Review: Q Parallelism & Simon's algorithm

$$(|0\rangle + |1\rangle + |2\rangle + \ldots) \otimes |0\rangle \;\rightarrow (|0\rangle \otimes |f(0)\rangle + |1\rangle \otimes |f(1)\rangle + |2\rangle \otimes |f(2)\rangle + \ldots)$$

- Massive "question-answer" entanglement
- But measurement creates some problem; need to be smart!

eg. measure 2nd $\Rightarrow f_0 \Rightarrow$ 1st register collapses to superposition of x such $f(x) = f_0$

- Simon's algorithm clearly illustrates this

Algorithm for Simon's Problem

1. ~~Set a counter~~ $i = 1$.
2. Prepare $\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle|\mathbf{0}\rangle$.
3. Apply U_f, to produce the state
$$\sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle|f(\mathbf{x})\rangle$$
4. (optional[2]) Measure the second register.
5. Apply $H^{\otimes n}$ to the first register.
6. Measure the first register and record the value $\mathbf{w}_i$.
7. If the dimension of the span of $\{\mathbf{w}_i\}$ equals $n - 1$, then go to Step 8, otherwise increment i and go to Step 2.
8. Solve the linear equation $\mathbf{W}\mathbf{s}^T = \mathbf{0}^T$ and let $\mathbf{s}$ be the unique non-zero solution.
9. Output $\mathbf{s}$.

$\perp s$

- Promised that a "hidden" string $s=s_1s_2..s_n$ such that $f(x)=f(y)$ if and only if $x=y$ or $x=y\oplus s$ (bitwise XOR)

➔ Find string s

$\rightarrow$ 2nd sec : f_0 $\rightarrow$ 1st superposition $\underbrace{(|x_0\rangle + |x_0 + s\rangle)}(|f_0\rangle)$

$H^{\otimes n}$

$\rightarrow \sim \sum_{z \perp s} |z\rangle$

$\downarrow$ measure $z \perp s$

Week 2: From foundation to science-fiction teleportation: Bell inequality, teleportation of states and gates, entanglement swapping, remote state preparation, superdense coding, and superdense teleportation

Quantum entangled states have correlations stronger than classical states $\longrightarrow$ Bell inequality

are useful as well $\rightarrow$ e.g. teleportation

A simple equality and an inequality

We have seen measurement of observables X, Y, Z or any one-qubit operator

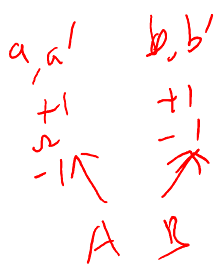
$$\vec{r}\cdot\vec{\sigma}, \quad \text{where}\, \vec{\sigma} \equiv (X, Y, Z), \quad |\vec{r}| = 1$$

gives an eigenvalue randomly, which is ±1 in this case.

- It is interesting that for four variables a, a', b, b' which can be ±1, we have:

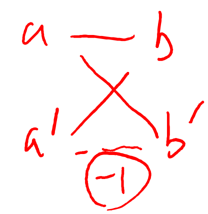
$$ab + ab' + a'b - a'b' = a(b + b') + a'(b - b') = \pm 2$$

- Thus, for any probability distribution p(a,a',b,b') we have [using **E** to denote expectation]



$$-2 \le \underbrace{\mathbf{E}(ab + ab' + a'b - a'b')}_{E(ab)} \equiv \sum_{a,a',b,b} \underbrace{\rho(a, a', b, b')}_{\text{distribution}}(ab + ab' + a'b - a'b') \le 2$$

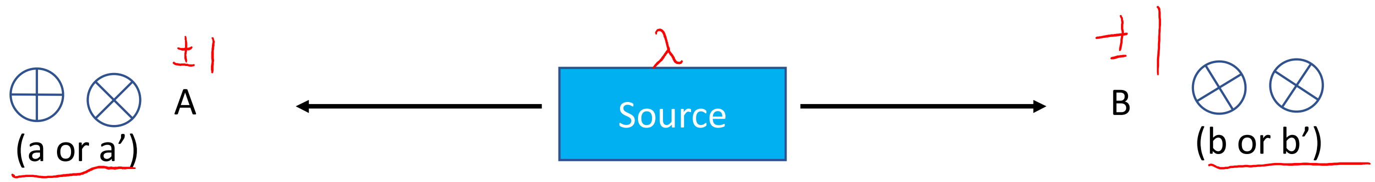
In the context of measuring **two choices of observables** at **two** locations A: a & a', B: b & b', we have the so-called Clauser-Horne-Shimony-Holt (CHSH) inequality:



$$-2 \le \mathbf{E}(a, b) + \mathbf{E}(a, b') + \mathbf{E}(a', b) - \mathbf{E}(a', b') \le 2$$

