# PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

Today 10/12:

1. Reminder: Homework 4 due Sunday 11:59pm 10/18
2. Today: Ising anyons and quantum computation; Kitaev chain and Majorana zero modes
3. Week 8: Magic state distillation and surface code

# Simple review question

Which model has anyons 1, e, m and f? e.g *e and m fuse to f*

Which anyon model has anyons 1 and τ ? $\tau \times \tau = 1 + \tau$

Which anyon model has anyons 1, ψ, σ? $\psi \times \psi = 1, \ \psi \times \sigma = \sigma, \ \sigma \times \sigma = 1 + \psi$

# Review: Ising anyons

- Anyons: 1, ψ, σ
- Fusion: $1 \times 1 = 1,\ 1 \times \psi = \psi,\ 1 \times \sigma = \sigma$

$$\psi \times \psi = 1,\ \psi \times \sigma = \sigma,\ \sigma \times \sigma = 1 + \psi$$

- Physical picture: 1 is condensate of Cooper pairs, ψ Bogoliubov fermion, σ Majorana zero mode bound to a vortex

Cooper pair

$\psi - \psi$

$\psi$ excitation lost energy to break Cooper pair

zero mode

- Qubits?

$\sigma + \sigma$

$$\sigma \times \sigma \times \sigma = (1+\psi) \times \sigma = 2 \cdot \sigma$$

$$\sigma \times \sigma \times \sigma \times \sigma = 2 \cdot 1 + 2 \cdot \psi$$

$$\sigma \times \sigma \times \sigma \times \sigma \times \sigma = 4 \cdot \sigma$$

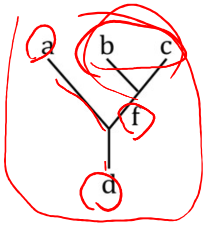
$$(\sigma \times \sigma \times \sigma \times \sigma \times \sigma) \times \sigma = 4 \cdot \sigma \times \sigma = 4 \cdot 1 + 4 \cdot \psi$$

➔ one qubit (not so practical, as final fusion product is σ )

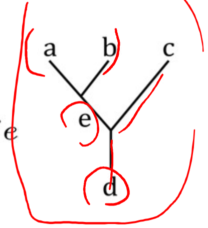
$$\{|(\sigma\sigma)\sigma;1\sigma;\sigma\rangle, |(\sigma\sigma)\sigma;\psi\sigma;\sigma\rangle\}$$

➔ 2n σ can encode n-1 qubits (assume fused to vacuum)

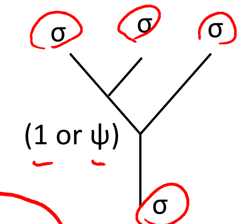
# Basis change, exchange and gates




$$= \sum_e (F^d_{abc})_{fe}$$



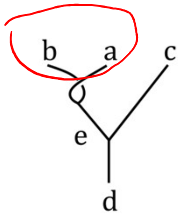

Basis: $\{|(\sigma\sigma)\sigma;1\sigma;\sigma\rangle, |(\sigma\sigma)\sigma;\psi\sigma;\sigma\rangle\}$



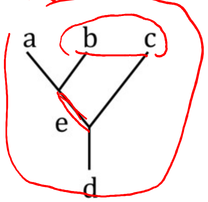

$$F = F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow$$

Hadamard

Z <—> X




$$= R^e_{ab}$$



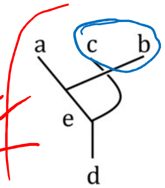

active action

$$R = \begin{pmatrix} R^1_{\sigma\sigma} & 0 \\ 0 & R^{\psi}_{\sigma\sigma} \end{pmatrix} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$$

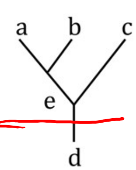
~ phase gate S

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

if we want to ~~...~~ braid b & c




$$= (F^d_{abc})^{-1} R (F^d_{abc})$$




$$F^{-1} R F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$F^{-1} R^2 F = e^{-i\pi 4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

~ NOT gate X

braiding 2 & 3 (b & c)

