-trivial problem. General principles of quantum field theory impose the following restrictions on this definition: (i) The hamiltonian should be positive with respect to physical vacuum. It should be possible to treat the spectrum in terms of particles. (ii) The S-matrix should be unitary and analytic. (iii) The answers one obtains should reproduce the perturbation series for small coupling constants. (iv) The theory should possess the correct quasiclassical limit. (v) The essential symmetries of the classical theory should survive after quantization. The natural way to extend the quantum theory to large coupling constants is to put it onto the lattice, the lattice spacing Δ playing a role of an ultraviolet cutoff ($\Delta \sim \Lambda^{-1}$).

Recently much attention has been given to the quantization of completely integrable field models in two space-time dimensions. It is necessary to preserve

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complete integrability in putting such a model onto the lattice because the natures of integrable and non-integrable models differ essentially. The common approach to quantization of completely integrable models is given by QISM [1]. The complete integrability in this method means the existence of an R-matrix which gives direct information on the structure of the action-angle variables of the model. It is natural to require that the structure of the action-angle variables should not be changed by lattice regularization. So we demand that the R-matrices of the initial model and of its lattice version be the same. The important thing for solving the model by QISM is to express the hamiltonian by means of "trace identities" in terms of the transfer matrix of the model. If it is done QISM allows one to calculate the spectrum of the hamiltonian by applying the algebraic Bethe ansatz.

In sect. 2 the integrable lattice versions of the classical NS and SG models are considered which possess the same classical r-matrices [2–4] as the corresponding continuous models. We formulate the method for obtaining local conservation laws by means of trace identities and define the corresponding classical hamiltonians using this method. Though their dependence on local lattice variables is rather complicated, these hamiltonians become the hamiltonians of the NS and SG models in the continuous ($\Delta \to 0$) limit. One should not, however, quantize this hamiltonian because the quantum model thus obtained is not integrable. To construct the integrable quantum models one must at first restore the quantum R-matrix from the classical r-matrix by means of the Yang–Baxter relations (see e.g. [4]). Then the knowledge of the R-matrix permits one to define the hamiltonian of the lattice quantum integrable model using quantum trace identities [5, 6]. This hamiltonian is in general quasilocal, i.e. all sites of the lattice interact but the interaction strength decreases exponentially with the distance between them. At the quasiclassical limit the hamiltonian turns into the local classical lattice hamiltonian. It appears that this approach allows one to satisfy all the requirements for regularization of quantum field theory models. In sect. 3 we construct the lattice quantum NS model and discuss its connection with the generalized XXX model. We show that for the lattice NS model the quantum hamiltonian can be made local. In sect. 4 the lattice quantum sine-Gordon model is considered, the monodromy matrix being expressed by elementary functions. The connection with the (spin $\frac{1}{2}$)$\otimes Z_N$ models is discussed. In this paper we use essentially the notations of papers [1, 4, 7].

2. Classical lattice models and trace identities

(a) The usual NS model is defined by the hamiltonian

$$H = \int (\partial_x \psi^* \partial_x \psi + \kappa \psi^* \psi^* \psi \psi)\, \mathrm{d}x \tag{1}$$

and by basic Poisson brackets $\{\psi(x), \psi^*(y)\} = i\delta(x-y)$ (we suppose $-L \leq x \leq L$). Our aim now is to construct the lattice version of this model [6] preserving the integrability (the LNS model). Consider the one-dimensional lattice with N sites and a lattice constant Δ ($N\Delta = 2L$). The elementary monodromy matrix $L(n|\lambda)$ at the nth site of the lattice (corresponding shift is equal to Δ) is defined as follows:

$$L(n|\lambda) = \begin{vmatrix} 1 - \frac{1}{2}i\lambda\Delta + \frac{1}{2}\kappa\chi_n^*\chi_n & -i\sqrt{\kappa}\chi_n^*\rho_n \\ i\sqrt{\kappa}\rho_n\chi_n & 1 + \frac{1}{2}i\lambda\Delta + \frac{1}{2}\kappa\chi_n^*\chi_n \end{vmatrix} . \tag{2}$$

Here λ is a complex spectral parameter;

$$\rho_n = (1 + \tfrac{1}{4}\kappa\chi_n^*\chi_n)^{1/2} , \qquad \{\chi_m, \chi_n^*\} = i\delta_{mn}\Delta ,$$

so that $\chi_n = \psi(x)\Delta$ in the limit $\Delta \to 0$. Notice that $L(n|\lambda)$ becomes the usual infinitesimal monodromy matrix [1, 2] of the NS model. The r-matrix for $L(n|\lambda)$ (2) is the same as in the continuous case:

$$r(\lambda, \mu) = \kappa(\lambda - \mu)^{-1}\Pi , \tag{3}$$

$$\{L(n|\lambda) \underset{,}{\otimes} L(n|\mu)\} = [L(n|\lambda) \otimes L(n|\mu), r(\lambda, \mu)] . \tag{4}$$

