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Modified Sine-Gordon Equations

Vasili.Semenov@StonyBrook.edu
1 631 632-8931

Basic Circuit Components

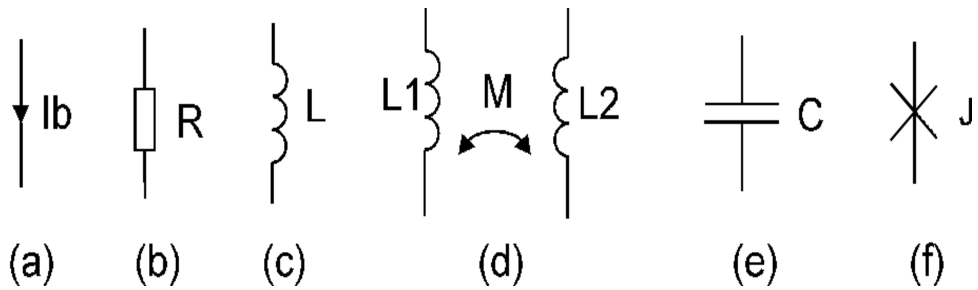


Fig. 1. Notations of basic components

Current source (a)

$I = I_b$

Resistor (b)

$V = R \cdot I$

Inductor (c)

$\Phi = L \cdot I$

Transformer (d)

$\Phi_1 = L_1 \cdot I_1 + M \cdot I_2$

$\Phi_2 = M \cdot I_1 + L_2 \cdot I_2$

Capacitor (e)

$$I = C \cdot \frac{dV}{dt}$$

Josephson junction (f)

$$I = Ic \cdot \sin(\varphi)\,,\ V = \frac{\Phi 0}{2\pi} \cdot \frac{d\varphi}{dt}$$

Characteristic energy

$E_J = \Phi_0 I_c / 2\pi \quad U = E_J \cdot (1 - \cos(\varphi))$

Josephson inductance:

$$LJ(\varphi) = \frac{\Phi_0}{2\pi \cdot I_c \cdot \cos(\varphi)}$$

Long Josephson Junction and Sine-Gordon Equation

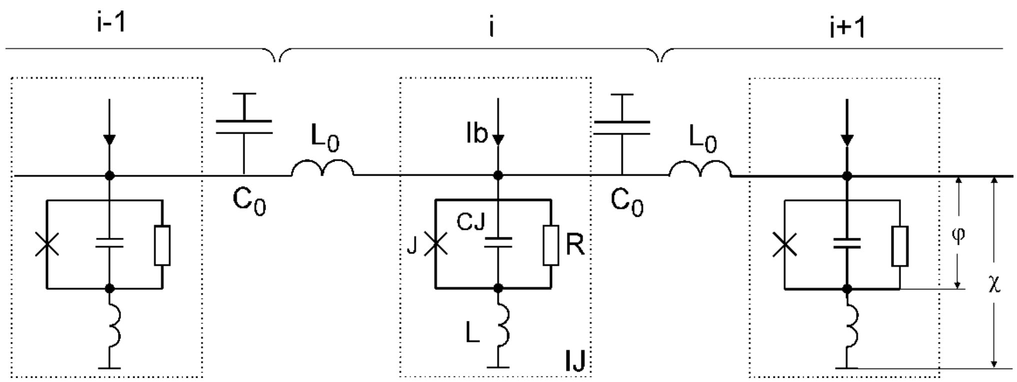


Fig. 2. Long Josephson junction.

Simplified current Kirchhoff equations for Long Josephson Junction (longJJ) looks like:

$$C_0 \frac{d^2\Phi_i}{dt^2} - \frac{\Phi_{i-1} - 2\Phi_i + \Phi_{i+1}}{L_0} + Ic \sin\varphi_i = 0 \,,$$

where

$$\Phi_i = \frac{\Phi_0}{2\pi}\varphi_i$$

This is a discrete Sine-Gordon equation.

