

PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

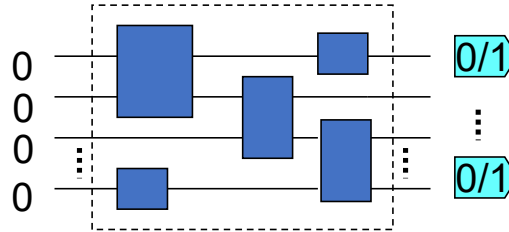
Today 10/21:

1. HW5 due 11:59pm Nov. 1st (selection of presentation topics, and exercises on surface code and Ising anyon gates)
2. Today: Quantum computing by measurement
(a.k.a. [Measurement-based quantum computation](#))

(Frameworks of) Quantum Computation

✓ Previously covered:

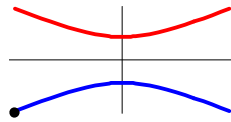
I. Circuit:



✓ Major scheme by most labs: IBM, Intel Rigetti, IonQ, Alibaba

✓ Done with this:

II. Adiabatic:

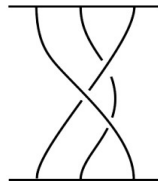


$$H(t) = \left(1 - \frac{t}{T}\right) H_{\text{initial}} + \frac{t}{T} H_{\text{final}}$$

✓ Approach by D-Wave

✓ Previously covered:

III. Topological:

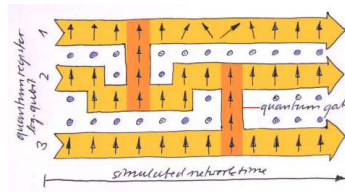


quantum gates = braiding anyons

✓ Approach by Microsoft, Google uses a hybrid of III and I (circuit version of IV)

Today:

IV. Measurement-based:



local measurement is the only operation needed

✓ Used in photonic systems, such as PsiQuantum

References on Measurement-based QC

[My article for Oxford Research Encyclopedia in Physics \(posted to Blackboard\)](#)

Jozsa, R. (2006). An introduction to measurement based quantum computation. NATO Science Series, III: Computer and Systems Sciences. Quantum Information Processing-From Theory to Experiment, 199, 137–158.

Briegel, H. J. (2009), "Cluster States", In Compendium of Quantum Physics - Concepts, Experiments, History and Philosophy. Springer. pp. 96–105. (eds. D. Greenberger, K. Hentschel, F. Weinert).

Briegel, H. J., Browne, D. E., Dür, W., Raussendorf, R., & Van den Nest, M. (2009). Measurement-based quantum computation. Nature Physics, 5(1), 19–26.

Raussendorf, R., & **Wei, T.-C.** (2012). Quantum computation by local measurement. Annu. Rev. Condens. Matter Phys., 3(1), 239–261.

Kwek, L. C., Wei, Z., & Zeng, B. (2012). Measurement-based quantum computing with valence-bond solids. International Journal of Modern Physics B, 26(02), 1230002.

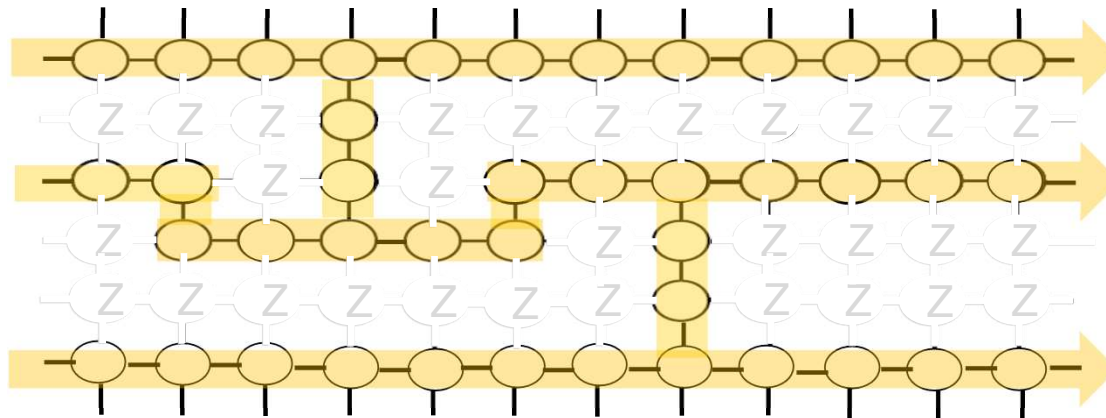
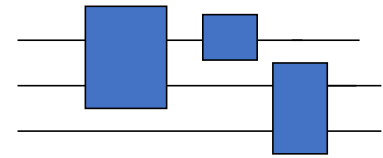
Wei, T.-C. (2018). Quantum spin models for measurement-based quantum computation. Advances in Physics: X, 3(1), 1461026.

Browne, D. E. & Briegel, H. J. (2019). One-Way Quantum Computation. In Quantum Information (eds D. Bruß and G. Leuchs), Wiles and Sons Ltd.