CHSH-Bell inequality (I_{2222})

CHSH generalized John Bell's idea (his original Bell inequality). The assumption is that a source emits e.g. a pair of photons



The choice of measurement axis (a or a') at A or (b or b') at B **cannot affect the outcome of the other side**. Nevertheless, **outcomes can be correlated** and described by some unknown-to-us distribution (depending on some hidden variable λ). This is also called the "Local hidden variable" theory

$$E_L(a,b) \equiv \int d\lambda\, \rho(\lambda) A(a,\lambda) B(b,\lambda)$$

distribution

where $A(a,\lambda) = \pm 1$ and $B(b,\lambda) = \pm 1$ are predetermined results for the measurement settings a for A and b for B depending on the local hidden variable λ; ρ(λ) is its distribution. Locality requires that the outcome A(a,λ) does not depend on setting b and that of B(b, λ) does not depend on setting a.

Violation of CHSH-Bell inequality

$a \quad a'$

$b \quad b'$

By averaging over the local hidden variable, we still have

$$|E_L(a,b) + E_L(a,b') + E_L(a',b) - E_L(a',b')| \le 2. \qquad (1)$$

repeat measurement process many times

Quantum mechanics can violate this inequality. To be specific, the operators to be measured are the Pauli operators $\vec{\sigma}$. Let $E_Q(a,b) \equiv \langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle$ denote expectation of repeated measurement along axes of unit vectors $\vec{a}$ and $\vec{b}$, respectively. Define

classical: $ab + ab' + a'b - a'b'$

classical $|\quad| \le 2$

$$\rightarrow \; 2B \equiv \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b} + \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b'} + \vec{\sigma}\cdot\vec{a'}\otimes\vec{\sigma}\cdot\vec{b} - \vec{\sigma}\cdot\vec{a'}\otimes\vec{\sigma}\cdot\vec{b'}.$$

For a singlet state $|\psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$,

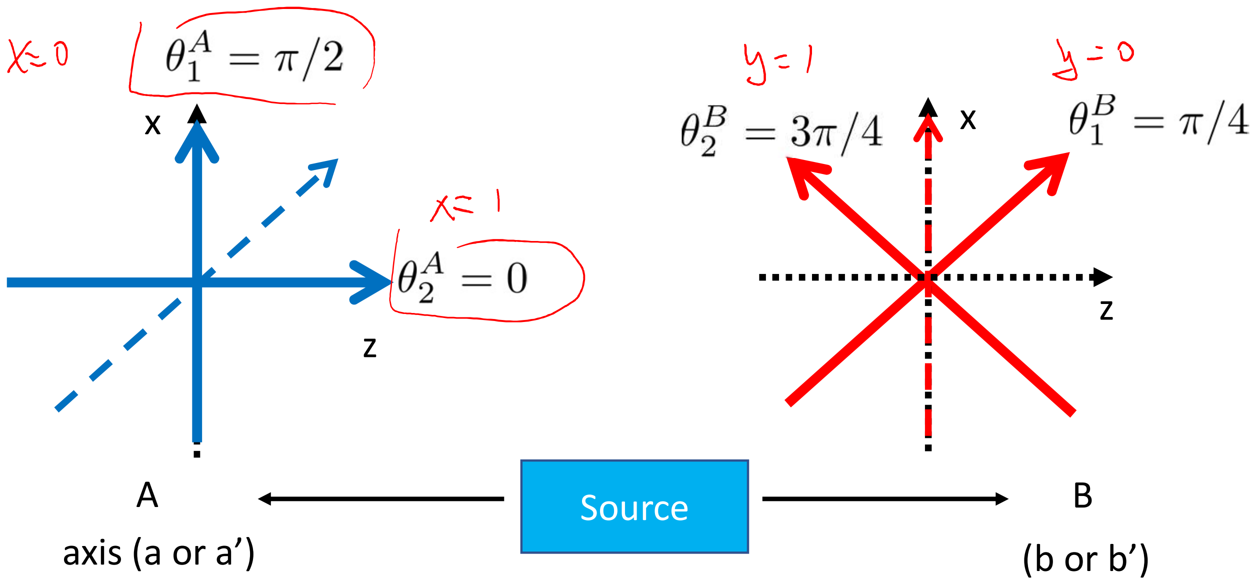
$$\max_{a,a',b,b'} |\langle\psi|2B|\psi\rangle| = 2\sqrt{2}, \quad \gtrsim 2.828\ldots > 2$$