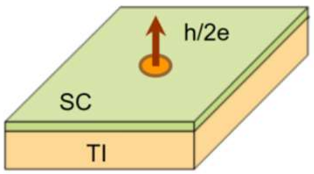
$(F^{-1} R F)$

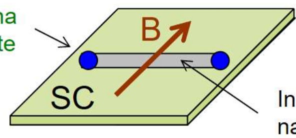

➔ It turns out that Ising anyons can implement only Clifford gates! ☹

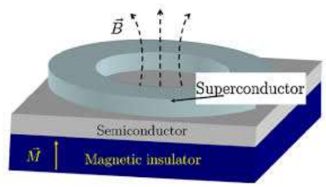
Other non-topologically protected methods for non-Clifford gates

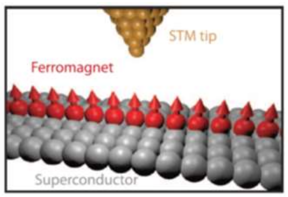
# Possible realizations

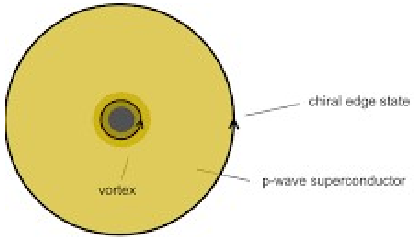
## Topological superconductors



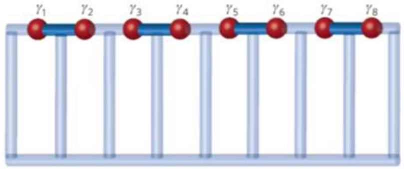



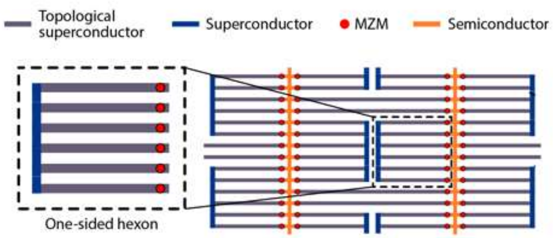




- ❑ TQC using Majorana zero modes




Alicea et al.,Nat Phys (2011)




Karzig et al.,PRB (2017)

# Two qubits using six Ising anyons

$$\sigma \times \sigma = 1 + \psi$$

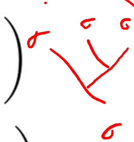
$$\sigma \times \sigma \times \sigma = 2 \cdot \sigma$$

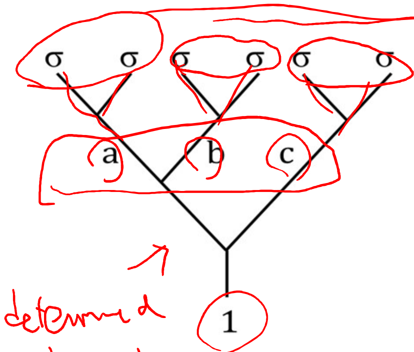
$$\sigma \times \sigma \times \sigma \times \sigma = 2 \cdot 1 + 2 \cdot \psi$$

$$\sigma \times \sigma \times \sigma \times \sigma \times \sigma \times \sigma = 4 \cdot 1 + 4 \cdot \psi$$

2n σ can encode
n-1 qubits (assume fused to 1)

vs.






σ σ → 1

➔ Basis states:

$$|00\rangle = |\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle,$$
$$|10\rangle = |\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle,$$
$$|01\rangle = |\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle,$$
$$|11\rangle = |\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle,$$

$$R = \begin{pmatrix} R^{1}_{\sigma\sigma} & 0 \\ 0 & R^{\psi}_{\sigma\sigma} \end{pmatrix} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$$

$$F = F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$F^{-1}R^2F = e^{-i\pi 4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

➢ Single-qubit gates:

$$X_1 = R_{23}^2 = F^{-1}R^2F \otimes \mathbb{I}, \qquad Z_1 = R_{12}^2 = R^2 \otimes \mathbb{I},$$
$$X_2 = R_{45}^2 = \mathbb{I} \otimes F^{-1}R^2F, \qquad Z_2 = R_{56}^2 = \mathbb{I} \otimes R^2$$
$$U_{H,1} = R_{12}R_{23}R_{12} = RF^{-1}RFR \otimes \mathbb{I},$$
$$U_{H,2} = R_{56}R_{45}R_{56} = \mathbb{I} \otimes RF^{-1}RFR,$$