Here Π is the transposition 4×4 matrix, i.e. $\Pi(A\otimes B)\Pi = B\otimes A$ for any 2×2 matrices A, B. The matrix $L(n|\lambda)$ has the following important properties (σ_i are usual Pauli matrices):

$$\sigma_1 L^*(n|\lambda^*)\sigma_1 = L(n|\lambda) , \tag{5}$$

$$\det L(n|\lambda) \equiv d_c(\lambda) = \tfrac{1}{4}\Delta^2(\lambda - \nu)(\lambda - \nu^*), \qquad \nu \equiv -2i/\Delta . \tag{6}$$

The monodromy matrix $T(\lambda)$ for the interval $[-L, L]$ is defined as usual:

$$T(\lambda) = L(N|\lambda) \cdots L(2|\lambda)L(1|\lambda) , \tag{7}$$

and has similar properties:

$$\sigma_1 T^*(\lambda^*)\sigma_1 = T(\lambda), \qquad \det T(\lambda) = d_c^N(\lambda) .$$

Our next step is to construct the local lattice hamiltonian which is integrable and has the correct continuous limit (1). As is known, the logarithmic derivatives of the trace of the monodromy matrix $\tau(\lambda) = \operatorname{tr} T(\lambda)$ [$\{\tau(\lambda), \tau(\mu)\} = 0$ due to (4)] are integrals of motion, the hamiltonian being among them. The local integrals of motion can be obtained by taking logarithmic derivatives of $\tau(\lambda)$ at points $\lambda = \nu$, $\lambda = \nu^*$, where $\det L(n|\lambda)$ (6) vanishes. Let us prove this fact. It follows from $\det L(n|\nu) = 0$, $\det L(n|\nu^*) = 0$ that $L(n|\lambda)$ is proportional to one-dimensional projectors at these points, i.e. it can be represented in the following form:

$$L_{ik}(n|\nu) = \alpha_i(n)\beta_k(n) . \tag{8}$$

Here $\alpha(n)$ and $\beta(n)$ are two-component vectors. Using this representation one can easily put $\tau(\nu)$ [as well as $\tau(\nu^*) = \tau^*(\nu)$, due to (5)] into the factorized form:

$$\tau(\nu) = \prod_{n=1}^{N} (\beta_{i_n}(n+1)\alpha_{i_n}(n)) , \qquad \beta(N+1) \equiv \beta(1) \tag{9}$$

(summing over the repeated index i_n is implied). So $\ln \tau(\nu)$ is local and corresponds to the interaction of the two nearest neighbours:

$$\ln \tau(\nu) = \sum_{n=1}^{N} \ln (\beta_{i_n}(n+1)\alpha_{i_n}(n)) .$$

Let us now calculate the first logarithmic derivative of $\tau(\nu)$:

$$\begin{aligned} \tau^{-1}(\nu)\tau'(\nu) &= \tau^{-1}(\nu) \sum_{k=1}^{N} \tau_k(\nu) , \\ \tau_k(\nu) &= \mathrm{tr}\,\{L(N|\nu) \cdots L'(k|\nu) \cdots L(1|\nu)\} . \end{aligned} \tag{10}$$

Consider first the contribution of $\tau_2(\nu)$, denoting $K \equiv L(N|\nu) \cdots L(4|\nu)$:

$$\tau_2(\nu) = \mathrm{tr}\,\{KL(3|\nu)L'(2|\nu)L(1|\nu)\} = (\beta_i(1)K_{ik}\alpha_k(3))(\beta_l(3)L'_{lm}(2|\nu)\alpha_m(1)) .$$

As $\tau(\nu)$ can be put into the similar form

$$\tau(\nu) = (\beta_i(1)K_{ik}\alpha_k(3))(\beta_l(3)L_{lm}(2|\nu)\alpha_m(1)) ,$$

one obtains

$$\begin{aligned} \tau^{-1}(\nu)\tau_2(\nu) &= (\beta_i(3)L_{ik}(2|\nu)\alpha_k(1))^{-1}(\beta_l(3)L'_{lm}(2|\nu)\alpha_m(1)) \\ &= \{\mathrm{tr}\,(L(3|\nu)L(2|\nu)L(1|\nu))\}^{-1}\,\mathrm{tr}\,(L(3|\nu)L'(2|\nu)L(1|\nu)) . \end{aligned}$$

Other $\tau_k(\nu)$ can be reduced to the form of $\tau_2(\nu)$ by the cyclic permutation of L's inside the trace in (10); one has finally

$$\tau^{-1}(\nu)\tau'(\nu) = \sum_{k=1}^{N} [\mathrm{tr}\,(L(k+1|\nu)L(k|\nu)L(k-1|\nu)]^{-1}\,\mathrm{tr}\,(L(k+1|\nu)L'(k|\nu)L(k-1|\nu) . \tag{11}$$

So $\tau^{-1}(\nu)\tau'(\nu)$ is local and describes the interaction of three nearest neighbours on the lattice. One can also check that higher logarithmic derivatives $\mathrm{d}^m(\ln \tau(\lambda))/\mathrm{d}\lambda^m|_{\lambda=\nu,\nu^*}$ are also local describing the interaction of $(m+2)$ nearest neighbours (the explicit form of these derivatives is given in [6]).