which can be achieved for the settings $\theta_a = \pi/2$, $\theta'_a = 0$, $\theta_b = \pi/4$, and $\theta'_b = 3\pi/4$, where the angles are measured from the z-axis in the $z-x$ plane.

Violation of Bell inequality

- Measurement along axes 1 and 2 of A & B are used to check violation of Bell inequality

note: $\langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle = -\vec{a}\cdot\vec{b}$



- The bound 2√2 is the Tsirelson bound. Deriving maximal violation and measurement settings for an arbitrary state is a math problem; see Horodecki et al.

Phys. Lett. A 200, 340 (1995) and Phys. Lett. A 210, 223 (1996).

Exercise

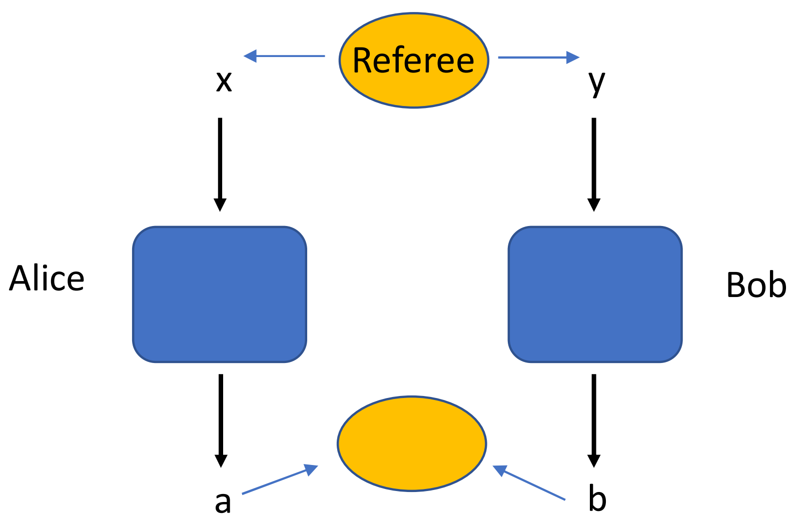
$$|\psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} = (|01\rangle - |10\rangle)/\sqrt{2}$$

$$\langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle = -\vec{a}\cdot\vec{b}$$

Verify that for the choices of measurement axes described earlier, we have

$$|\langle\psi|2B|\psi\rangle| = 2\sqrt{2}$$

Related inequality and a game between two players



(1) Referee gives separately bits x and y to Alice and Bob.

(2) Alice and Bob have to produce a bit respectively (a and b) without communication. They win if “x AND y = a⊕b”

x \ y	0	1
0	0	0
1	0	1

always output a=0, b=0

win 3/4

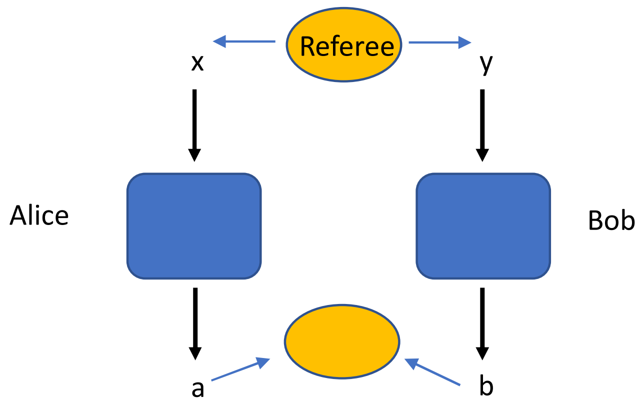
x = y = 1
& → 1 fail
a ⊕ b = 0

The probability P(a,b|x,y) is referred to as a “Box”. The question we are interested in is how to maximize the probability of winning, depending on the models for P.

