Some Open Problems in Exactly Solvable Models

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ABSTRACT. Some open problems in Exactly Solvable Models are presented.

1. Introduction

In the recent years there has been considerable progress in investigating rigorously the correlation functions of physical models solvable by Bethe Ansatz. These investigations based on various methods (Algebraic Bethe Ansatz, quantum Knizhnik-Zamolodchikov equation, quantum groups, etc.) allowed for the very first time to obtain information about the short and large distance asymptotic behavior of correlation functions of models not equivalent to free fermions. Of course there is still a lot of work to be done and a lot of unsolved problems. Also with the advent of quantum computation and quantum information new objects of interest appeared like the entanglement entropy. The entanglement entropy is a special correlation function and it can be computed exactly in some cases. In this paper we are going to present four unsolved problems which we believe that are of considerable interest. Of course they do not exhaust the list of open problems in exactly solvable models. Also the list of references is by no means complete and should be considered only as a starting point for a complete tour of the literature on the subject. This article is an updated version of [1].

2. Norm of the eigenfunctions of the Hubbard model

An important step in computing the correlation functions of models solvable by Bethe Ansatz is the calculation of the norm of Bethe ansatz wavefunctions. The first conjecture expressing the norm of the Bethe wavefunction in terms of a determinant was put forth by M.Gaudin in [3] in the case of what is called now the Gaudin model. The conjecture was extended later [4] in the case of the XXX and XXZ spin chains and Bose gas with δ-function interaction.

The first proof of Gaudin's conjecture was given in [5] using the formalism of Algebraic Bethe Ansatz. In the subsequent years the results of [5] were generalized in the case of systems solvable with Nested Bethe Ansatz [6, 7].

The first open problem that we present is the conjecture proposed in [2] for the norm of the eigenfunctions of the one dimensional Hubbard model. Let us be more specific. The Hamiltonian of the one-dimensional Hubbard model on a periodic

L-site chain may be written as

$$(2.1) \qquad H = -\sum_{j=1}^{L} \sum_{a=\uparrow,\downarrow} (c_{j,a}^{+} c_{j+1,a} + c_{j+1,a}^{+} c_{j,a}) + U \sum_{j=1}^{L} (n_{j\uparrow} - \tfrac{1}{2})(n_{j\downarrow} - \tfrac{1}{2}) \,.$$

Here $c_{j,a}^{+}$ and $c_{j,a}$ are creation and annihilation operators of electrons of spin a at site j , $n_{j,a} = c_{j,a}^{+} c_{j,a}$ is the corresponding particle number operator, and U is the coupling constant. Periodicity is ensured by the condition $c_{L+1,a} = c_{1,a}$. The coordinates and the spin of the electrons are denoted by $x_j = 1, \cdots, L$ and $a_j = \uparrow, \downarrow$. The Hubbard Hamiltonian conserves the number of electrons N and the number of down spins M, therefore he eigenstates of the Hamiltonian (2.1) may be represented as

$$|N, M\rangle = \frac{1}{\sqrt{N!}} \sum_{x_1,\ldots,x_N=1}^{L} \sum_{a_1,\ldots,a_N=\uparrow,\downarrow} \psi(x_1,\ldots,x_N; a_1,\ldots,a_N) \, c_{x_N a_N}^{+} \cdots c_{x_1 a_1}^{+} |0\rangle$$

where $\psi(x_1,\ldots,x_N; a_1,\ldots,a_N)$ is the Bethe ansatz wave function which depends on two sets of quantum numbers, $\{k_j \,|\, j = 1,\ldots,N\}$ and $\{\lambda_l \,|\, l = 1,\ldots,M\}$, which can be complex. The quantum numbers k_j and λ_j are called charge and spin rapidities and satisfy the Lieb-Wu equations

$$e^{ik_j L} = \prod_{l=1}^{M} \frac{\lambda_l - \sin k_j - iU/4}{\lambda_l - \sin k_j + iU/4} \,, \ j = 1,\ldots,N \,,$$

$$\prod_{j=1}^{N} \frac{\lambda_l - \sin k_j - iU/4}{\lambda_l - \sin k_j + iU/4} = \prod_{m=1,m\neq l}^{M} \frac{\lambda_l - \lambda_m - iU/2}{\lambda_l - \lambda_m + iU/2} \,, \ l = 1,\ldots,M \,.$$