QC by Local Measurement---an overview picture

[Raussendorf & Brigel '01]

- There is a highly entangled state on a 2D array of qubits. First carve out entanglement structure on **cluster state** by local Pauli Z measurement



- Then:

- (1) Measurement along each wire simulates one-qubit evolution (gates)
- (2) Measurement near & on each bridge simulates two-qubit gate (CNOT)



2D or higher dimensions are needed for universal QC

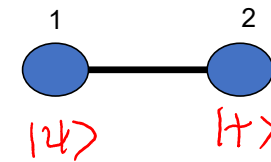
Unitary operation by measurement?

□ Intuition: entanglement as resource!

❖ Controlled-Z (CZ) gate from Ising interaction

$$CZ_{12} = e^{-i\frac{\pi}{4}(1-\sigma_Z^{(1)})(1-\sigma_Z^{(2)})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

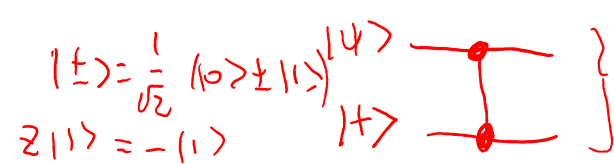
$11 \rightarrow -11$



❖ Entanglement is generated:

$$(a|0\rangle + b|1\rangle) |+\rangle \xrightarrow{CZ} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$

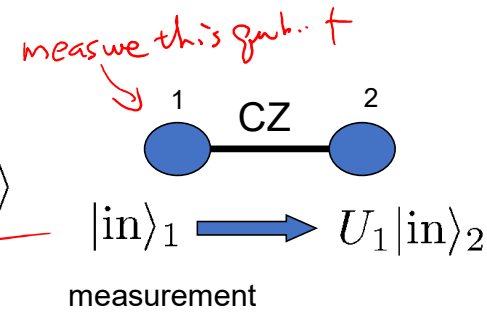
$a|0\rangle|+\rangle + b|1\rangle|+\rangle \rightarrow a|0\rangle|+\rangle + b|1\rangle|-\rangle$



Unitary operation by measurement?

□ Intuition: entanglement as resource!

$$(a|0\rangle + b|1\rangle)|+\rangle \xrightarrow{\text{CZ}} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$



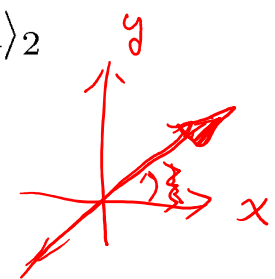
❖ Measurement on 1st qubit in basis

$$|\pm\xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle)/\sqrt{2}$$

with outcome denoted by $\pm = (-1)^{s=0,1}$

$$\hat{O}(\xi) = \cos(\xi)\sigma_x + \sin(\xi)\sigma_y$$

$\begin{matrix} \text{III} & \text{III} \\ \text{X} & \text{Y} \end{matrix}$



$$\hat{O}(\xi)|\pm\xi\rangle = \pm|\pm\xi\rangle$$

→ Second qubit becomes

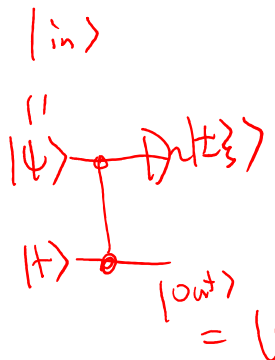
$${}_1\langle\pm\xi|\psi\rangle_{12} \sim a e^{i\xi/2}|+\rangle_2 \oplus b e^{-i\xi/2}|-\rangle_2 = H e^{i\xi Z/2} Z^s (a|0\rangle_2 + b|1\rangle_2)$$

$$\text{Prob}(\pm) = |\langle\pm\xi|\psi\rangle|^2$$

$$e^{i\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

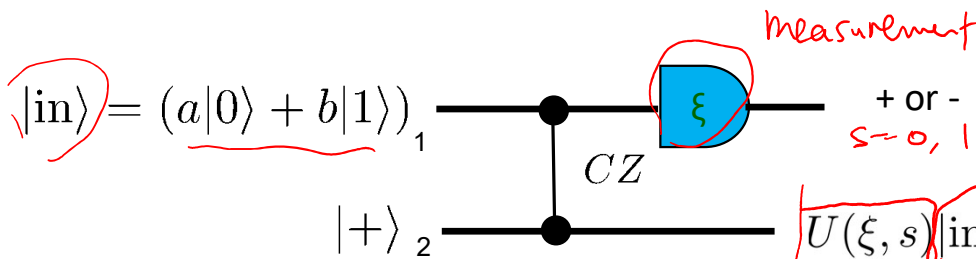
→ A unitary gate is induced: $U(\xi, s) \equiv H e^{i\xi Z/2} Z^s$

$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



Saw this before (on 9/2): A variant---gate teleportation*

Controlled-Z gate and single-qubit measurement induces rotation



$$CZ_{12} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Handwritten notes: $00 \rightarrow 00$, $01 \rightarrow 01$, $10 \rightarrow 10$, $11 \rightarrow \ominus 11$. A circled X is also present.

$$U(\xi, s) = H e^{i\xi Z/2} Z^s$$

Handwritten notes: "rotation", "Hadamard", "or 1", "1 Z".