1. For classical no-signaling theory: the max winning probability is <= 3/4
2. For quantum mechanics: the max winning probability is (2+sqrt(2))/4 ~ 0.8535

x = 0, 1 choice of a/a'
y = 0, 1 = b/b'
+1 ⇒ 0, −1 ⇒ 1

Popescu-Rohrlich (PR) box



Regard x and y ϵ{0,1} as representing two different measurement settings and a and b ϵ{0,1} as representing two different outcomes (i.e +1 and -1).

distribution / strategy

Popescu and Rohrlich propose a box that always achieves "x AND y = a⊕b" :

x=0, y=0

P(0,0|0,0)=P(1,1|0,0)=P(0,0|0,1)=P(1,1|0,1)
=P(0,0|1,0)=P(1,1|1,0)=P(0,1|1,1)=P(1,0|1,1)=1/2 ⇒ win 100%
(all other combinations are zero)

assign fixed a outcome prob does not change

This box obeys no-signaling conditions: P(0,0|0,0)+P(0,1|0,0)=P(0,0|0,1)+P(0,1|0,1), etc., but it enables violation of Bell-CHSH inequality and gives B=4.
[➔PR box has correlation stronger than quantum mechanics]

PR Box > QM > classical

even if b is differ

Do poll 8/31-(2)

Polls

Polling 7: 8/31-(2)

Edit

Polling is closed — 19 voted

1. What is correct about the CHSH-Bell inequality?

Classical (Hidden-Variable) Theory can achieve a value of 2	(0)	0%
Quantum Mechanics can violate it and achieve a value of 2 sqrt(2)	(3)	16%
Popescu-Rohrlich (PR) box can violate it and achieve a value of 4	(0)	0%
All of above	(16)	84%

Share Results

Re-launch Polling

GHZ state: violation at a single shot

We can generalize the Bell state Φ^+ to three particles and arrive at the Greenberger-Horne-Zeilinger state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

- ❑ Consider four commuting observables: (i) $X \otimes X \otimes X$, (ii) $Y \otimes Y \otimes X$, (iii) $Y \otimes X \otimes Y$, (iii) $X \otimes Y \otimes Y$

$$X \otimes X \otimes X|\mathrm{GHZ}\rangle = (+1)|\mathrm{GHZ}\rangle$$
$$Y \otimes Y \otimes X|\mathrm{GHZ}\rangle = (-1)|\mathrm{GHZ}\rangle$$
$$Y \otimes X \otimes Y|\mathrm{GHZ}\rangle = (-1)|\mathrm{GHZ}\rangle$$
$$X \otimes Y \otimes Y|\mathrm{GHZ}\rangle = (-1)|\mathrm{GHZ}\rangle$$

$|111\rangle + |000\rangle$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad |0\rangle \to i|1\rangle$

$|1\rangle \to -i|0\rangle$

$-|111\rangle - |000\rangle$

- ➢ For classical local theory, one attributes this to local properties:

  $x_1x_2x_3=+1$, $y_1y_2x_3=-1$, $y_1x_2y_3=-1$, $x_1y_2y_3=-1$ (where $x,y= \pm 1$)

  $1 = 1 \times (-1) \times (-1) \times (-1) = -1$?!

- ➢ But this gives contradiction when we multiply all four equalities together:

  1= -1 ! (experiments show QM is correct)

Quantum entangled states are useful

Quantum Teleportation

One of the most incredible tasks that an entangled pair allows is quantum teleportation. For illustration, we use the state Φ^+ to explain this. Suppose we have an arbitrary state ψ of particle 1 at A, who share the entanglement with B via Φ^+_{23}