In logarithmic form the Lieb-Wu equations take the form

$$k_j L - i \sum_{l=1}^{M} \ln \left(\frac{iU + 4(\lambda_l - \sin k_j)}{iU - 4(\lambda_l - \sin k_j)} \right) = 2\pi n_j^c \,,$$

$$i \sum_{j=1}^{N} \ln \left(\frac{iU + 4(\lambda_l - \sin k_j)}{iU - 4(\lambda_l - \sin k_j)} \right) - i \sum_{m=1}^{M} \ln \left(\frac{iU + 2(\lambda_l - \lambda_m)}{iU - 2(\lambda_l - \lambda_m)} \right) = 2\pi n_l^s \,.$$

In the previous equations n_j^c is integer, if M is even and half odd integer, if M is odd. Similarly, n_l^s in is integer, if $N - M$ is odd, and half odd integer, if $N - M$ is even.

It was shown in [**2**] that the Lieb-Wu equations can be obtained as extremum conditions for the action

$$\mathcal{S} = \sum_{j=1}^{N} (k_j \sin k_j + \cos k_j) L + \sum_{j=1}^{N} \sum_{l=1}^{M} \Theta_U(\lambda_l - \sin k_j) - \frac{1}{2} \sum_{l,m=1}^{M} \Theta_{2U}(\lambda_l - \lambda_m)$$

$$- 2\pi \sum_{j=1}^{N} n_j^c \sin k_j - 2\pi \sum_{l=1}^{M} n_l^s \lambda_l \quad .$$

where $\Theta_U(x) = i \int_0^x dy \, \ln \left(\frac{iU+4y}{iU-4y} \right)$.

Now we can state the conjecture:

CONJECTURE 2.1. [**2**] *The square of the norm of the Hubbard wave function ψ is given by*

$$\|\psi\|^2 = (-1)^{M'} N! \left(\frac{U}{2}\right)^M \prod_{j=1}^N \cos k_j \prod_{1 \le j < k \le M} \left(1 + \frac{U^2}{4(\lambda_j - \lambda_k)^2}\right) det \begin{pmatrix} \dfrac{\partial^2 \mathcal{S}}{\partial s^2} & \dfrac{\partial^2 \mathcal{S}}{\partial s \partial \lambda} \\ \dfrac{\partial^2 \mathcal{S}}{\partial \lambda \partial s} & \dfrac{\partial^2 \mathcal{S}}{\partial \lambda^2} \end{pmatrix} .$$

where $s = \sin k_j$ and M' is the number of complex conjugated k_j's in a given solution of Lieb-Wu equation. The matrix appearing in the previous expression has dimension $(N + M) \times (N + M)$ and has four blocks with matrix elements

$$\left(\frac{\partial^2 \mathcal{S}}{\partial s^2}\right)_{mn} = \frac{\partial^2 \mathcal{S}}{\partial s_m \partial s_n} = \delta_{m,n} \left\{\frac{L}{\cos k_n} + \sum_{l=1}^M \frac{U/2}{(U/4)^2 + (\lambda_l - s_n)^2}\right\}, \ m, n = 1, \ldots, N \ ,$$

$$\left(\frac{\partial^2 \mathcal{S}}{\partial \lambda \partial s}\right)_{mn} = \left(\frac{\partial^2 \mathcal{S}}{\partial s \partial \lambda}\right)_{nm} = \frac{\partial^2 \mathcal{S}}{\partial \lambda_m \partial s_n} = -\frac{U/2}{(U/4)^2 + (\lambda_m - s_n)^2} \ , m = 1, \ldots, M, \ n = 1, \ldots, N,$$