Single nSQUID

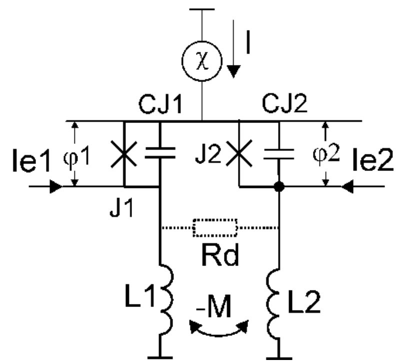


Fig. 3. Equivalent circuit of nSQUID

$$\chi = (\varphi_1 + \varphi_2)/2,\ \varphi = (\varphi_2 - \varphi_1)/2$$

$$\frac{U(\chi,\varphi)}{\Phi_0 I_C/2\pi} = \left(\frac{\chi^2}{1-k} + \frac{(\varphi-\varphi_e)^2}{1+k}\right) - 2\cos\chi\cos\varphi$$

$$\varphi_e \sim (Ie1 - Ie2)/(1+k)$$

$$k = |M|/L$$

String of nSQUIDs

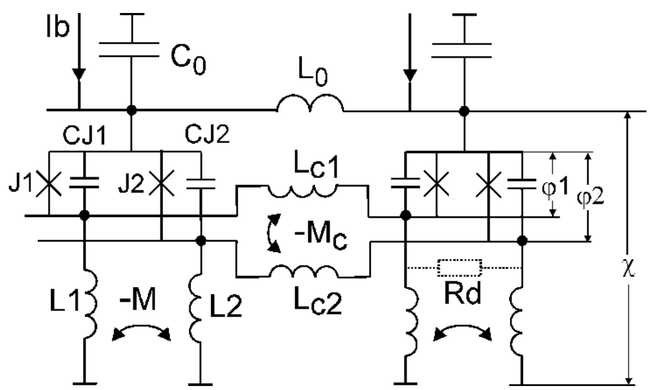


Fig. 4. Equivalent circuit of a string of nSQUID.
Values of parameters: Ic=10μA, L1=L2=L=26 pH, -M=20 pH, [ASC08], L0= , C0=;
Length=180 um, width=24 um (A1 layer with spec. C =5.8 fF/um2)

The most important static and dynamic effects in the sting could be described by the following simplified set of equations:

$$C_1 \frac{d^2\chi_i}{dt^2} - \frac{\chi_{i-1} - 2\chi_i + \chi_{i+1}}{L_0} + \frac{\Phi_0}{2\pi} I_c sin\chi_i \cdot cos\varphi_i = 0, \quad \text{(A)}$$

$$C_2 \frac{d^2\varphi_i}{dt^2} - \frac{\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}}{L_1} + \frac{\Phi_0}{2\pi} I_c sin\varphi_i \cdot cos\chi_i = 0. \quad \text{(B)}$$

for 2 sets of variables $\chi_i \cong (\varphi 1 + \varphi 2)/2$ and $\varphi_i = (\varphi 2 - \varphi 1)/2$. Values of circuit components shown in Fig. 4 are represented in the equations via 2 capacitive (C1 and C2) and 3 inductive (L0, L1 and L2) constants.

Equations (A) and (B) are quite similar. The only difference is the third term in Eq. B that prevents an infinite grows of variables φ_i.

At low L2 ($\lambda_{L2} \equiv \frac{2\pi}{I_c L_2} < 1$) the system allows for φ_i only trivial solutions $\varphi_i = 0$. In this case $\cos(\varphi_i) = 1$ and (A) is reduced to well investigated discrete Sine-Gordon equations. At low L0 (($\lambda_{L_0} \equiv \frac{2\pi}{I_c L_0} << 1$)) static solutions of the equations could be approximated by linear functions with arbitrary slope A and constant B:

$$\chi_i \sim A \cdot i + B,$$

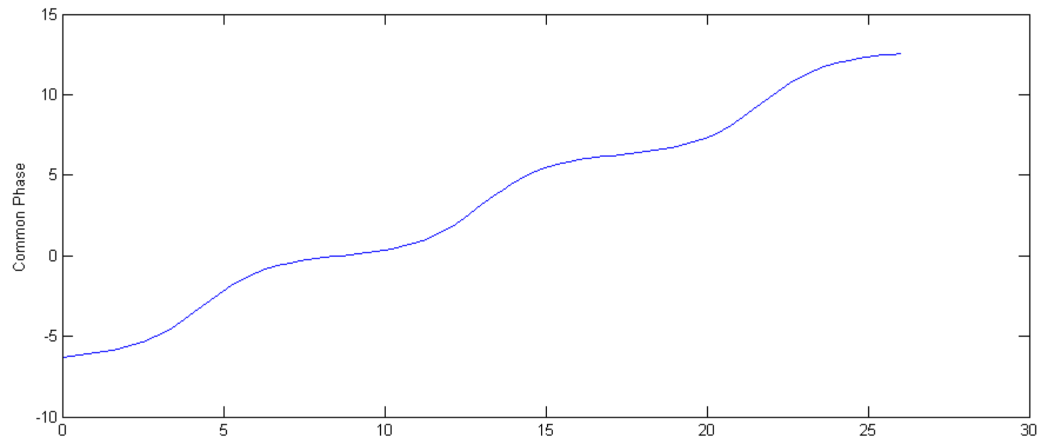
where A has a meaning of an "average" magnetic field applied to the string. Each soliton (or vortex) in the used units carries flux 2π so we can estimate the vortex density as $A/2\pi$. We can also say that one vortex occupies $2\pi/A$ cells. (We believe that for our application one vortex should optimally occupy 8.5 cells.)