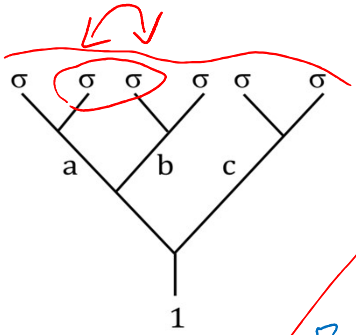
"real" act as Hadamard

➢ Two-qubit CZ gate:

$$U_{CZ} = R_{12}^{-1}R_{34}R_{56}^{-1}.$$

3 braiding

# Equivalent representations




➔ Basis states:

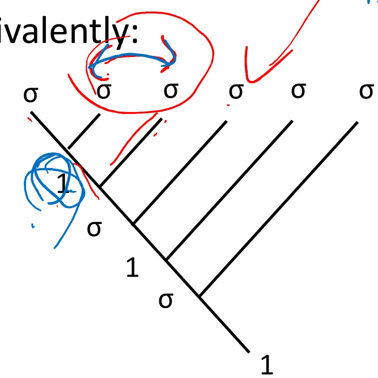
$$
\begin{aligned}
|00\rangle &= |\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,,\\
|10\rangle &= |\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,,\\
|01\rangle &= |\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,,\\
|11\rangle &= |\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,,
\end{aligned}
$$

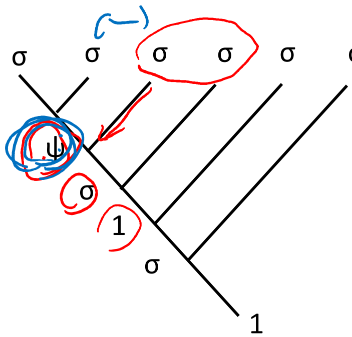
| | a | b | c |
|---|---|---|---|
| **\|00 ⟩** | 1 | 1 | 1 |
| **\|10 ⟩** | $\psi$ | $\psi$ | 1 |
| **\|01 ⟩** | 1 | $\psi$ | $\psi$ |
| **\|11 ⟩** | $\psi$ | 1 | $\psi$ |

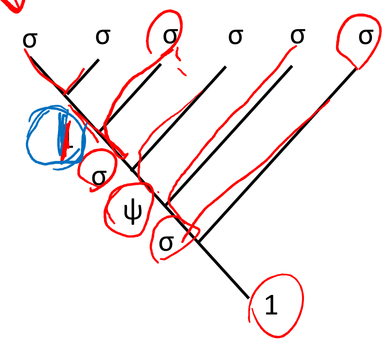
Braiding 2&3 induces a matrix that mixes $|00\rangle$ & $|10\rangle$

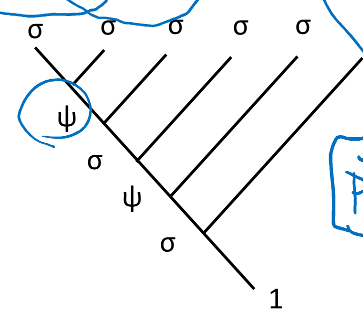
also $|01\rangle$ & $|11\rangle$

Equivalently:




$\boxed{F^{-1} R F} \otimes \mathbb{1}$

$|00\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;1\rangle|\sigma\sigma;1\rangle$ $\quad |10\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle$ $\quad |01\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle$ $\quad |11\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle$

$\sigma\sigma \quad \sigma\sigma$

$\psi \quad \psi$

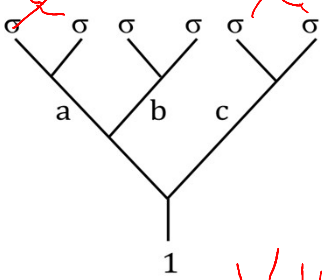
$\psi \times \sigma = \sigma$ $\qquad \sigma \times \sigma = 1$

➔ Useful for understanding gate operations

# Gates in pictures

$X_1$ $X_2$ $Z_2$

$Z_1$




$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ braiding 1&2 ; braiding 5&6 $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ → braid 2x ⇒ $Z_2$

$S_2$

$S_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

➔ Basis states:

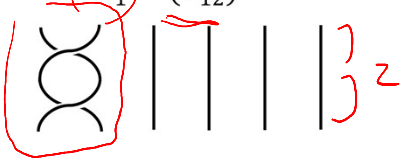
$$|00\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;1\rangle|\sigma\sigma;1\rangle,$$
$$|10\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle,$$
$$|01\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle,$$
$$|11\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle,$$

braiding 3 & 4 $\begin{pmatrix} 1 & & & \\ & i & & \\ & & i & \\ & & & 1 \end{pmatrix}$

| | a | b | c |
|---|---|---|---|
| **\|00⟩** | 1 | 1 | 1 |
| **\|10⟩** | $\psi$ | $\psi$ | 1 |
| **\|01⟩** | 1 | $\psi$ | $\psi$ |
| **\|11⟩** | $\psi$ | 1 | $\psi$ |

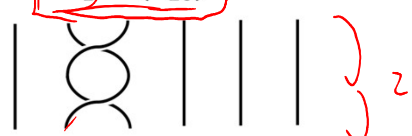
## Single-qubit gates:

$Z_1 = (R_{12})^2$



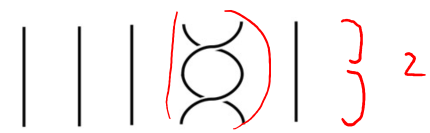

$$R = \begin{pmatrix} R^1_{\sigma\sigma} & 0 \\ 0 & R^\psi_{\sigma\sigma} \end{pmatrix}$$
$$= e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$$

$X_1 = (R_{23})^2$




$$F^{-1}R^2F = e^{-i\pi 4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X_2 = R^2_{45} = \mathbb{I} \otimes F^{-1}R^2F$$




$R \sim \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

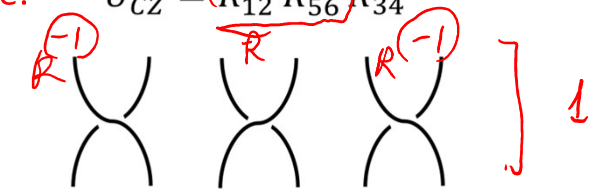
$R^{-1} \sim \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$

mirror sym in diagram

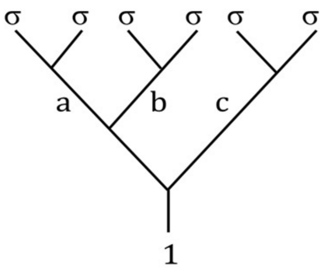
$U_{H,1} = R_{12}R_{23}R_{12}$ $\quad$ $U_{H,2} = R_{56}R_{45}R_{56}$


$$SHSHS = e^{i\pi/4}H$$

→ Two-qubit CZ gate: $U_{CZ} = R^{-1}_{12}R^{-1}_{56}R_{34}$



# Equivalent representations



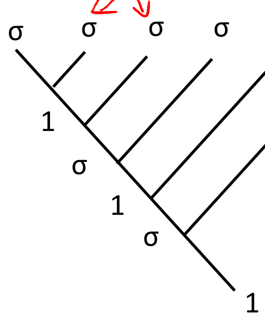

➔ Basis states:

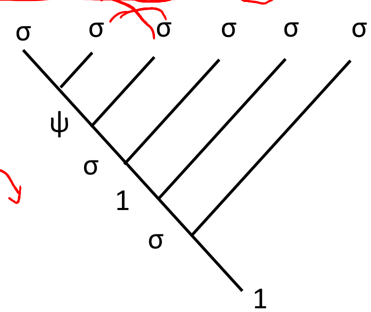
$$
\begin{aligned}
|00\rangle &= |\sigma\sigma;1\rangle|\sigma\sigma;1\rangle|\sigma\sigma;1\rangle, & 1 \times 1 \times 1 &= 1 \\
|10\rangle &= |\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle, & 1 \times i \times (-i) &= 1 \\
|01\rangle &= |\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle, & -i \times i \times 1 &= 1 \\
|11\rangle &= |\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle, & (-i) \times 1 \times (-i) &= -1
\end{aligned}
$$