(3rd rule QM)

The measurement basis ξ is defined via

$$|\pm \xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle) / \sqrt{2}$$

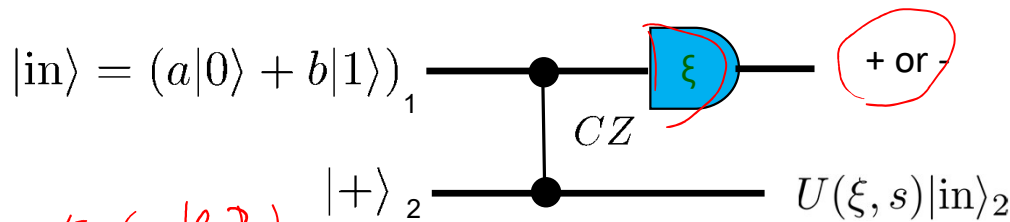
or the observable:

$$\cos(\xi)\sigma_x + \sin(\xi)\sigma_y = \begin{pmatrix} 0 & e^{-i\xi} \\ e^{i\xi} & 0 \end{pmatrix}$$

Handwritten labels: σ_x (III), σ_y (III), X , Y .

$$|\pm \xi\rangle = \pm |+\xi\rangle$$

Saw this before (on 9/2): Derivation*



The measurement basis ξ is defined via

$$|\pm \xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle)/\sqrt{2}$$

$$CZ_{12} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

gate (rule 2)

$$\textcircled{1} (a|0\rangle + b|1\rangle) \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow a|0\rangle(|0\rangle + |1\rangle) + b|1\rangle(|0\rangle - |1\rangle)$$

$$= a|0\rangle|+\rangle + b|1\rangle|-\rangle \Rightarrow |\Psi_{12}\rangle$$

② measurement: proj to eigenstates $|\pm \xi\rangle = \frac{1}{\sqrt{2}}(e^{-i\xi/2}|0\rangle + e^{i\xi/2}|1\rangle)$

$$\langle \pm \xi | \Psi_{12} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\xi/2} & \pm e^{-i\xi/2} \\ 1 & 1 \end{pmatrix} (a|0\rangle|+\rangle + b|1\rangle|-\rangle)$$

$$= \frac{1}{\sqrt{2}} (a e^{i\xi/2} |+\rangle_2 \pm b e^{-i\xi/2} |-\rangle_2) = \frac{1}{\sqrt{2}} H e^{i\xi/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (a|0\rangle + b|1\rangle)$$

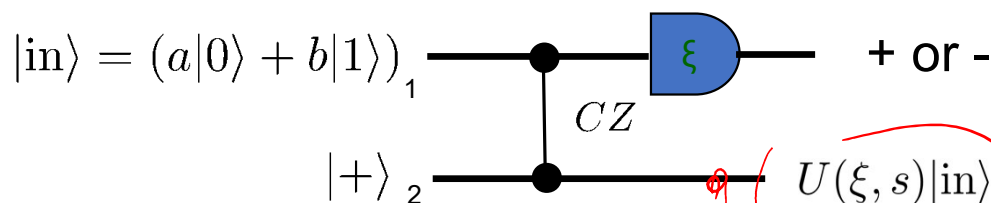
$$= \frac{1}{\sqrt{2}} H e^{i\xi/2} Z^s (a|0\rangle + b|1\rangle)$$

$s = (-1)^{\pm 1}$

Simulating arbitrary one-qubit gates

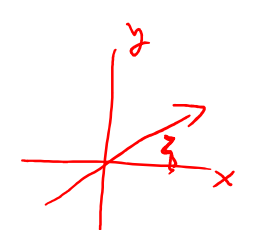
□ In terms of circuit:

[Raussendorf & Wei, Ann Rev Cond-Mat '12]