Now define the hamiltonian of the classical LNS model as follows:

$$H = \frac{i}{12\kappa}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^{-1}}\right)^3 \ln\left[\left(1+\frac{\lambda}{\nu}\right)^{-N}\tau(\lambda)\right]\Bigg|_{\lambda=\nu} + \mathrm{C.C.} \tag{12}$$

This hamiltonian is real and local [see (5)]. It corresponds to the interaction of five nearest neighbours. One can easily prove by direct calculation that as $\Delta \to 0$ it becomes the continuous hamiltonian (1). One may see the explicit form of (12) in the local lattice fields in [6].

(b) The SG model is defined by the hamiltonian

$$H = \int \left[\tfrac{1}{2}p^2 + \tfrac{1}{2}(\partial_x u)^2 + \frac{m^2}{\beta^2}(1-\cos \beta u)\right] \mathrm{d}x \tag{13}$$

and by Poisson brackets $\{p(x), u(y)\} = \delta(x-y)(-L \leq x, y \leq L)$. The corresponding lattice model (the LSG model) is constructed by means of the following elementary monodromy matrix $L(n|\lambda)$ [8]:

$$L(n|\lambda) = \begin{pmatrix} \pi_n^{-1/2}\varphi(u_n)\pi_n^{-1/2} & \frac{1}{4}m\Delta(\lambda\chi_n^- - \lambda^{-1}\chi_n^+) \\ \frac{1}{4}m\Delta(\lambda^{-1}\chi_n^- - \lambda\chi_n^+) & \pi_n^{+1/2}\varphi(u_n)\pi_n^{+1/2} \end{pmatrix}, \tag{14}$$

$$\begin{gathered} \chi_n^\pm = \exp\{\pm\tfrac{1}{2}i\beta u_n\}, \qquad \pi_n^\pm = \exp\{\pm\tfrac{1}{4}i\beta p_n\}, \\ \varphi(u) = (1+2S\cos\beta u)^{1/2}, \qquad S = (\tfrac{1}{4}m\Delta)^2. \end{gathered} \tag{15}$$

The Poisson brackets of local lattice fields p_n, u_n are $\{p_m, u_n\} = \delta_{mn}$. The phase space of the model is the direct product of N tori which become cylinders in the continuous limit ($u_n \simeq u(x)$; $p_n \simeq p(x)\Delta$). The matrix $L(n|\lambda)$ satisfies relation (4) with the SG classical r-matrix:

$$r(\lambda,\mu) = \frac{\gamma}{\operatorname{sh}\alpha}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \operatorname{ch}\alpha & -1 & 0 \\ 0 & -1 & \operatorname{ch}\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

$$\lambda/\mu \equiv \exp\alpha, \qquad \gamma = \tfrac{1}{8}\beta^2,$$

and has the following properties:

$$\sigma_2 L^*(n|\lambda^*)\sigma_2 = L(n|\lambda), \tag{17}$$

$$\det L(n|\lambda) \equiv d_c(\lambda) = 1 + (\lambda^2 + \lambda^{-2})S. \tag{18}$$

The local hamiltonian (three neighbours on the lattice interact) is constructed by means of trace identities in terms of $\tau(\lambda) = \operatorname{tr} T(\lambda)$:

$$H = \frac{m^2\Delta}{8\gamma}\sum_{+,-}\frac{\partial}{\partial\lambda^{\pm 2}}\ln[\tau(\lambda)\tau_0^{-1}(\lambda)]|_{\lambda^2=-b^{\pm1}}. \tag{19}$$

Here $\tau_0(\lambda)$ is a trace of the monodromy matrix at $p_n = u_n = 0$; $b = 2S(1+\sqrt{1-4S^2})^{-1}$. Notice that $d_c(\lambda^2 = -b^{\pm1}) = 0$ which ensures the locality of H. The hamiltonian is also real which can be easily proved by using (17). Its explicit form in terms of p_n, u_n is given in ref. [8]. So we have described the classical completely integrable lattice NS and SG models. Notice that these models can be solved by means of the inverse scattering method [9]. They have the same structure of the action angle variables as the corresponding continuous models.