$$\left(\frac{\partial^2 \mathcal{S}}{\partial \lambda^2}\right)_{mn} = \frac{\partial^2 \mathcal{S}}{\partial \lambda_m \partial \lambda_n} = \delta_{m,n} \left\{\sum_{j=1}^N \frac{U/2}{(U/4)^2 + (\lambda_n - s_j)^2} - \sum_{l=1}^M \frac{U}{(U/2)^2 + (\lambda_n - \lambda_l)^2}\right\}$$
$$+ \frac{U}{(U/2)^2 + (\lambda_m - \lambda_n)^2}, \ m, n = 1, \ldots, M.$$

The conjecture was checked for small values of M and N, $M = 0$ and arbitrary N, and in the limit of $U \to \infty$ with M, N arbitrary. However at the moment we do not know of a general proof.

Open problem: Prove the previous conjecture expressing the norm of the eigenfunctions of the Hubbard model.

3. The Heisenberg Spin Chain and Number Theory

Consider the antiferromagnetic spin 1/2 Heisenberg XXX spin chain with the hamiltonian

$$(3.1) \qquad \mathcal{H}_{XXX} = \sum_{j=-\infty}^{\infty} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z\right) ,$$

where $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are the Pauli matrices and we will denote σ^0 the 2×2 unit matrix

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Bethe succeeded in diagonalizing this Hamiltonian [**8**] by means of what we call today the coordinate Bethe Ansatz, and the unique antiferromagnetic ground state in the thermodynamic limit was investigated by Hulthen [**9**]. The correct description of the excitations in terms of magnons of spin 1/2 was first obtained by Takhtajan and Faddeev in [**10**].

We are interested in the following correlation functions: the longitudinal spin-spin correlation function $\langle S_i^z S_{i+n}^z \rangle$ and the *emptiness formation probability* $P(n) = \langle GS| \prod_{j=1}^n P_j |GS\rangle$ where $S_j^z = \sigma_j^z/2$ and $P_j = S_j^z + 1/2$. $P(n)$ gives the probability of having a ferromagnetic string of length n in the antiferromagnetic ground state.

In 2001 H.Boos and one of the authors computed exactly the emptiness formation probability $P(n)$ for small strings ($n = 1, \ldots, 4$). The starting point was the multiple integral representation derived in [14] based on the vertex operator approach [59]. In [15] it was shown that the integrand in the multiple integral representation for the emptiness formation probability can be reduced to a canonical form which then can be integrated with the results:

$$
\begin{aligned}
P(1) &= \frac{1}{2} = 0.5\,, \\
P(2) &= \frac{1}{3} - \frac{1}{3}\ln 2\,, \\
P(3) &= \frac{1}{4} - \ln 2 + \frac{3}{8}\zeta(3) = 0.007624158\,, \\
P(4) &= \frac{1}{5} - 2\ln 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\ln 2 \cdot \zeta(3) - \frac{51}{80}\zeta^2(3) \\
&\quad - \frac{55}{24}\zeta(5) + \frac{85}{24}\ln 2 \cdot \zeta(5) = 0.000206270\,.
\end{aligned}
$$

In the previous expressions ζ is the Riemann zeta function defined as [11]

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Re(s) > 1\,.
$$

Alternatively the Riemann zeta function can be expressed in terms of the alternating zeta series ζ_a

$$
\zeta(s) = \frac{1}{1 - 2^{1-s}}\zeta_a(s)\,, \quad s \neq 1\,,
$$

where

$$
\zeta_a(s) = \sum_{n>0} \frac{(-1)^{n-1}}{n^s} = -\mathrm{Li}_s(-1)\,.
$$

with $\mathrm{Li}_s(x)$ the polylogarithm. In some cases it is preferable to work with the alternating zeta function due to the fact that unlike the Riemann zeta function who has a pole at $s = 1$, $\zeta_a(1) = \ln 2$.