$$|\psi\rangle_1 = a|0\rangle + b|1\rangle$$

$$|\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

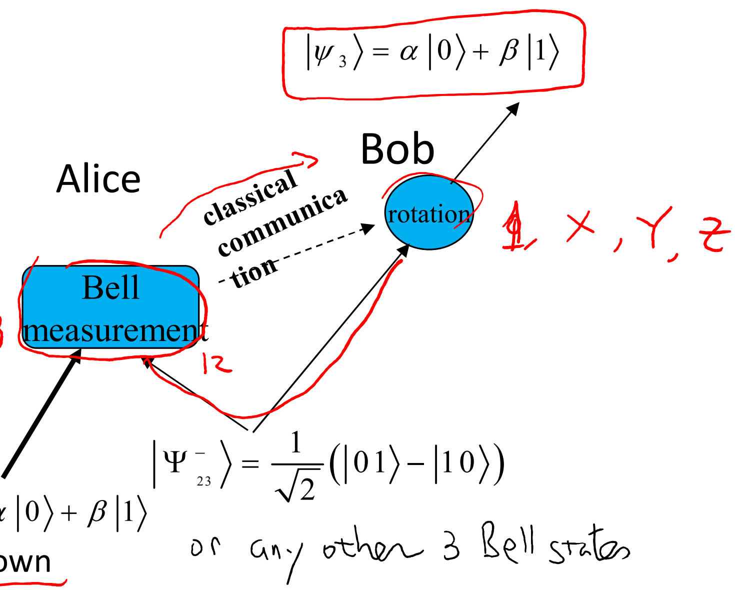
$$|\psi\rangle_1 \otimes |\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle) \otimes (|00\rangle + |11\rangle)$$

$= ($ [illegible] 1 $)_{12}$ $($ state $1)_3$

$+ (\; 2 \;) \; (\; 2 \;)$

$\vdots$

We derive it with equations.



Quantum Teleportation (analysis)

$$|\psi\rangle_1 \otimes |\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle) \otimes (|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)$$

$$= \frac{1}{2}\{a(|\Phi^+\rangle + |\Phi^-\rangle) \otimes |0\rangle + a(|\Psi^+\rangle + |\Psi^-\rangle) \otimes |1\rangle + b(|\Psi^+\rangle - |\Psi^-\rangle) \otimes |0\rangle + b(|\Phi^+\rangle - |\Phi^-\rangle) \otimes |1\rangle\}$$

$$= \frac{1}{2}\{\underbrace{|\Phi^+\rangle}_{(1)} \otimes \underbrace{(a|0\rangle + b|1\rangle)}_{I|\psi\rangle} + \underbrace{|\Phi^-\rangle}_{(2)} \otimes \underbrace{(a|0\rangle - b|1\rangle)}_{Z|\psi\rangle} + \underbrace{|\Psi^+\rangle}_{(3)} \otimes \underbrace{(a|1\rangle + b|0\rangle)}_{X|\psi\rangle} + \underbrace{|\Psi^-\rangle}_{(4)} \otimes \underbrace{(a|1\rangle - b|0\rangle)}_{iY|\psi\rangle}\}$$

$$|\Phi^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\Psi^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

$00 = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}$

$11 = \frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}}$

$01 = (|\Psi^+\rangle + |\Psi^-\rangle)/\sqrt{2}$

$10 = \frac{|\Psi^+\rangle - |\Psi^-\rangle}{\sqrt{2}}$

$iY = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

state

measurement

four outcomes

each $P = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

The unknown information a & b is preserved in the third particle, but depending on the outcome of the 'Bell-state' measurement in the basis of $\Phi^\pm$ & $\Psi^\pm$