3. The quantum lattice NS model

The elementary monodromy matrix $L(n|\lambda)$ in quantum case is defined by the same formula (2). Now χ_n and χ_n^* are, however, quantum operators with commutation relations $[\chi_m, \chi_n^*] = \delta_{mn}\Delta$. The star denotes the complex conjugation of c-numbers and the hermitian conjugation of quantum operators. The main property of $L(n|\lambda)$ is that it satisfies the relation

$$R(\lambda, \mu)L(n|\lambda)\otimes L(n|\mu) = L(n|\mu)\otimes L(n|\lambda)R(\lambda, \mu), \tag{20}$$

with the R-matrix [6] of the continuous NS model:

$$R(\lambda, \mu) = \Pi - i\kappa(\lambda - \mu)^{-1}E. \tag{21}$$

E and Π are identity and transposition 4×4 matrices. Eq. (20) results in $[\tau(\lambda), \tau(\mu)] = 0$ and ensures the complete integrability of the models whose hamiltonians are constructed by means of trace identities (notice that the trace $\tau(\lambda)$ of the monodromy matrix $T(\lambda)$ in the matrix space is called "transfer matrix" in the quantum case, $\tau(\lambda)$ being, of course, a scalar quantum operator). The quantum elementary monodromy matrix $L(n|\lambda)$ possesses properties which are similar to the classical ones. Eq. (5) is not changed; the quantum analogue of eq. (6) is

$$L(n|\lambda)\sigma_2 L^{\mathrm{T}}(n|\lambda + i\kappa)\sigma_2 = d_{\mathrm{q}}(\lambda), \tag{22}$$

$$d_{\mathrm{q}}(\lambda) = \tfrac{1}{4}\Delta^2(\lambda - \nu)(\lambda - \nu^* + i\kappa), \qquad \nu \equiv -2i/\Delta. \tag{23}$$

This last property is an example of the generalization of the notion of the determinant of monodromy matrix to the quantum case, see [6]. This generalization is crucial for obtaining the needed trace identities with local or at least quasilocal properties.

Consider now the trace identities in more detail. Taking eq. (22) at points $\lambda = \nu$; $\lambda = \nu^* - i\kappa$ where $d_{\mathrm{q}}(\lambda) = 0$, one easily obtains the following representations which are generalizations of (8):

$$L_{ik}(n|\nu) = \alpha_i(n)\beta_k(n), \qquad L_{ik}(n|\nu^* - i\kappa) = \tilde{\alpha}_i(n)\tilde{\beta}_k(n), \tag{24}$$

$$L_{ik}(n|\nu + i\kappa) = \delta_k(n)\gamma_i(n), \qquad L_{ik}(n|\nu^*) = \tilde{\delta}_k(n)\tilde{\gamma}_i(n), \qquad (i, k = 1, 2). \tag{25}$$

Here α, β, γ, δ are two-component vectors with operator (non-commuting) components. We call the representation (24) "the direct projector representation", similarly we call the representation (25) "the inverse projector representation". These representations are essentially different in the quantum case due to the non-commutativity of the vector components.

Now one has to define the hamiltonian of the LNS model by means of trace identities. One can define H by means of the logarithmic derivatives of the transfer matrix $\tau(\lambda)$ at points $\lambda = \nu$, $\lambda = \nu^*$ by a direct generalization of the classical trace identities (12) [6]. The model thus constructed turns into the usual quantum NS model in the limit $\Delta \to 0$. The hamiltonian of this lattice model is, however, only quasilocal. The reason for this non-locality is that $L(n|\lambda)$ can not be represented simultaneously in the forms (24), (25) at some point λ.

Here we will nevertheless construct a quantum local hamiltonian which also becomes (1) when $\Delta \to 0$. To do this let us modify the monodromy matrix, making $L(n|\lambda)$ different at even and odd sites of the lattice:

$$L(n|\lambda) = \begin{vmatrix} -\frac{1}{2}i\lambda\Delta + Z_n + \frac{1}{2}\kappa\chi_n^*\chi_n & i\sqrt{\kappa}\chi_n^*\rho_n \\ i\sqrt{\kappa}\rho_n\chi_n & \frac{1}{2}i\lambda\Delta + Z_n + \frac{1}{2}\kappa\chi_n^*\chi_n \end{vmatrix}, \tag{26}$$

$$\rho_n = \sqrt{Z_n + \tfrac{1}{4}\kappa\chi_n^*\chi_n}\,, \qquad Z_n = 1 + (-1)^n\tfrac{1}{4}\kappa\Delta\,.$$

It is convenient to rewrite the monodromy matrix $T(\lambda) = L(N|\lambda)\cdots L(1|\lambda)$ with L defined by (26) in terms of the "doubled" elementary monodromy matrix $l(n|\lambda)$ $(n = 1, \ldots, \frac{1}{2}N)$:

$$l(n|\lambda) = L(2n|\lambda)L(2n-1|\lambda)\,, \tag{27}$$

$$T(\lambda) = l(N/2|\lambda)\cdots l(2|\lambda)l(1|\lambda)\,. \tag{28}$$

The following properties of $l(n|\lambda)$ can be easily established:

$$R(\lambda, \mu)l(n|\lambda)\otimes l(n|\mu) = l(n|\mu)\otimes l(n|\lambda)R(\lambda, \mu)\,, \tag{29}$$