The result for $P(1)$ is obvious from symmetry and $P(2)$ can be obtained from the ground state energy computed by Hulthen [9]. $P(3)$ can be obtained from Takahashi's result [12] for the nearest neighbor correlation (see also [13]). The formulae for $P(5)$ and $P(6)$ obtained in [16] and in [17] showed a similar structure.

The techniques developed in [15] and [17] were also used in the computations of the spin-spin correlation functions with the results:

$$\langle S_j^z S_{j+1}^z \rangle = \frac{1}{12} - \frac{1}{3}\zeta_a(1) = -0.147715726853315\,,$$

$$\langle S_j^z S_{j+2}^z \rangle = \frac{1}{12} - \frac{4}{3}\zeta_a(1) + \zeta_a(3) = 0.060679769956435\,,$$

$$\langle S_j^z S_{j+3}^z \rangle = \frac{1}{12} - 3\zeta_a(1) + \frac{74}{9}\zeta_a(3) - \frac{56}{9}\zeta_a(1)\zeta_a(3) - \frac{8}{3}\zeta_a(3)^2$$
$$- \frac{50}{9}\zeta_a(5) + \frac{80}{9}\zeta_a(1)\zeta_a(5) = -0.050248627257235\,,$$

$$\langle S_j^z S_{j+4}^z \rangle = \frac{1}{12} - \frac{16}{3}\zeta_a(1) + \frac{290}{9}\zeta_a(3) - 72\zeta_a(1)\zeta_a(3) - \frac{1172}{9}\zeta_a(3)^2 - \frac{700}{9}\zeta_a(5)$$
$$+ \frac{4640}{9}\zeta_a(1)\zeta_a(5) - \frac{220}{9}\zeta_a(3)\zeta_a(5) - \frac{400}{3}\zeta_a(5)^2$$
$$+ \frac{455}{9}\zeta_a(7) - \frac{3920}{9}\zeta_a(1)\zeta_a(7) + 280\zeta_a(3)\zeta_a(7)$$
$$= 0.034652776982728\,.$$

Again the nearest neighbor correlator was obtained from Hulthen result [9] and the second-neighbor correlator was obtained by Takahashi in 1977 [12] using the strong coupling expansion of the ground state energy of the half-filled Hubbard chain. The next nearest correlators and other type of correlation functions up to 8 lattice sites were succesively calculated in [18],[19],[20],[21],[22]. In 2001 the following conjecture was put forward in the light of the previous results

CONJECTURE 3.1. ([15],[17]) *Any correlation function of the XXX spin chain can be represented as a polynomial in* $\ln 2$ *and values of Riemann zeta function at odd arguments with rational coefficients .*

Even though the conjecture was finally proved in 2006 by H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama [23] we still not have explicit expressions for all the rational coefficients that enter in the expressions for the correlation functions. Recently Jun Sato [24] obtained a formula for the linear terms and some of the higher terms. In order to present the results let us introduce $c_1, c_2, \cdots$ by

$$c_1 = \frac{\zeta_a(1) - \zeta_a(3)}{3}\,,$$
$$c_2 = \frac{\zeta_a(3) - \zeta_a(5)}{3}\,,$$
$$c_3 = \frac{\zeta_a(5) - \zeta_a(7)}{3}\,, \cdots$$

and define $c_0 = (1/4 - \zeta_a(1))/3$. Then Sato's formula reads

$$\langle S_j S_{j+n} \rangle = \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} \binom{2k+1}{k} c_k - 24 \binom{n-1}{2} c_1(c_0 + c_1) + \text{ higher terms }.$$

The formula reproduces the results up to $\langle S_j S_{j+7} \rangle$. Now we can state our second open problem:

Open problem: Obtain an efficient description of the rational coefficients which appear in the expression for the correlation functions of the XXX spin chain as a polynomial in alternating zeta series at odd arguments.