At larger L2 ($\lambda_{L2} \equiv \frac{2\pi}{I_c L_2} > 1$) and at $\cos(\chi_i) \sim -1$ equations (B) allows positive and negative nontrivial solutions. (It is easy to show that the trivial solution $\varphi_i = 0$ becomes unstable.)

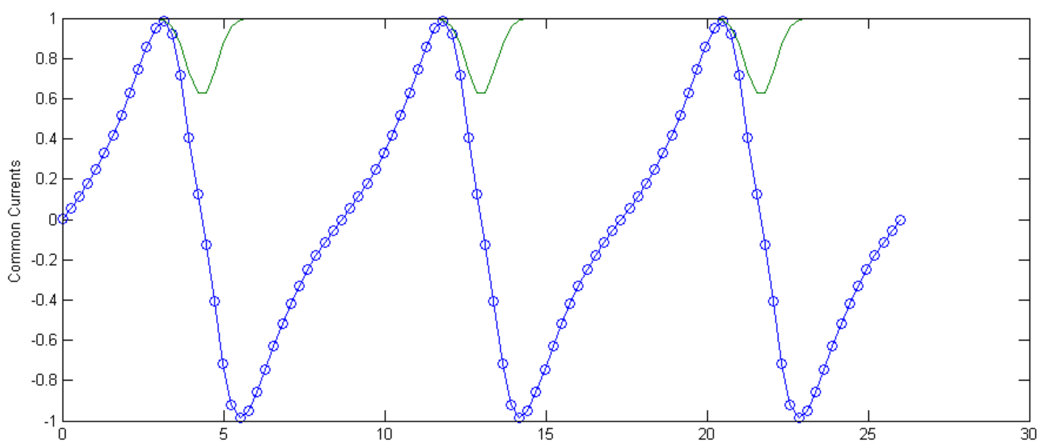
I guess, it would be difficult to find analytical static solutions. However, they could be easily found numerically at any set of parameters. It looks that dynamic solutions could be derived from the static one. (This is true for Sine-Gordon equations.) Main dynamic effect is "relativistic" distortion of vortices. The distortion could be rather strange because equations (A) and (B) have different speeds of light. Maybe the third term in Eq. B breaks the expected similarity of new equations with old Sine-Gordon equation.

Anyway, this is my story. There are new equations that are similar but not identical to Sine-Gordon equation. The applied value of new equation is derived from the fact that there are two symmetric (positive and negative) solutions for φ. However, $\cos\varphi_i$ term in Eq. (A) hides the sign of φ.

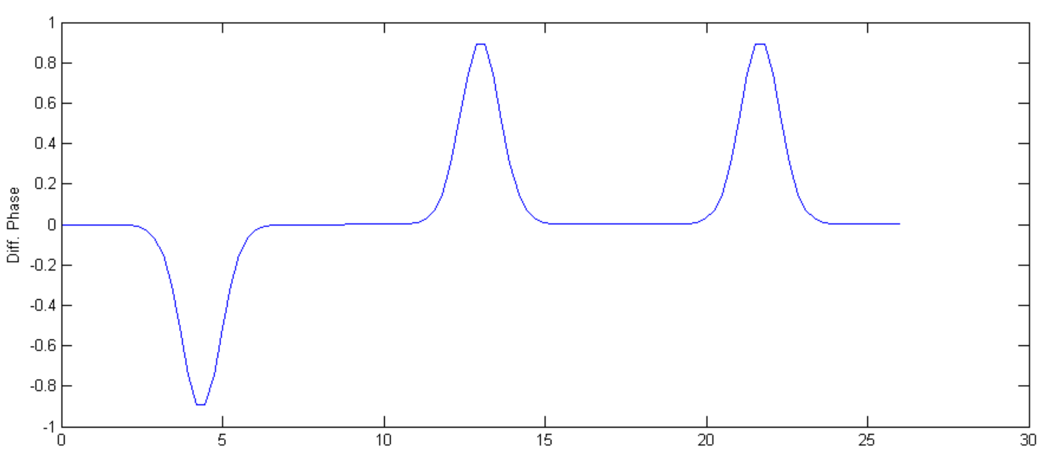
MATLAB parameters: lSQ=0.3; lA=10; lQ=0.6.