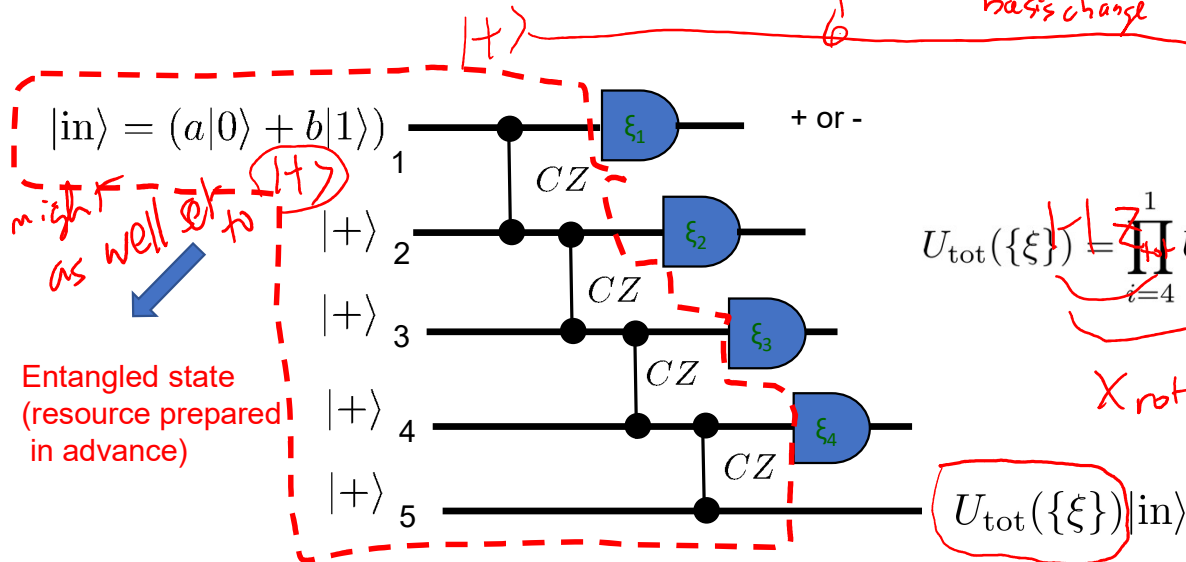


$$U(\xi, s)|in\rangle_2$$

$$U(\xi, s) = H e^{i\xi Z/2} Z^s$$



□ Can cascade this a few times:



might as well set to $|+\rangle$

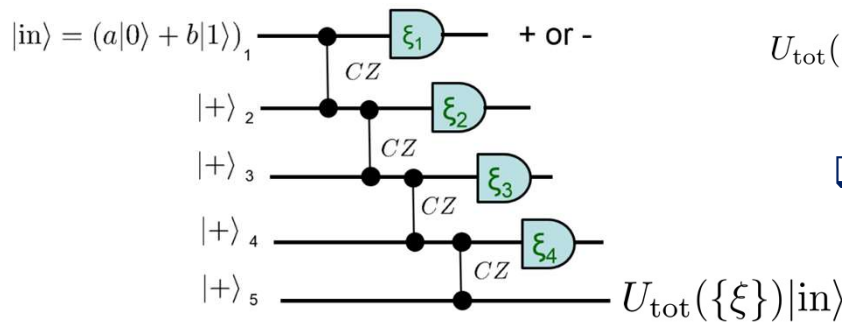
Entangled state (resource prepared in advance)

$$U_{\text{tot}}(\{\xi\}) = \prod_{i=1}^4 U(\xi_i, s_i)$$

X_{rot} Z_{rot} X_{rot} \Rightarrow Euler rotation
any single qubit rotation

$U(\xi'_1, s'_1) U(\xi_2, s_2) |in\rangle$
basis change angle = 0

Arbitrary one-qubit gate



$$U_{\text{tot}}(\{\xi\}) = \prod_{i=1}^4 U(\xi_i, s_i) \quad U(\xi, s) = H e^{i\xi Z/2} Z^s$$

□ Consider: $\xi_1=0$ & construct arbitrary rotation

$H Z$
 $= X H$

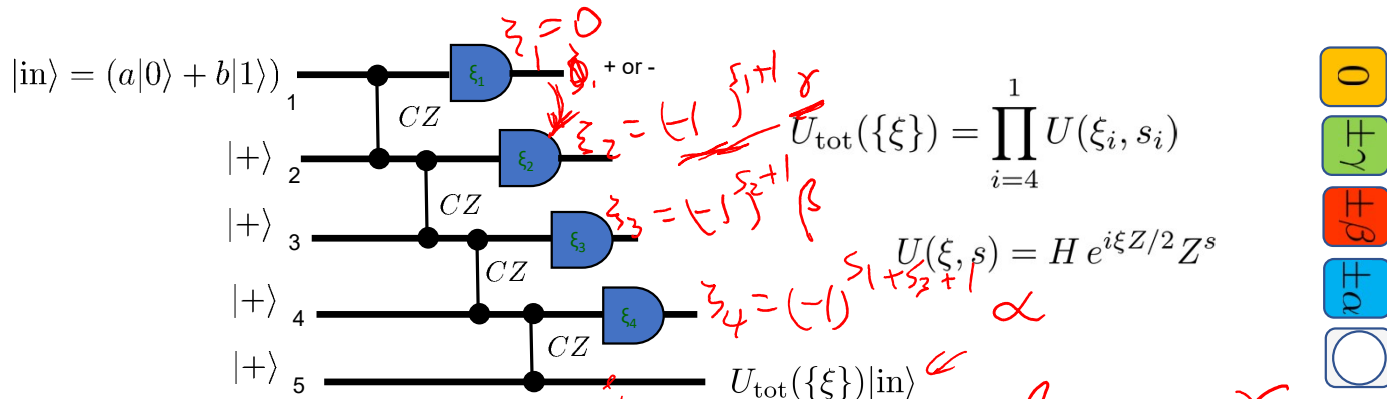
$$U_{\text{tot}}(\{\xi\}) = (H e^{i\xi_4 Z/2} Z^{s_4}) (H e^{i\xi_3 Z/2} Z^{s_3}) (H e^{i\xi_2 Z/2} Z^{s_2}) (H Z^{s_1})$$