4. Asymptotic Behaviour of Time and Temperature Dependent Correlation Functions

Computing the large time and distance asymptotics of correlation functions is one of the most difficult tasks in the field of Exactly Solvable Models. In the last twenty years intense studies on models equivalent with free fermions showed that the following strategy is applicable. The first step is obtaining a representation of the correlation functions in terms of a Fredholm determinant either by direct summation of form factors or other methods. This determinant representation is fundamental in obtaining a integrable system of partial differential equations which completely characterizes the correlators. The last step is the large time and distance analysis of the matrix Riemann-Hilbert problem associated with the integrable system from which the asymptotics of the correlation functions can be extracted. Let us present the results in the case of the the isotropic XY model [25] (called alternatively the XX0 spin chain) in transverse magnetic field with the hamiltonian

$$\mathcal{H}_{XX0} = -\sum_{j=-\infty}^{\infty} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + h\sigma_j^z \right)$$

where σ are the Pauli matrices and h is the magnetic field. We are interested in the asymptotic behavior of the time and temperature correlation function

$$(4.1) \qquad \langle \sigma_1^+(t)\sigma_{n+1}^-(0)\rangle_T \equiv g(n,t) = \frac{Tr\left\{ \left(e^{-\mathcal{H}_{XX0}/T}\right)\sigma_1^+(t)\sigma_{n+1}^-(0)\right\}}{Tr\left(e^{-\mathcal{H}_{XX0}/T}\right)}$$

when n and t are large and $h \in [0,2)$. Making use of the above mentioned strategy the authors of [28] showed that $g(n,t)$ decays exponentially but the rate of decay depends on the direction ϕ defined as $\cot\phi = n/4t$ when $n, t \to \infty$. The asymptotics in the space-like and time-like regions are:

- Space-like directions $0 \le \phi < \pi/4$

$$g(n,t) \to C\exp\left\{ \frac{n}{2\pi}\int_{-\pi}^{\pi} dp\ \ln\left|\tanh\left[\frac{h-2\cos p}{T}\right]\right|\right\}$$

- Time-like directions $\pi/4 < \phi \le \pi/2$

$$g(n,t) \to Ct^{(2\nu_+^2+2\nu_-^2)}\exp\left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi} dp\ |n-4t\sin p|\ \ln\left|\tanh\left[\frac{h-2\cos p}{T}\right]\right|\right\}$$

with

$$\nu_+ = \frac{1}{2\pi}\left|\tanh\left(\frac{h-2\cos p_0}{T}\right)\right| \qquad \nu_- = \frac{1}{2\pi}\left|\tanh\left(\frac{h+2\cos p_0}{T}\right)\right| \qquad \frac{n}{4t} = \sin p_0$$

We should note that at zero temperature the asymptotics of the correlation functions were evaluated in [26] and [27] and for $\phi = \pi/2$ the leading factor was computed in [30]. Similar results, making use of the same strategy, were obtained in the case of one dimensional impenetrable bosons [60] and the impenetrable electron gas [31, 32, 33, 34].

Unfortunately the method outlined fails in the case of models who are not equivalent with free fermions. In this case it is not possible to obtain a representation of the correlation functions in terms of Fredholm determinants. An alternative way which seems promising is the study of multiple integral representations for correlators. Very recently Kitanine, Kozlowski, Maillet, Slavnov and Terras proposed

in [**35**] a general method to obtain the large distance asymptotics from the first principles using the Algebraic Bethe Ansatz. Their method is applicable to a large class of integrable models but in order to be more precise we are going to focus on the XXZ spin chain. We remind the hamiltonian

$$\mathcal{H}_{XXZ} = - \sum_{j=-\infty}^{\infty} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_i^z \sigma_{j+1}^z + h \sigma_j^z \right)$$

where Δ is the anisotropy and h the magnetic field.