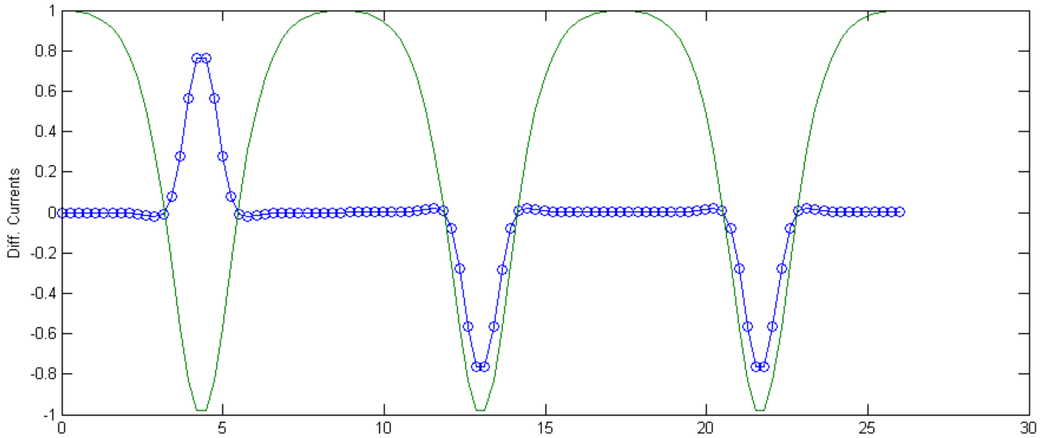
Common Phase



Common current (blue) and its effective “critical” current (green).

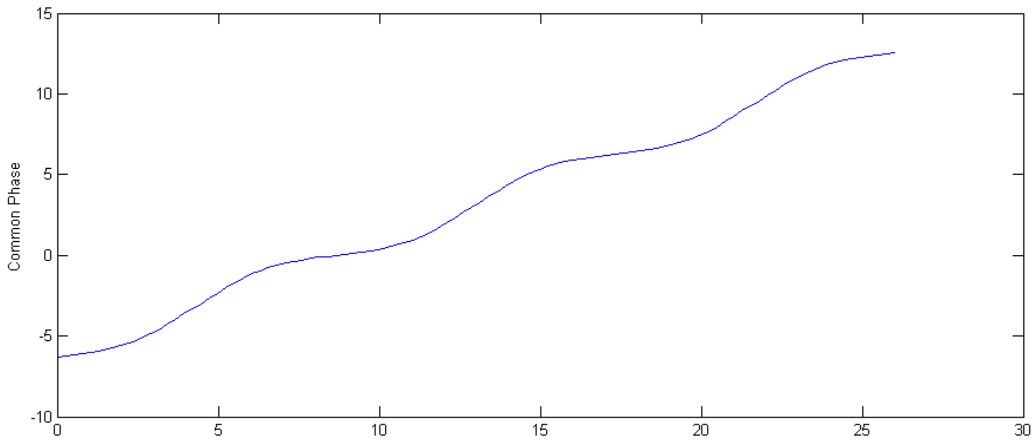


Differential phase.