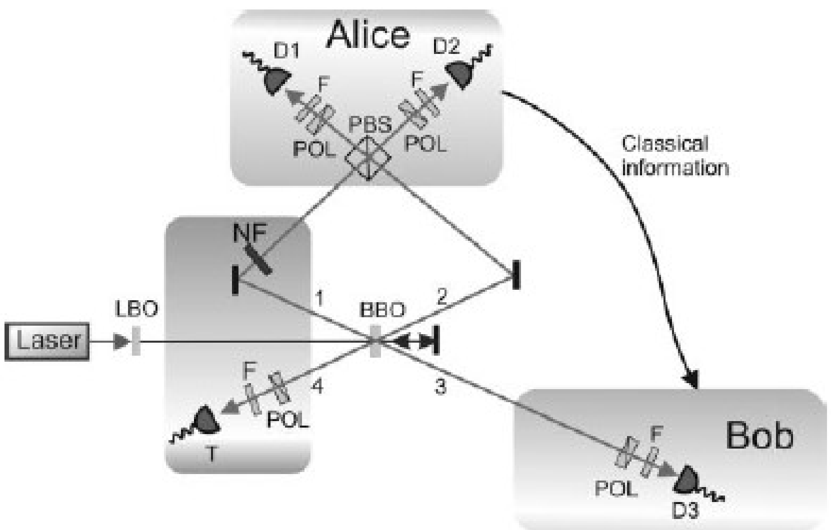
$X^2 = Y^2 = Z^2 = 1$

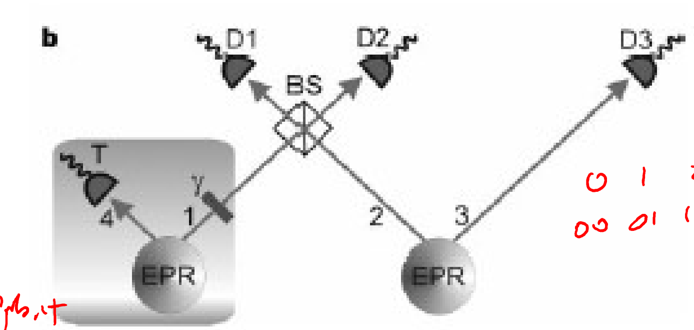
Four possible outcomes, Alice informs Bob: (1) Φ^+ → apply identity (nothing); (2) Φ^- → apply Z to particle 3; (3) Ψ^+ → apply X to particle 3; (4) Ψ^- → apply –iY to particle 3

→ Recover ψ at particle 3

Teleportation experiment

[Pan et al. ’03, Bouwmeester et al. ’97]





0 1 2 3

00 01 10 11

n qubit

n pairs

each requires 2 bits

⇒ total classical comm is 2n bits

Exercise: Teleportation for qudits

For two d-level qudits, there are dxd basis states and also dxd entangled basis states:

$$|\Psi_{nm}\rangle = \sum_{j=0}^{d-1} e^{i2\pi jn/d} |j\rangle \otimes |(j+m) \operatorname{mod} d\rangle$$

For an arbitrary qudit state $|\psi\rangle_1 = \sum_k c_k |k\rangle$

The shared entanglement that we will use is $|\Psi_{00}\rangle_{23}$

Suppose Alice performs measurement on particles 1&2 in the basis defined by Ψ_{nm}

If she obtains the outcome nm, what action needs Bob to perform to recover ψ?

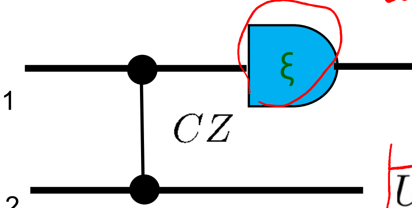
$$U_{nm} = \sum_k e^{i2\pi kn/d} |k\rangle\langle (k+m) \operatorname{mod} d|$$

FYI: A variant---gate teleportation*

Controlled-Z gate and single-qubit measurement induces rotation

measurement

$|\text{in}\rangle = (a|0\rangle + b|1\rangle)_1$



CZ

$|+\rangle_2$

+ or -

s = 0, 1

$$CZ_{12} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

00 → 00
01 → 01
10 → 10
11 → −11

X

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

rotation

$U(\xi, s)|\text{in}\rangle_2$

$$U(\xi, s) = H e^{i\xi Z/2} Z^s$$

Hadamard

0 or 1

1 Z

(3rd rule QM)

The measurement basis ξ is defined via

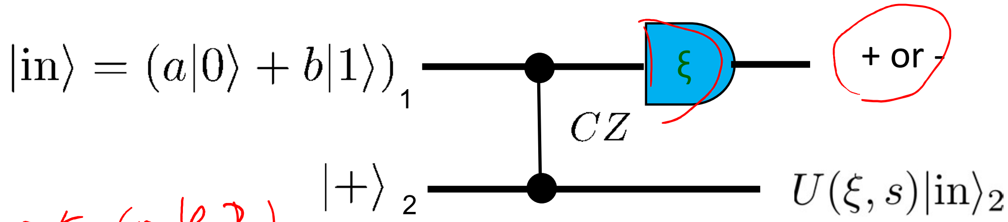
$$|\pm\xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle)/\sqrt{2}$$

or the observable:

$$\cos(\xi)\underset{\underset{X}{\equiv}}{\sigma_x} + \sin(\xi)\underset{\underset{Y}{\equiv}}{\sigma_y} = \begin{pmatrix} 0 & e^{-i\xi} \\ e^{i\xi} & 0 \end{pmatrix}$$