with the same R-matrix (21) [it is also true for $L(n|\lambda)$, (26)]. Then instead of (17), (22), (23) we have

$$\sigma_1 l^*(n|\lambda^*)\sigma_1 = l(n|\lambda)\,, \tag{30}$$

$$l(n|\lambda)\sigma_2 l^{\mathrm{T}}(n|\lambda + i\kappa)\sigma_2 = D_{\mathrm{q}}(\lambda)\,, \tag{31}$$

$$D_{\mathrm{q}}(\lambda) = (\tfrac{1}{2}\Delta)^4(\lambda - \nu_1)(\lambda - \nu_2)(\lambda - \nu_3)(\lambda - \nu_4)\,, \tag{32}$$

$$\nu_1 = -\frac{2i}{\Delta} + \frac{i\kappa}{2}\,, \qquad \nu_2 = \frac{2i}{\Delta} - \frac{i\kappa}{2} = \nu_1^*\,, \qquad \nu_3 = -\frac{2i}{\Delta} - \frac{i\kappa}{2}\,, \qquad \nu_4 = \frac{2i}{\Delta} - \frac{3i\kappa}{2}\,.$$

Taking eq. (31) at zeros of $D_{\mathrm{q}}(\lambda)$, one obtains that at $\lambda = \nu_K$ $(K = 1, \ldots, 4)$ the matrix $l(n|\lambda)$ can be represented as a direct projector (24); at points $\lambda = \nu_K + i\kappa$ $(K = 1, \ldots, 4)$ it is an inverse projector (25). It follows then that at points

$$\nu_1 = \nu_3 + i\kappa\,, \qquad \nu_2 = \nu_4 + i\kappa \;(= \nu_1^*)\,, \tag{33}$$

$l(n|\lambda)$ has simultaneously both representations:

$$l_{ik}(n|\nu_a) = \alpha_i^a(n)\beta_k^a(n) = \delta_k^a(n)\gamma_i^a(n)\,, \qquad (a = 1, 2)\,. \tag{34}$$

This is a crucial point for the locality of the logarithmic derivatives of the transfer matrix at $\lambda = \nu_a$. Let us explain the locality for the first logarithmic derivative at the point $\lambda = \nu_1 \equiv \nu$ of $\tau(\lambda) = \mathrm{tr}\ T(\lambda)$:

$$\tau^{-1}(\nu)\tau'(\nu) = \tau^{-1}(\nu)\sum_{k=1}^{N/2}\tau_k(\nu)\,,$$

$$\tau_k(\nu) = \mathrm{tr}\,[l(\tfrac{1}{2}N|\nu)\cdots l'(k|\nu)\cdots l(1|\nu)]\,. \tag{35}$$

Notice that matrices l can be cyclically transposed in the trace due to the commutativity of quantum operators at different sites ("ultralocality"). So it is enough to compute the contribution $\tau_2(\nu)$ which can be put into the form:

$$\tau_2(\nu)=\operatorname{tr}[Kl(3)l'(2)l(1)]=K_{ik}l_{kj}(3)l'_{jm}(2)l_{mi}(1)\,,\qquad K=l(\tfrac{1}{2}N)\cdots l(4)\,. \tag{36}$$

Making use of (34) one obtains

$$\tau_2(\nu)=(\delta_i(1)K_{ik}\alpha_k(3))(\beta_j(3)l'_{jm}(2)\gamma_m(1)) \tag{37}$$

($\delta_i(1)$ can be put to the left due to ultralocality). The transfer matrix itself can be put into a similar form,

$$\tau(\nu)=(\delta_i(1)K_{ik}\alpha_k(3))(\beta_j(3)l_{jm}(2)\gamma_m(1))\,,$$

and one has

$$\begin{aligned}\tau^{-1}(\nu)\tau_2(\nu)&=[\beta_j(3)l_{jm}(2)\gamma_m(1)]^{-1}[\beta_j(3)l'_{jm}(2)\gamma_m(1)]\\&=[\operatorname{tr} l(3|\nu)l(2|\nu)l(1|\nu)]^{-1}\operatorname{tr} l(3|\nu)l'(2|\nu)l(1|\nu)\,,\end{aligned}$$

and finally

$$\tau^{-1}(\nu)\tau'(\nu)=\sum_{k=1}^{N/2}[\operatorname{tr} l(k+1|\nu)l(k|\nu)l(k-1|\nu)]^{-1}\operatorname{tr} l(k+1|\nu)l'(k|\nu)l(k-1|\nu)\,. \tag{38}$$

So the first logarithmic derivative is indeed local. Expression (38) can be shown to describe non-polynomial interactions of four nearest neighbours. The proof of the locality for higher logarithmic derivatives is essentially the same as in the classical case.