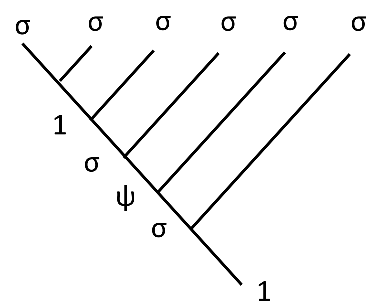
$R_{12}^{-1}$ $R_{34}$ $R_{56}^{-1}$

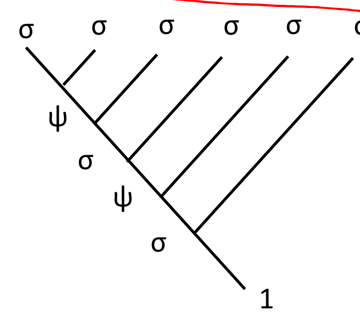
we get $CZ = (1, 1, 1, -1)$

Equivalently:



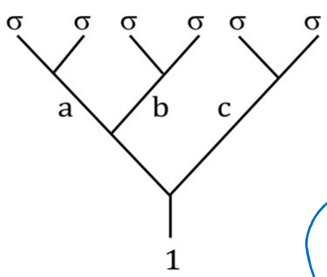




$|00\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;1\rangle|\sigma\sigma;1\rangle$ $|10\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle$ $|01\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle$ $|11\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle$

# Equivalent representations

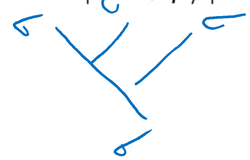



$F^{\sigma}_{\sigma\sigma\sigma}$

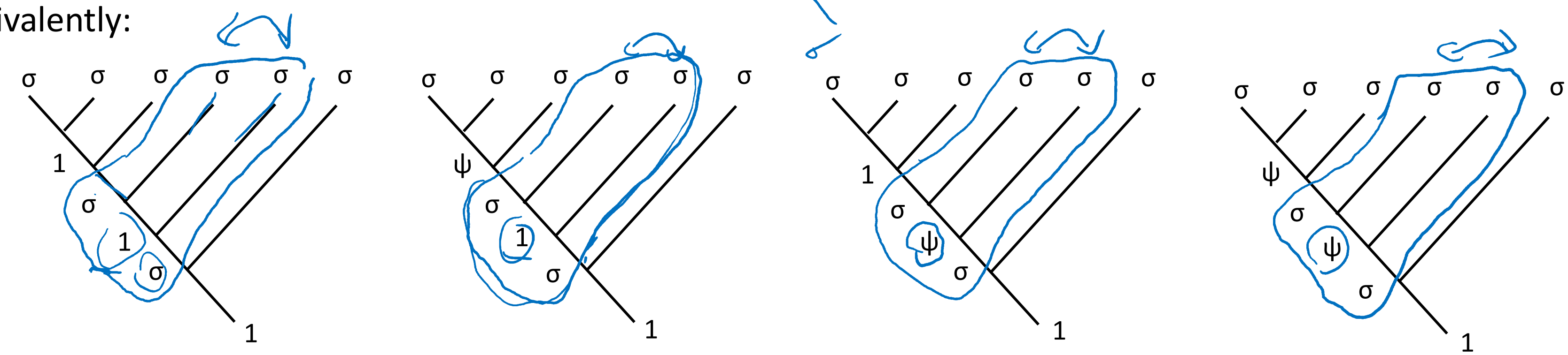
➔ Basis states:

$$
\begin{aligned}
|00\rangle &= |\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,, \\
|10\rangle &= |\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,, \\
|01\rangle &= |\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,, \\
|11\rangle &= |\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,,
\end{aligned}
$$

| | a | b | c |
|---|---|---|---|
| **$\|00\rangle$** | 1 | 1 | 1 |
| **$\|10\rangle$** | $\psi$ | $\psi$ | 1 |
| **$\|01\rangle$** | 1 | $\psi$ | $\psi$ |
| **$\|11\rangle$** | $\psi$ | 1 | $\psi$ |



Equivalently:




$|00\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;1\rangle|\sigma\sigma;1\rangle$ $|10\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle$ $|01\rangle = |\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle|\sigma\sigma;\psi\rangle$ $|11\rangle = |\sigma\sigma;\psi\rangle|\sigma\sigma;1\rangle|\sigma\sigma;\psi\rangle$

00 10 mixing 01 11 $\mathbb{1} \otimes F^{-1} R^2 F$

⇒ should make a homework exercise

# Topologically protected gates from Ising anyons are Clifford gates

Recall:

- Theorem 10.7: (Gottesman–Knill theorem) Suppose a quantum computation is performed which involves only the following elements: **state preparations in the computational basis, Hadamard gates, phase gates, controlled-NOT gates, Pauli gates, and measurements of observables in the Pauli group** (which includes measurement in the computational basis as a special case), together with the possibility of classical control conditioned on the outcome of such measurements.