$|\pm\xi\rangle = \pm|\pm\xi\rangle$

Derivation*



The measurement basis ξ is defined via

$$|\pm\xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle)/\sqrt{2}$$

$$\mathrm{CZ}_{12} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

gate (rule 2)

① $|\Psi\rangle_{12} = (a|0\rangle + b|1\rangle)\frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \longrightarrow a|0\rangle(|0\rangle+|1\rangle) + b|1\rangle(|0\rangle - |1\rangle)$

$= a|0\rangle\otimes|+\rangle + b|1\rangle\otimes|-\rangle \Rightarrow |\Psi_{12}\rangle$

② measurement : proj to eigenstates $|\pm\xi\rangle = \frac{1}{\sqrt{2}}(e^{-i\xi/2}|0\rangle + e^{i\xi/2}|1\rangle)$

$$\langle\pm\xi|_1 \cdot |\Psi_{12}\rangle = \frac{1}{\sqrt{2}}\left(e^{i\xi/2}\langle 0|_1 \pm e^{-i\xi/2}\langle 1|_1\right)\left(a|0\rangle_1 \otimes |+\rangle + b|1\rangle_1 \otimes |-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(a\,\underbrace{e^{i\xi/2}|+\rangle_2}_{H|0\rangle} \pm b\,\underbrace{e^{-i\xi/2}|-\rangle_2}_{H|1\rangle}\Big) = \frac{1}{\sqrt{2}} H e^{i\frac{\xi}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}(a|0\rangle \pm b|1\rangle)$$

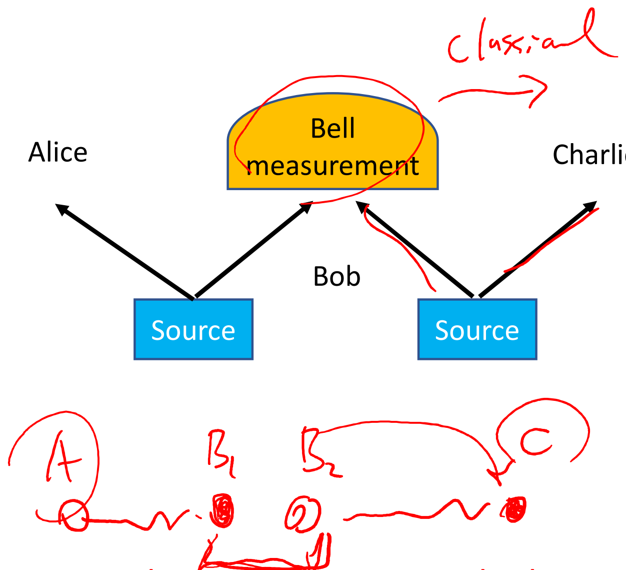
$s = (-1)$ or $(-1)^1$

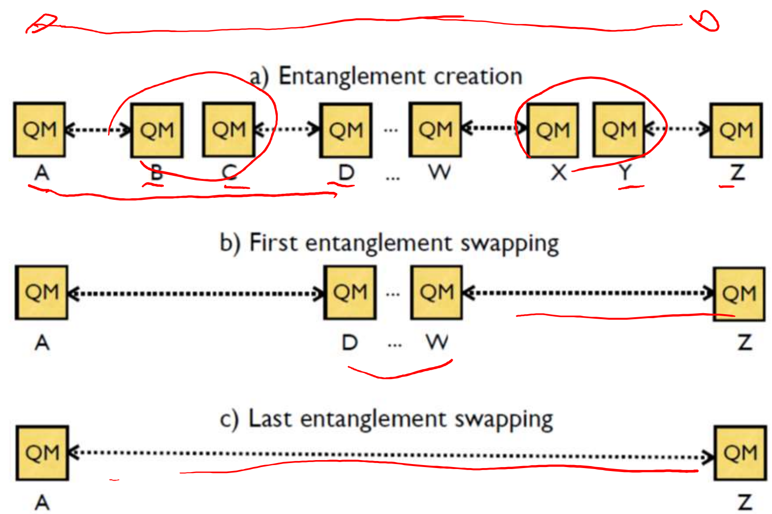
$$= \frac{1}{\sqrt{2}} H e^{i\frac{\xi}{2}Z} Z^s (a|0\rangle + b|1\rangle)$$

$= \frac{1}{\quad}$

Entanglement swapping (via teleportation)

Imagine that Alice and Bob share an entangled pair and Bob and Charlie share another entangled pair. By performing the Bell-state measurement on Bob's two particles, Bob 'teleports' his entanglement with Charlie to Alice (or equivalently, Bob 'teleports' his entanglement with Alice to Charlie). This results in shared entanglement between Alice and Charlie.