□ Propagating Z's to left and use $HZH=X$:

$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{i(-1)^{s_1+s_3}\xi_4 X/2} e^{i(-1)^{s_2}\xi_3 Z/2} e^{i(-1)^{s_1}\xi_2 X/2}$$

Euler rotation
 (α, β, γ)

Realizing arbitrary rotation



$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{i(-1)^{s_1+s_3}\xi_4 X/2} e^{i(-1)^{s_2}\xi_3 Z/2} e^{i(-1)^{s_1}\xi_2 X/2}$$

Take $\xi_1 = 0$, $\xi_2 = (-1)^{s_1+1}\gamma$, $\xi_3 = (-1)^{s_2+1}\beta$, $\xi_4 = (-1)^{s_1+s_3+1}\alpha$

we realize an Euler rotation:

$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{-i\alpha X/2} e^{-i\beta Z/2} e^{-i\gamma X/2}$$

→ Note: measurement basis can depend on prior results

→ $Z^{s_1+s_3} X^{s_2+s_4}$ has no effect

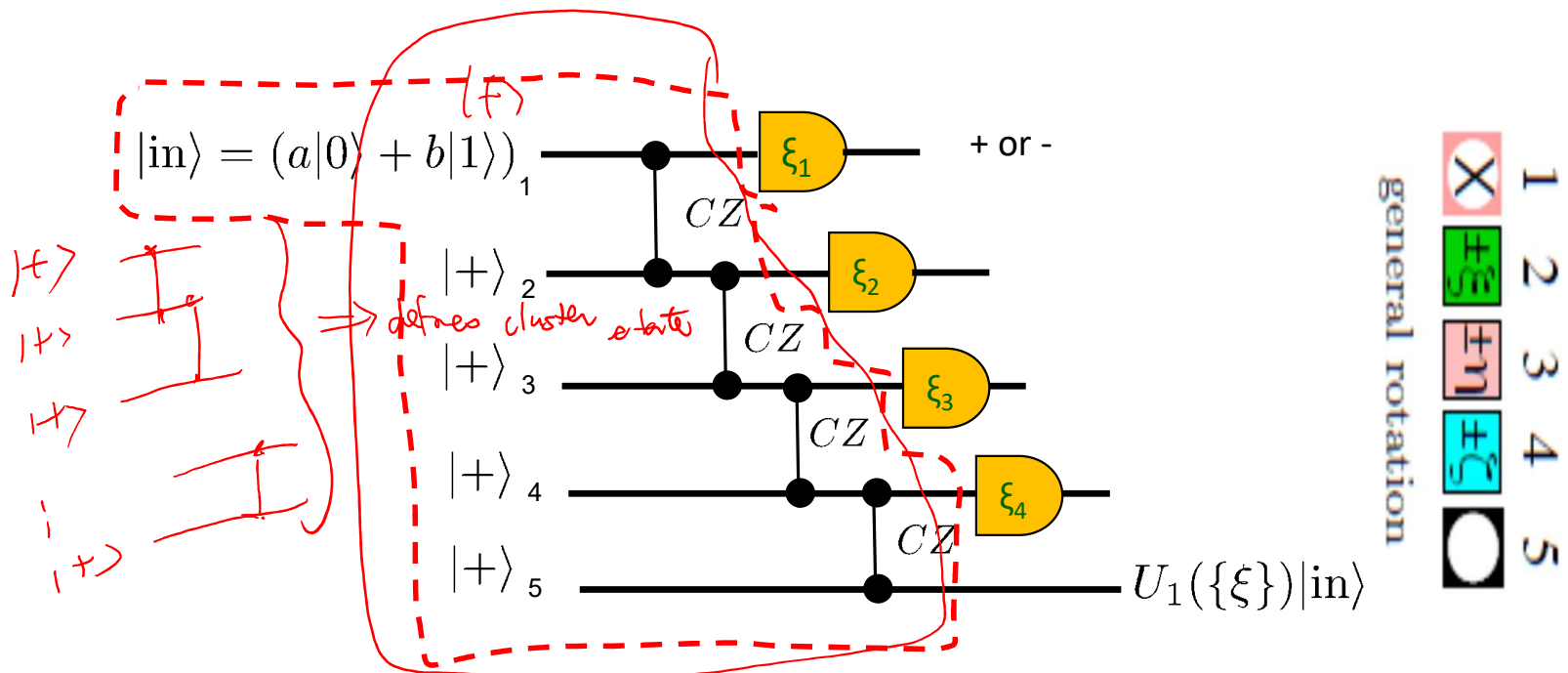
can be absorbed by modifying later measurement basis
know $s_2+s_4 \rightarrow$ correct X flip by hand