We have shown that $\ln\tau(\lambda)$ at $\lambda\sim\nu_a$ $(a=1,2)$ is a generating functional of local conservation laws. The hamiltonian of the LNS model is defined as follows:

$$\begin{aligned}H=&\left[\frac{i}{12\kappa}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^{-1}}\right)^3+\tfrac{1}{6}i\kappa\frac{\mathrm{d}}{\mathrm{d}\lambda^{-1}}\right]\ln[(1-\tfrac{1}{4}\kappa\Delta+\tfrac{1}{2}i\lambda\Delta)^{-N/2}\\&\times(1+\tfrac{1}{4}\kappa\Delta+\tfrac{1}{2}i\lambda\Delta)^{-N/2}\tau(\lambda)]_{|\lambda=\nu_1}+\text{h.c.}\end{aligned} \tag{39}$$

This hamiltonian is hermitian and local [due to eq. (30)]. It describes the interaction of eight nearest neighbours on the lattice and becomes the hamiltonian (1) in the continuous limit. This limit is easy to obtain from the results [5] on the trace identities for the continuous NS model, due to the fact that as $\Delta\to 0$ the matrix $l(n|\lambda)$ is [up to $O(\Delta^2)$] a product of two usual infinitesimal operators $L(n|\lambda)$ [1] at adjacent sites.

The local LNS model can be solved now by QISM in quite a standard way. Applying the algebraic Bethe ansatz one finds the eigenvalues Λ of the transfer matrix $\tau(\lambda)$:

$$\Lambda = (1-\tfrac{1}{4}\kappa\Delta-\tfrac{1}{2}i\lambda\Delta)^{N/2}(1+\tfrac{1}{4}\kappa\Delta-\tfrac{1}{2}i\lambda\Delta)^{N/2}\prod_{k=1}^{m}\frac{\lambda-\lambda_k+i\kappa}{\lambda-\lambda_k}$$
$$+(1-\tfrac{1}{4}\kappa\Delta+\tfrac{1}{2}i\lambda\Delta)^{N/2}(1+\tfrac{1}{4}\kappa\Delta+\tfrac{1}{2}i\lambda\Delta)^{N/2}\prod_{k=1}^{m}\frac{\lambda-\lambda_k-i\kappa}{\lambda-\lambda_k}, \tag{40}$$

where the "momenta" λ_k satisfy the system of equations

$$\left[\frac{(1-\frac{1}{4}\kappa\Delta-\frac{1}{2}i\lambda_l\Delta)(1+\frac{1}{4}\kappa\Delta-\frac{1}{2}i\lambda_l\Delta)}{(1-\frac{1}{4}\kappa\Delta+\frac{1}{2}i\lambda_l\Delta)(1+\frac{1}{4}\kappa\Delta+\frac{1}{2}i\lambda_l\Delta)}\right]^{N/2}=\prod_{\substack{k\neq l\\k=1}}^{m}\left(\frac{\lambda_l-\lambda_k-i\kappa}{\lambda_l-\lambda_k+i\kappa}\right). \tag{41}$$

The eigenvalues of the hamiltonian can be easily obtained from (39), (40) as:

$$E=\sum_k h(\lambda_k),$$
$$h(\lambda_k)=\left[\frac{i}{12\kappa}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^{-1}}\right)^3+\tfrac{1}{6}i\kappa\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^{-1}}\right)\right]\ln\left(\frac{\lambda-\lambda_k-i\kappa}{\lambda-\lambda_k}\right)\bigg|_{\lambda=\nu_1}+\mathrm{C.C.}, \tag{42}$$
$$h(\lambda)\xrightarrow[\Delta\to 0]{}\lambda^2.$$

It follows from (42) that the spectrum of the hamiltonian is additive and can be treated in terms of "particles".

So we have constructed the lattice quantum NS model. It must be emphasized that our method of obtaining local trace identities based on considering the zeros of the "quantum determinant" is different from the usual one for the models of XYZ type where the dimension of the auxiliary matrix space is the same as that of the local "quantum" (spin) space at the nth site of the lattice [10].

To conclude this section we construct the generalized XXX model starting from the quantum matrix $L(n|\lambda)$ (2). It is not surprising that the LNS model is closely connected with the generalized XXX model. It is already known [11] that the classical NS model is gauge equivalent to the Heisenberg ferromagnet. We define the elementary monodromy matrix of the generalized XXX model as follows:

$$L^x(n|\lambda)=-2\Delta^{-1}\sigma_3 L(n|\lambda)\equiv i\lambda+\kappa t_i^{(n)}\sigma_i. \tag{43}$$

The matrix $L(n|\lambda)$ here is defined by (2); the operators $t_i^{(n)}$ ($i=1,2,3$; $n=1,2,\ldots,N$) are

$$t_1^{(n)}=i\kappa^{-1/2}\Delta^{-1}[\chi_n^*\rho_n+\rho_n\chi_n],$$
$$t_2^{(n)}=\kappa^{-1/2}\Delta^{-1}[\rho_n\chi_n-\chi_n^*\rho_n],\qquad \rho_n=(1+\tfrac{1}{4}\kappa\chi_n^*\chi_n)^{1/2}, \tag{44}$$
$$t_3^{(n)}=-2\kappa^{-1}\Delta^{-1}[1+\tfrac{1}{2}\kappa\chi_n^*\chi_n],\qquad [\chi_n,\chi_m^*]=\delta_{mn}\Delta.$$