- Entanglement swapping is the basic protocol to establish entanglement between distant nodes (such as the Duan-Luken-Cirac-Zoller with atomic ensemble quantum memory)

Remote state preparation (*)

It uses shared entanglement, e.g. the singlet state (which is antisymmetric):

$$\boxed{|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)}$$

From the antisymmetry, one sees that for any single qubit state ψ and its orthogonal $\psi^\perp$:

$$|\Psi^-\rangle = e^{i\theta}\frac{1}{\sqrt{2}}(|\psi\rangle \otimes |\psi^\perp\rangle - |\psi^\perp\rangle \otimes |\psi\rangle)$$

If Alice performs measurement on her particle in the basis $\{\psi,\psi^\perp\}$, with probability 1/2, she obtains $\psi^\perp$ and thus **prepares** Bob's state in ψ, and similarly with probability 1/2 prepares Bob's state in $\psi^\perp$

$(\psi, \psi^\perp)$

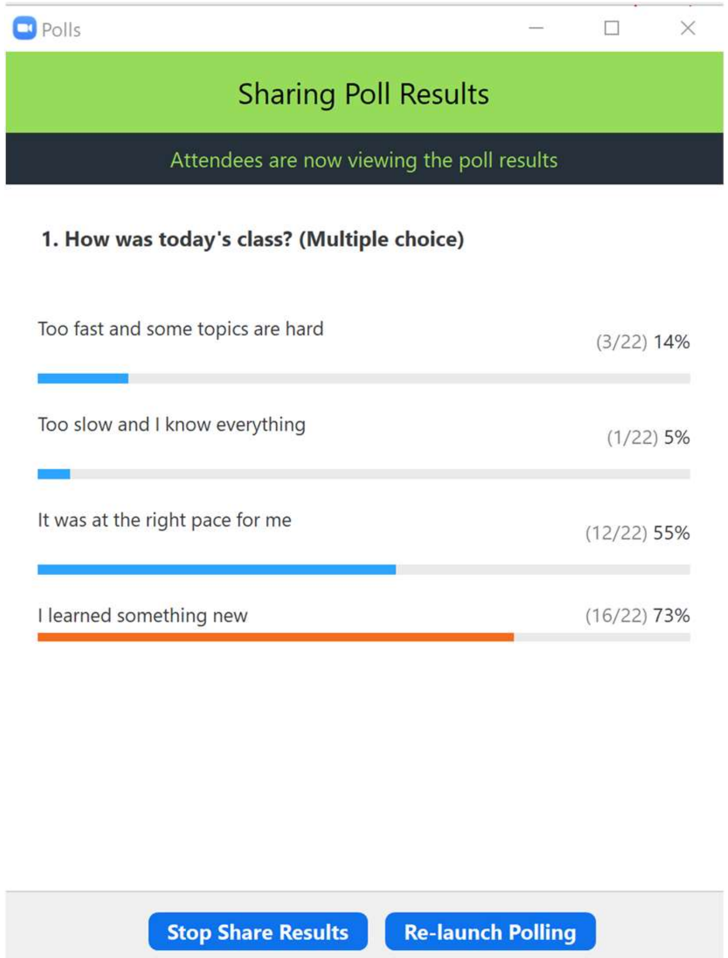
In the latter case, it is in general impossible for Bob to transform from in $\psi^\perp$ to ψ, except for 'equatorial states'

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)}$$

A: $\psi \rightarrow$ B $\psi^\perp$

$\psi^\perp \rightarrow \psi$

rotate


Polls
Sharing Poll Results
Attendees are now viewing the poll results
1. How was today's class? (Multiple choice)
Too fast and some topics are hard
(3/22) 14%
Too slow and I know everything
(1/22) 5%
It was at the right pace for me
(12/22) 55%
I learned something new
(16/22) 73%
Stop Share Results
Re-launch Polling