→ Such a computation may be efficiently simulated on a classical computer.

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

CNOT

$\parallel$

$(I \otimes H)\, CZ\, (I \otimes H)$

closer

- One naïve solution (non-topological ☹): bring two anyons closer to induce interaction and energy shift;
  → Some physical-system dependent improvements

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\Delta E t} \end{pmatrix}$$

not protected

- Magic state distillation for non-Clifford gates (later)

# Initialization and Readout in Ising-anyon qubits

$$
\begin{aligned}
|00\rangle &= |\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,, \\
|10\rangle &= |\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,, \\
|01\rangle &= |\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle\,, \\
|11\rangle &= |\sigma\sigma;\psi\rangle\,|\sigma\sigma;1\rangle\,|\sigma\sigma;\psi\rangle\,,
\end{aligned}
$$

- **Initialization/Preparation** of e.g. |00> state: Assume that σ anyons are created pairwise from the vacuum with no other anyons present

- **Readout**

2 → fuse

1: no change in energy

ψ: change in energy

5 6

fuse

⇓

detect energy

no change: "1" → logical 0

change: "ψ" → logical 1

  ✓ Z-basis measurement of the first qubit: detecting the fusion outcome of anyons 1 and 2. If no change in energy is detected, then → 0 state; if observe change in energy, then → 1 state

  ✓ Z-basis measurement of the 2nd qubit: detecting the fusion outcome of anyons 5 and 6

  ✓ X-basis measurement of qubit 1: detecting the fusion outcome of anyons 2 and 3.
  X-basis measurement of qubit 2: detecting the fusion outcome of anyons 4 and 5.
  (or one can apply appropriate Hadamard gate before Z measurement)

# Kitaev chain

**HW3 Problem 3**.
$$H = -\sum_{x=1}^{N-1} (\hat{c}_x^\dagger \hat{c}_{x+1} + \hat{c}_x \hat{c}_{x+1} + \hat{c}_{x+1}^\dagger \hat{c}_x + \hat{c}_{x+1}^\dagger \hat{c}_x^\dagger) = -i \sum_{x=1}^{N-1} \hat{\gamma}_{B,x} \hat{\gamma}_{A,x+1}$$


show this

$$\hat{c}_x = (\hat{\gamma}_{B,x} + i\hat{\gamma}_{A,x})/2 \qquad \hat{c}_x^\dagger = (\hat{\gamma}_{B,x} - i\hat{\gamma}_{A,x})/2$$

➔ Unpaired zero modes at ends (effectively one non-local Dirac fermion)

no trivial topological phase

➔ By Jordan-Wigner transformation, it is mapped to a ferromagnetic chain: $\boxed{H = -\sum_j \sigma_j^y \sigma_{j+1}^y}$

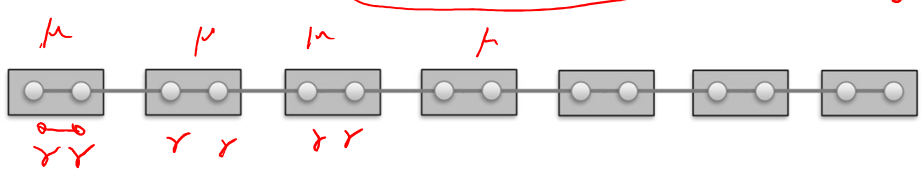
Ising chain

$\to \uparrow_y \uparrow_y \uparrow_y \uparrow_y$

$\sim \downarrow_y \downarrow_y \cdots \downarrow_y$

trivial phase

❑ Example of trivial phase: $H = \sum_{x=1}^{N-1} -\mu\,(c_x^\dagger c_x - \frac{1}{2})$ $\to \sim \sum \gamma_{Bx} \gamma_{Ax}$