The matrix $L^x(n|\lambda)$ satisfies eq. (20) with the same R-matrix (21) as $L(n|\lambda)$ (2). Notice that the transfer matrices of the generalized XXX model and of the LNS model are similar as quantum operators if the number of sites N is even. Operators $t_i^{(n)}$ generate a representation of the SU(2) algebra in the Fock space:

$$[t_i^{(n)}, t_k^{(n)}] = i\varepsilon_{ikl} t_l^{(n)} . \tag{45}$$

This representation is infinite dimensional and in general irreducible, the square of the total "spin" being equal to

$$t^2 = S(S+1) , \qquad S = -2/\kappa\Delta . \tag{46}$$

One can see from this formula that when $S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ the infinite-dimensional representation becomes reducible and the $(2S+1)$-dimensional irreducible representation separates. Generators $t_i^{(n)}$ acting in this representation may be then realized as $(2S+1)\times(2S+1)$ matrices and the monodromy matrix $L^x(n|\lambda)$ with these $t_i^{(n)}$ corresponds to the completely integrable lattice model. The local quantum space of this model at the nth site is a spin space ($S = \frac{1}{2}, 1, \ldots$). This realization of the SU(2) algebra in Fock space was first constructed in [12].

4. The lattice quantum SG model

The construction of the quantum integrable LSG model is necessary for the quantization of the SG model as discussed in the introduction. The quantum R-matrix of the SG-model is well-known [1]:

$$R(\lambda, \mu) = \begin{pmatrix} \operatorname{sh}(\alpha + i\gamma) & 0 & 0 & 0 \\ 0 & i\sin\gamma & \operatorname{sh}\alpha & 0 \\ 0 & \operatorname{sh}\alpha & i\sin\gamma & 0 \\ 0 & 0 & 0 & \operatorname{sh}(\alpha + i\gamma) \end{pmatrix} , \qquad \exp(\alpha) = \frac{\lambda}{\mu} . \tag{47}$$

It can be derived from the classical r-matrix (16). The elementary monodromy matrix $L(n|\lambda)$ which satisfies eq. (20) with the R-matrix (47) is given by the same formulae (14), (15) as in the classical case, the variables p_n and u_n being now, however, quantum operators with canonical commutation relations:

$$[u_n, p_m] = i\delta_{mn} . \tag{48}$$

Notice that the ordering of operators π, χ in (14) is now essential. The property (17) also holds in the quantum case. The quantum analogue of (18) is the following relation:

$$\begin{gathered} L(n|\lambda)\sigma_2 L^{\mathrm{T}}(n|\lambda\, \mathrm{e}^{-i\gamma})\sigma_2 = d_{\mathrm{q}}(\lambda) , \\ d_{\mathrm{q}}(\lambda) = 1 + S(\lambda^2 \mathrm{e}^{-i\gamma} + \lambda^{-2} \mathrm{e}^{i\gamma}) , \qquad S = (\tfrac{1}{4} m\Delta)^2 . \end{gathered} \tag{49}$$

When $\Delta \to 0$ the monodromy matrix $L(n|\lambda)$ turns into the infinitesimal matrix $L_n(\lambda)$ which was used in the pioneering paper [13].

We define the hamiltonian of the quantum LSG model in terms of the transfer matrix $\tau(\lambda)$ by making the natural generalization of the corresponding definition (19) of the classical LSG model:

$$H=\frac{m^2\Delta}{16\sin\gamma}\sum_{+,-}\left(\frac{\partial}{\partial\lambda^{\pm2}}F(\lambda^2=\kappa_\pm)+\frac{\partial}{\partial\lambda^{\pm2}}G(\lambda^2=\nu_\pm)\right), \tag{50}$$

$$F(\lambda^2)=\ln[d_q^{-N/2}(\lambda)\tau(\lambda)], \qquad G(\lambda^2)=\ln[d_q^{-N/2}(\lambda^{-1})\tau(\lambda)],$$

$$\nu_\pm=-b^{\pm1}\exp(i\gamma), \qquad \kappa_\pm=\nu_\pm^*, \qquad \gamma=\tfrac{1}{8}\beta^2, \qquad b=2S(1+\sqrt{1-4S^2})^{-1},$$

$$d_q(\nu_\pm)=0.$$

One can easily prove using (17) that H is hermitian ($H=H^*$). This hamiltonian is not, however, local as in the case of the classical model. It is quasilocal: all the sites on the lattice interact but the strength decreases as $(\frac{1}{2}\sin\gamma)^{|n-m|}$ with the distance. Let us explain this statement. It is easily seen from (49) that $L(n|\nu_\pm)$ has a representation in the form of a direct projector (24) and $L(n|\kappa_\pm)$ in the form of an inverse projector (25). As the point $\nu_\pm$, $\kappa_\pm$ are all different there exists no point where both representations are valid simultaneously. This is the reason for H being non-local. In the quasiclassical limit the difference between the direct projector and the inverse projector vanishes and locality is restored. The hamiltonian (50) turns into the classical hamiltonian (19) in this limit. The fact that the classical phase space is a direct product of tori leads to the commutativity of quantum matrix $L(n|\lambda)$, $T(\lambda)$, $\tau(\lambda)$ and H with the local operators $\exp\{4\pi i p_n/\beta\}$ and $\exp\{8\pi i u_n/\beta\}$.