future angle needs to adapted



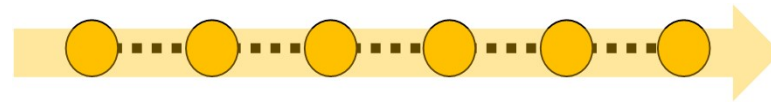
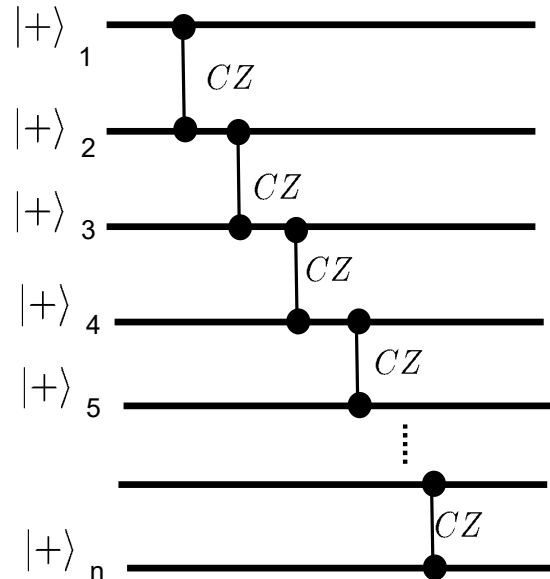
(± 1)
 $a|0\rangle + b|1\rangle$
final measurement is always 0/1

Linear cluster state: resource for simulating arbitrary one-qubit gates



- May as well take $|\text{in}\rangle = |+\rangle$; the whole state before measurement ξ 's is a highly entangled state \rightarrow 1D cluster state

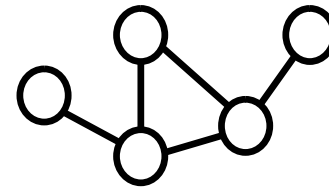
1D Cluster state: simulate 1-qubit gate



➤ Local measurement induces discrete evolution of quantum state (one qubit)

➤ Cluster/graph states: on any graph:

$$|G\rangle = \bigotimes_{\langle i,j \rangle \in \text{Edge}} CZ_{ij} (|+\rangle|+\rangle \cdots |+\rangle)$$

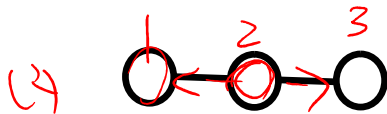


Working out cluster states

cluster state (graph) $\equiv \prod_{\langle ij \rangle} CZ_{ij} |+\rangle \dots |+\rangle$

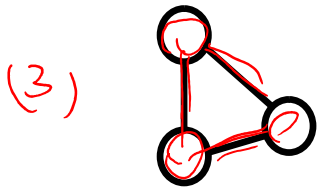


$|\psi_{(1)}\rangle = CZ_{12} |++\rangle = |0\rangle|+\rangle + |1\rangle|-\rangle = |+\rangle|0\rangle + |-\rangle|1\rangle$
 (ignoring $\frac{1}{\sqrt{2}}$ factor)
 $(\begin{smallmatrix} \downarrow \\ 0 \end{smallmatrix}) + |1\rangle |+\rangle \Rightarrow |0\rangle|+\rangle + |1\rangle|-\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$



$|\psi_{(2)}\rangle = CZ_{21} CZ_{23} |+++ \rangle = (|+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle) \frac{1}{\sqrt{2}}$

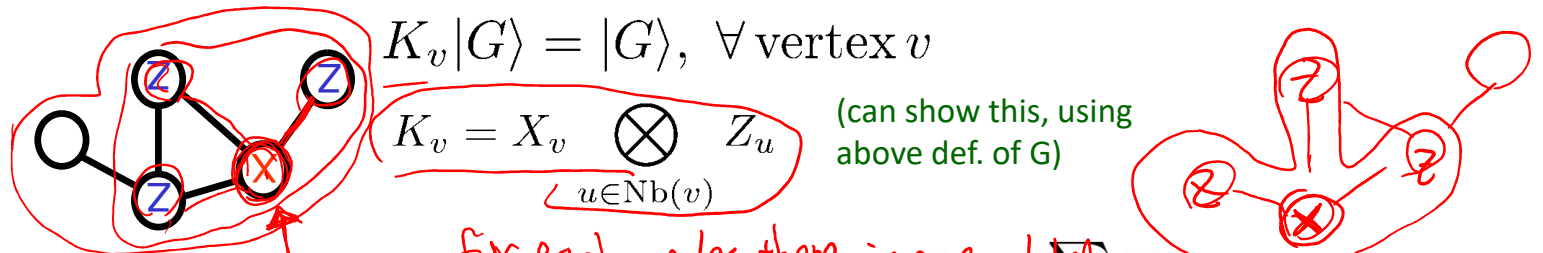
 $|\psi_{(2)}\rangle = CZ_{13} |\psi_{(2)}\rangle$