# Kitaev chain: general Hamiltonian

(p-wave) Cooper pairings

$$H = \sum_{x=1}^{N-1} -t(\hat{c}_x^\dagger \hat{c}_{x+1} + \hat{c}_{x+1}^\dagger \hat{c}_x) + \Delta(\hat{c}_x \hat{c}_{x+1} + \hat{c}_{x+1}^\dagger \hat{c}_x^\dagger) - \mu c_x^\dagger c_x$$

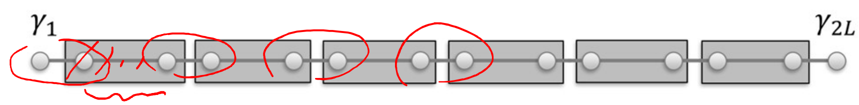
t

x  t  x+1

Parameters--- t: hopping, Δ: p-wave pairing, μ: chemical potential (e.g. gate voltage)

## ❑ Two phases:

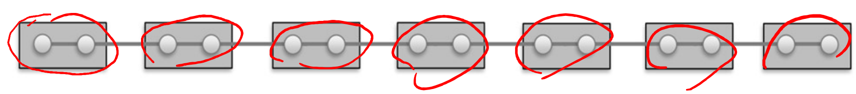
➢ when (|2t|>|μ|) → topological [localized modes decay exponentially]

[representative] 

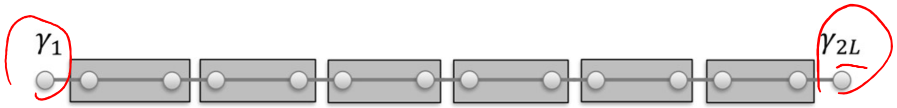

entglent

a gate voltage can be used to tune phases

➢ when (|2t|<|μ|) → topologically trivial

[representative] 

# Fermion parity and qubit encoding




$\hat{d} = (\gamma_1 + i\gamma_{2L})/2$ Fermion parity : $P_f = (1 - 2\hat{d}^\dagger \hat{d}) = -i\gamma_1\gamma_{2L}$ n = 0 n = 1

fermion # = $d^\dagger d$ , parity +1 −1

n

$$i\gamma_1\gamma_{2L}\,|\sigma\sigma;1\rangle = -|\sigma\sigma;1\rangle\,, \qquad i\gamma_1\gamma_{2L}\,|\sigma\sigma;\psi\rangle = +|\sigma\sigma;\psi\rangle$$

$-P_f$ $-P_f$

➔ Of different parity, thus not useful for qubit encoding

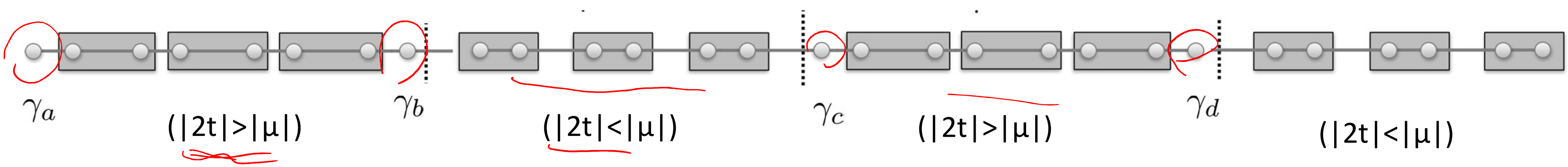
❑ Can use four σ's with total parity +1: $\sigma \times \sigma \times \sigma \times \sigma = 2 \cdot 1 + 2 \cdot \psi$

$$|0\rangle \equiv |\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,, \qquad |1\rangle \equiv |\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle$$

# Segments of Kitaev chains for qubits

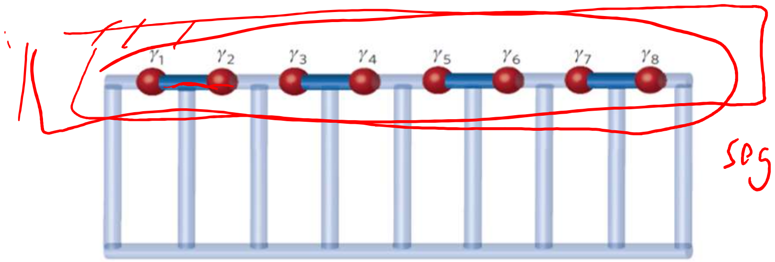
$$H = \sum_{x=1}^{N-1} -t(\hat{c}_x^\dagger \hat{c}_{x+1} + \hat{c}_{x+1}^\dagger \hat{c}_x) + \Delta(\hat{c}_x \hat{c}_{x+1} + \hat{c}_{x+1}^\dagger \hat{c}_x^\dagger) - \mu\, c_x^\dagger c_x$$