The starting point of the analysis performed in [**35**] is the master equation for the generating functional of the longitudinal spin-spin correlation function $\langle \sigma_1^z \sigma_{n+1}^z \rangle$ in finite volume obtained in [**41**]. In general taking the thermodynamic limit directly in the master equation is a very difficult task, however Kitanine *et al.,* were able to obtain a new expansion of the master equation in terms of multiple integrals of a special type called cycle integrals. The large distance asymptotic behavior of the cycle integrals can be obtained from the Riemann-Hilbert analysis of the Fredholm determinant of an integral operator with a generalized sine-kernel [**36**]. Then the multiple series corresponding for the generating function can be summed asymptotically by computing each term using the asymptotic behavior of the cycle integrals. Using this technique the large distance asymptotics of the longitudinal spin-spin correlation functions at zero temperature were computed.

We see that the principal ingredients are the multiple integral representation and the master equation. In the static case at zero temperature and zero magnetic field they were obtained by Jimbo, Miki, Miwa and Nakayashiki in [**37, 38**] and in [**39, 40**] at non-zero magnetic field. The generalization at finite temperature was obtained by Göhmann, Klümper, Seel and Hasenclever in [**44, 45**] and the multiple integral representation for the dynamical correlation functions at zero temperature were obtained in [**42**]. However in the case of time and temperature correlation functions we still do not have such formulae or know rigorously the asymptotic behavior. This brings us at our third open problem:

Compute rigorously the large time and distance asymptotic behavior of the spin spin correlation functions of the XXZ spin chain at finite temperature in the critical region $(-1 \leq \Delta < 1)$.

5. Entropy of Subsystems

In the recent years a large amount of effort was spent in investigating entanglement in a multitude of quantum systems. Besides being the fundamental resource in quantum computation and quantum information it is believed that a better understanding of entanglement will provide further insight in the theory of quantum phase transitions and in the study of strongly correlated quantum systems.

We are interested in one dimensional systems of interacting spins which posses a unique ground state $|GS\rangle$. The ground state can be considered as a bipartite system composed of a contiguous block of spins denoted by A and the rest of the spins denoted by B. The density matrix of the entire system is given by

$$\rho_{A\&B} = |GS\rangle\langle GS|\,,$$

and the density matrix of the subsystem A is obtained by tracing the B degrees of freedom

$$\rho_A = Tr_B(\rho_{A\&B})\,.$$

In [**46**] Bennet, Bernstein, Popescu and Schumacher discovered that the von Neumann entropy of the subsystem A

$$(5.1) \qquad S(\rho_A) = -Tr(\rho_A \ln \rho_A).$$

provides an efficient way of measuring entanglement. Another important measure of entanglement is the Renyi entropy defined as

$$S_R(\rho_A, \alpha) = \frac{1}{1-\alpha} \ln \mathrm{Tr}(\rho_A^\alpha).$$

If we consider the doubling scaling limit in which the size of the block of spins is much larger than one but much smaller than the length of the entire chain the results of numerous investigations using different techniques (conformal field theory, exact results, numerics, etc.) can be summarized as follows.

The entropy of subsystems of critical one dimensional models scales logarithmically with the size of the block of spins. If c is the central charge of the associated conformal field theory that describes the critical model for a block of n spins we have

$$(5.2) \qquad S(n) = \frac{c}{3} \ln n, \qquad n \to \infty.$$

This formula was first derived for the geometrical entropy (the analog of (5.1) for conformal field theory) by Holzhey, Larsen and Wilczek in [**47**] (see also [**48**],[**49**],[**54**]).

In the case of non-critical models the entropy of subsystems will increase with the size of the subsytem until it will reach a limiting value $S(\infty)$. This was first conjectured in [**48**] (based on numerical evidence) and proved analytically for the XY and AKLT spin chains in [**56**],[**57**] and [**51**]. In the case of the AKLT spin chain [**50**] $S(\infty) = 2$.

It is instructive to present the results obtained for the the XY spin chain in magnetic field with the hamiltonian

$$\mathcal{H}_{XY} = - \sum_{j=-\infty}^{\infty} \left((1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y + h\sigma_j^z \right),$$

where $0 < \gamma < 1$ is the anisotropy parameter an h is the magnetic field. The model was solved in [**25**] and [**52**]. The ground state is unique and in general there is a gap in the spectrum.