The quantum LSG model can be solved by means of QISM. The solution reduces to that given in [7, 8] if one changes the local pseudovacuum at two adjacent sites (eq. (12) in [7]). The new pseudovacuum is equal to

$$\varphi_n=\delta\left(u_{2n}-u_{2n-1}-\tfrac{1}{4}\beta+\frac{2\pi}{\beta}\right)\left[1-2S\cos\left(\beta\frac{u_{2n}+u_{2n-1}}{2}\right)\right]^{-1/2}. \tag{51}$$

The eigenstates Φ_m of the hamiltonian (50) can be found by applying the algebraic Bethe ansatz. The eigenvalues of H are real and additive:

$$H\Phi_m=\left(\sum_{k=1}^{m}h(\lambda_k)\right)\Phi_m, \qquad m=0,1,2,\ldots, \tag{52}$$

$$h(\lambda)=\tfrac{1}{4}m^2\Delta\sin\gamma\sum_{+,-}\frac{[1-2\lambda^{\pm2}b\cos\gamma-\lambda^{\pm4}b^2(1+2\cos2\gamma)]\lambda^{\pm2}}{(1+\lambda^{\pm2}b\,e^{3i\gamma})(1+\lambda^{\pm2}b\,e^{-3i\gamma})(1+\lambda^{\pm2}b\,e^{i\gamma})(1+\lambda^{\pm2}b\,e^{-i\gamma})}.$$

The values of λ_k in (52) is not arbitrary but must satisfy the system of equations

$$\left[\frac{d_q(\lambda_l)}{d_q(\lambda_l^{-1})}\right]^{N/2}=\prod_{\substack{k=1\\k\neq l}}^{m}\left(\frac{\lambda_k^2\,e^{-i\gamma}-\lambda_l^2\,e^{i\gamma}}{\lambda_k^2\,e^{i\gamma}-\lambda_l^2\,e^{-i\gamma}}\right), \qquad l=1,2,\ldots,m. \tag{53}$$

It can be shown that all the λ_k are different [14].

To conclude let us discuss the connection of the above model with the (spin $\frac{1}{2}$)$\otimes Z_P$ models. Notice that commutation relations

$$\overset{+}{\pi}_n \overset{+}{\chi}_n = e^{i\gamma} \overset{+}{\chi}_n \overset{+}{\pi}_n \tag{54}$$

at $\gamma/2\pi = Q/P$ (the integers Q and P are supposed to be relatively prime, and $Q < P$) admit the following constraint:

$$(\chi^+)^P = (\pi^+)^P = 1 . \tag{55}$$

In this way the field variables becomes discrete ones. It is possible to change the δ-function in (51) into the discrete δ-symbol. The pseudovacuum thus became normalizable. At $\gamma = 2\pi Q/P$, χ and π can be realised by the following $P\times P$ matrices:

$$\begin{gathered} \overset{+}{\chi}_{ab} = \delta_{ab} \exp\left[\frac{2\pi i}{P}(a-1)\right], \qquad \overset{+}{\pi}_{ab} = \delta_{a+Q,b}, \\ a, b = 1, \ldots, P, \qquad a+P \equiv a . \end{gathered} \tag{56}$$

In this way the LSG model becomes (spin $\frac{1}{2}$)$\otimes Z_P$ model.

5. Conclusion

We have constructed lattice versions of the NS and SG models preserving the complete integrability and other symmetries of the models (with the natural exception of Lorentz invariance). From the quantum field theoretical point of view the LSG model is of special interest when $\Delta \to 0$ and after mass renormalization. It serves as a definition of the continuous relativistic quantum SG model. The analysis of the solution of the model [15] shows that Lorentz invariance is restored in the continuum limit. At $\gamma = \frac{2}{3}\pi$ the naive physical vacuum becomes unstable. One has to fill the pseudovacuum also with bound states to obtain the real physical vacuum as was described in the paper [16]. Notice that the existence of a critical phenomenon at $\gamma = \frac{2}{3}\pi$ was predicted in [17].

Our regularization of the SG model seems to be the most natural one. In contrast to other lattice regularizations preserving integrability it is done in terms of boson fields. So we have shown in detail how the programme of lattice regularization of the quantum field theory, which was described in the introduction can be fulfilled.

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