- Use position-dependent chemical potential μ to tune the systems into segments



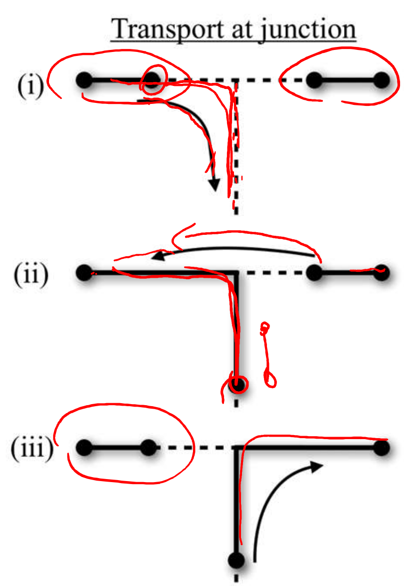

$$|0\rangle \equiv |\sigma\sigma;1\rangle\,|\sigma\sigma;1\rangle\,, \qquad |1\rangle \equiv |\sigma\sigma;\psi\rangle\,|\sigma\sigma;\psi\rangle$$

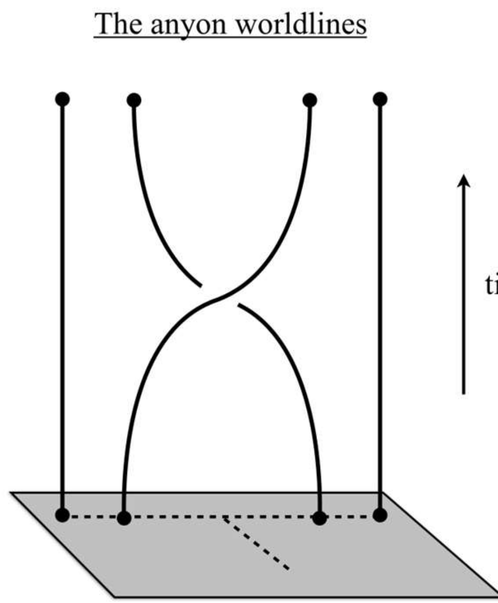
# Braiding Majorana “fermions” using T-junctions




Alicea et al.,Nat Phys (2011)

segments created by tuning local gates

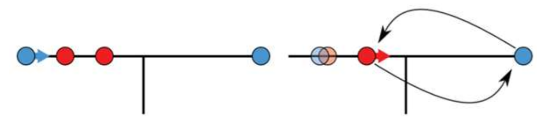



Lahtinen & Pachos, arxiv:1705.04103

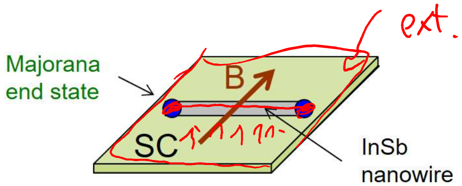
# Challenges using Majorana zero modes for QC

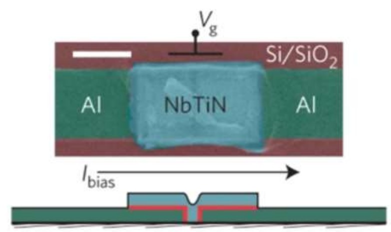
- ❑ Presence of unwanted anyons (or Majorana modes)
  → use Josephson-charging energy switching braiding protocol



- ❑ Leakage to an external reservoir: For closed system, Majorana qubits protected by fermion parity
  But contact with S-wave superconductor or can lead to tunneling of Cooper pair and Bogoliubov particles

ext. res.




➔ heterostructures for topological nanowires exhibit long "contaminating" times

1 minutes

- ❑ Finite-temperature issue → use engineered p-wave wire (rather the intrinsic topological one): energy gap protects the localized states on the domain walls from extended states

$e^{-\Delta E/T}$

- ❑ Note that there is strong evidence for Majorana modes, but braiding has not been realized