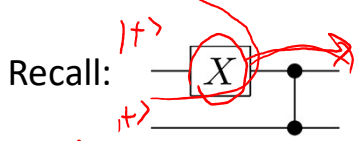
$= \left(\begin{matrix} \widehat{000} & + & \widehat{001} & + & \widehat{010} & + & \widehat{011} \\ + & \widehat{100} & + & \widehat{101} & + & \widehat{110} & + & \widehat{111} \end{matrix} \right) \frac{1}{\sqrt{8}}$

Graph states and the stabilizer group

- Cluster state: special case of general "graph" states



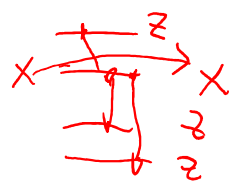
→ Uniquely define the state G, also via Hamiltonian $H = -\sum_v K_v$ operator
 for each vertex there is one stabilizer operator
 ⇒ The set (group) uniquely determines (the state) $n-n=0$



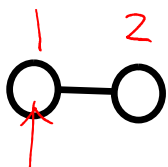
$H = -\sum_u K_u$


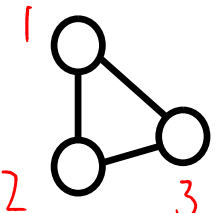
cf. $H_{\text{Toyc}} = -\sum_s A_s - \sum_p B_p$

$|\psi\rangle = CZ X_1 |+\rangle |+\rangle = X_1 Z_2 (Z |+\rangle |+\rangle) = X_1 Z_2 |\psi\rangle$



Example of graph Hamiltonians


 2 stabilizer elements: $X_1 Z_2 - X_2 Z_1$ $(S = \langle X_1 Z_1, X_2 Z_1 \rangle)$
 $H = -$ \uparrow boundary terms


 except boundary
 $H = - \sum_{i \notin \text{boundary}} Z_{i-1} X_i Z_{i+1} - X_1 Z_2 - X_2 Z_{2-1}$
 $Z \quad X \quad Z$
 $X \quad 1 \quad X \quad 1 \quad X \dots$
 $1 \quad X \quad 1 \quad X \quad 1 \quad X$
 \Downarrow symmetry protected $Z_2 \times Z_2$ topological phase

 $\rightarrow H = -(X_1 Z_2 Z_3 + X_2 Z_3 Z_1 + X_3 Z_1 Z_2)$

Simulating CNOT by measurement

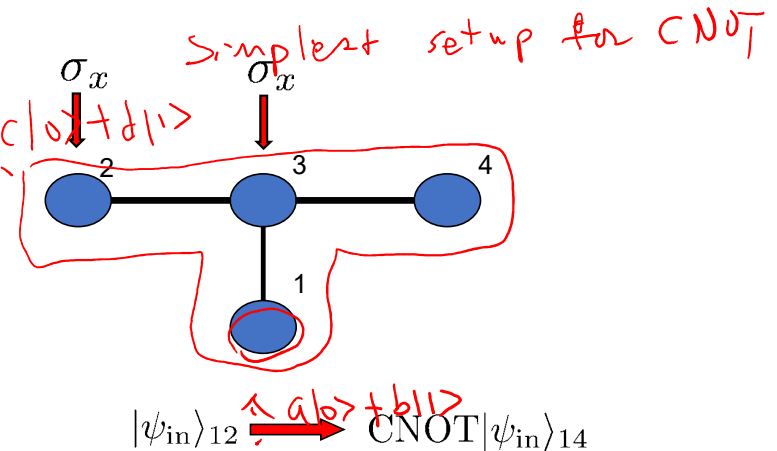
- Consider initial state

$$(a|0\rangle + b|1\rangle)_1 (c|0\rangle + d|1\rangle)_2 |+\rangle_3 |+\rangle_4$$

$$\xrightarrow{CZ_{23} CZ_{13} CZ_{34}} |\psi\rangle_{1234}$$

$$|\psi\rangle_{1234} = |0\rangle_3 (a|0\rangle_1 + b|1\rangle_1) (c|0\rangle_2 + d|1\rangle_2) |+\rangle_4$$

$$+ |1\rangle_3 (a|0\rangle_1 - b|1\rangle_1) (c|0\rangle_2 - d|1\rangle_2) |-\rangle_4$$



- Measurement on 2nd and 3rd qubits in basis $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$