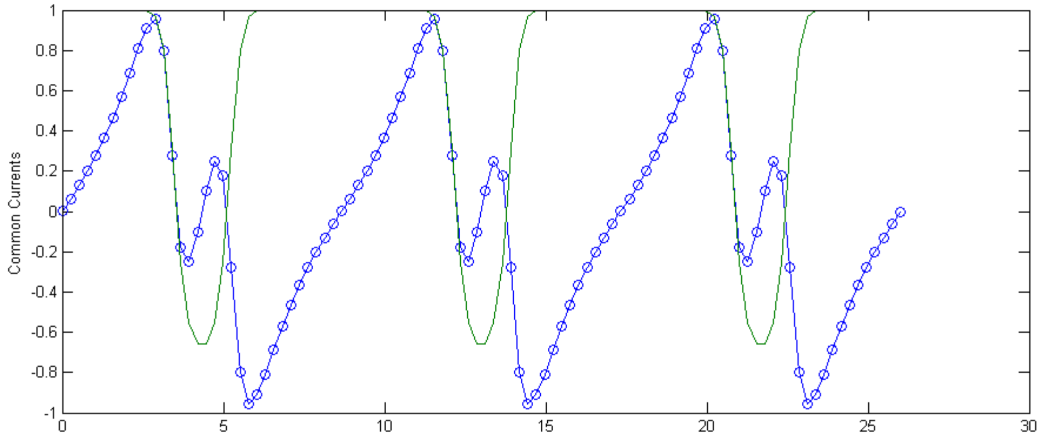


Differential current (blue) and its effective “critical” current.

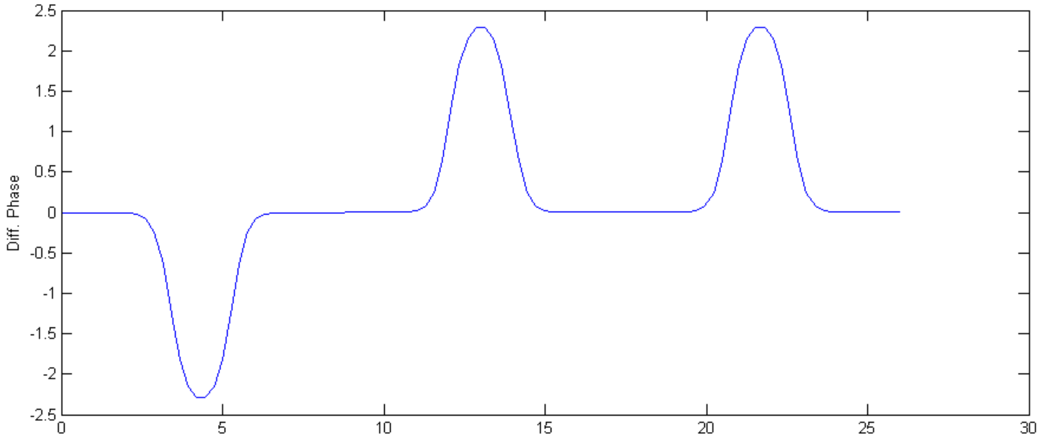
MATLAB parameters: lSQ=0.3; lA=30; lQ=0.3; nVRT=3; fA=-1.



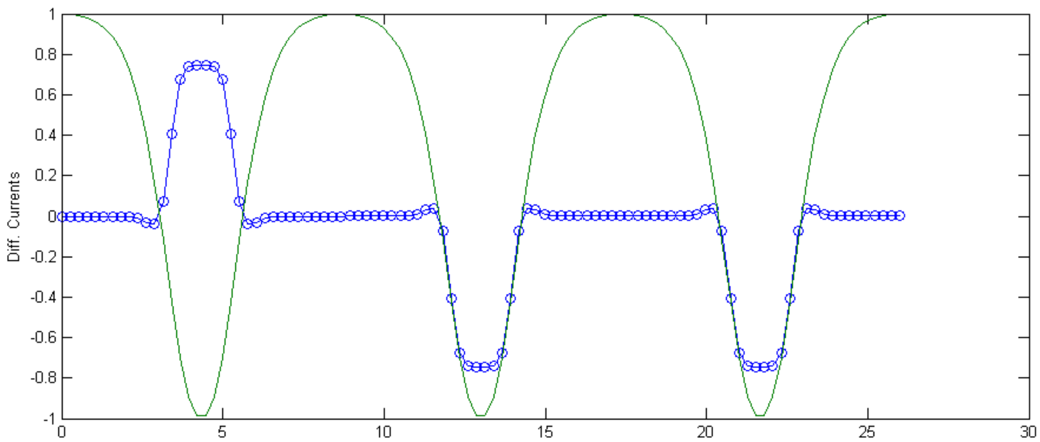
Common Phase



Common current (blue) and its effective "critical" current (green).



Differential phase.



Differential current (blue) and its effective "critical" current.