The limiting value of the entropy in the double scaling limit depends on the isotropy and magnetic field. The density matrix of a block of n neighboring spins in the ground state will be denoted by $\rho(n)$. We can distinguish three cases:

- Case Ia: moderate magnetic field $2\sqrt{1-\gamma^2} < h < 2$
- Case Ib: weak magnetic field including zero magnetic field $0 \leq h < 2\sqrt{1-\gamma^2}$
- Case II: strong magnetic field $h > 2$

The results for all the regions obtained in [**56**] can be presented compactly as

$$S(\infty) = \frac{\pi}{2} \int_0^\infty \ln \left(\frac{\theta_3(ix + \frac{\sigma\tau}{2})\theta_3(ix - \frac{\sigma\tau}{2})}{\theta_3^2(\frac{\sigma\tau}{2})} \right) \frac{dx}{\sinh^2(\pi x)}.$$

The modulus k of the theta function θ_3 is different in the three regions; $\tau = I(k')/I(k)$ where $I(k)$ is the complete elliptic integral of modulus k, $k' = \sqrt{1-k^2}$

is the complementary modulus and $\sigma = 1$ in Case I and $\sigma = 0$ in Case II. Independently, I. Peschel using the approach of [54] calculated and simplified the results in region (Ia) and (II) obtaining

$$S(\infty) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + \left(1 - \frac{k^2}{2}\right)\frac{4I(k)I(k')}{\pi}\right] + \ln 2 \quad \text{with} \quad k = \frac{\sqrt{(h/2)^2 + \gamma^2 - 1}}{\gamma}, \quad \text{(Ia)}$$

$$S(\infty) = \frac{1}{12}\left[\ln\frac{16}{k^2k'^2} + (k^2 - k'^2)\frac{4I(k)I(k')}{\pi}\right] \quad \text{with} \quad k = \frac{\gamma}{\sqrt{(h/2)^2 + \gamma^2 - 1}}. \quad \text{(II)}$$

The simplified result for the region (Ib) was obtained in [56]

$$S(\infty) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + \left(1 - \frac{k^2}{2}\right)\frac{4I(k)I(k')}{\pi}\right] + \ln 2 \quad \text{with} \quad k = \frac{\sqrt{1 - \gamma^2 - (h/2)^2}}{\sqrt{1 - (h/2)^2}}. \quad \text{(Ib)}$$

The isotropic case $\gamma = 0$ was treated in [53]. If the magnetic field is larger than 2 then the ground-state is ferromagnetic and the entropy is zero. When $0 < h < 2$ the ground state is again unique but the model is critical. The analytic solution obtained in [53]

$$S(\rho(n)) = \frac{1}{3}\ln(n\sqrt{4 - h^2}) - \int_0^\infty dt \left\{\frac{e^{-3t}}{3t} + \frac{1}{t\sinh^2(t/2)} - \frac{\cosh t/2}{2\sinh^3(t/2)}\right\},$$

confirmed the logarithmic scaling of the subsystem entropy. It is interesting to see that the effect of the magnetic field is an effective reduction of the size of the subsystem.

In the case of the XXZ spin chain with the Hamiltonian

$$\mathcal{H}_{XXZ} = -\sum_{j=-\infty}^{\infty}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_i^z\sigma_{j+1}^z\right),$$

the results obtained for this model are not complete. In the case of $\Delta > 1$ the ground-state is ferromagnetic so $S(\rho(n)) = 0$ and in the critical region ($-1 \le \Delta < 1$) the entropy scales logarithmically. As we have said, in the gapped antiferromagnetic case ($\Delta < -1$) we expect that the subsystem entropy will tend to a limiting value $S(\infty)$ but at this moment we are not aware of such a computation.

Open problem: Calculate the entropy of a a subsystem in the antiferromagnetic ground state of the XXZ spin chain when the number of spins in the block is large.

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