If outcome=++: an effective CNOT applied: (could be an exercise)

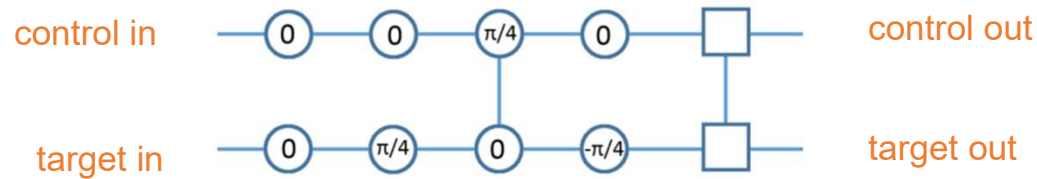
$$|\psi\rangle_{14} = {}_{23}\langle ++ | \psi \rangle_{1234} \sim \text{CNOT}_{14} (a|0\rangle_1 + b|1\rangle_1) (c|0\rangle_4 + d|1\rangle_4)$$

Can show: $|\psi_{\text{out}}\rangle \sim Z_1^{s_2} X_4^{s_3} Z_4^{s_2} \text{CNOT}_{14} |\text{in}\rangle_{14}$

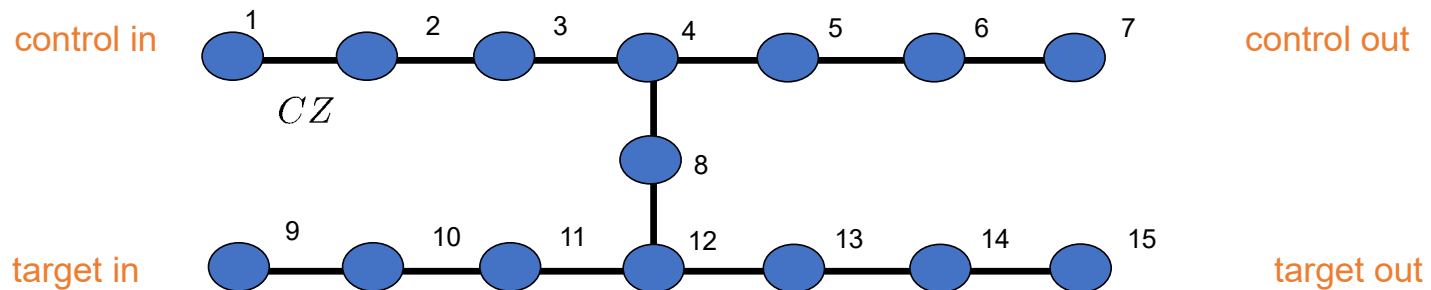
- Note the action of CZ gates can be pushed up front (a 4-qubit "cluster" state can be used to simulating CNOT)

CNOT: Other implementations

1.



2.



with measurement pattern:

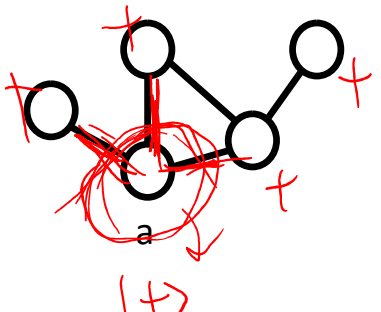
	1	2	3	4	5	6	7
control	X	Y	Y	Y	Y	Y	●
				Y	8		
target	X	X	X	Y	X	X	●
	9	10	11	12	13	14	15

CNOT-gate

Z measurement on graph state

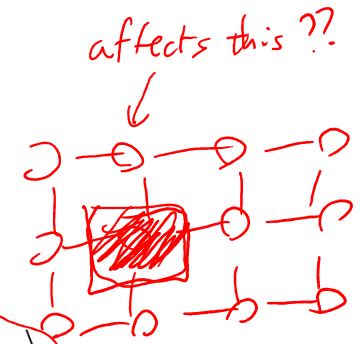
- The effect is just to remove the measured qubit, keeping the remaining entanglement structure

Think in terms of "construction stage" when we grow the graph



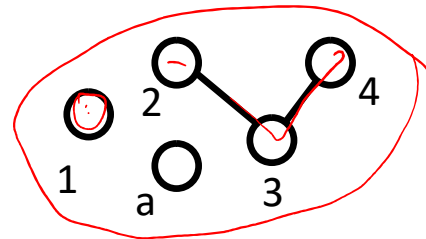
$$|\Psi_G\rangle = |0\rangle_a |\Psi_{G \setminus a}\rangle + |1\rangle_a \left(\prod_{b \in NB(a)} Z_b \right) |\Psi_{G \setminus a}\rangle$$

up to Z correction



affects this??

→ Graph after Z measurement on qubit a :



✓ If outcome = 0: $|0\rangle_a |+\rangle_1 |C\rangle_{234}$ $|C\rangle_{234}$: linear cluster state

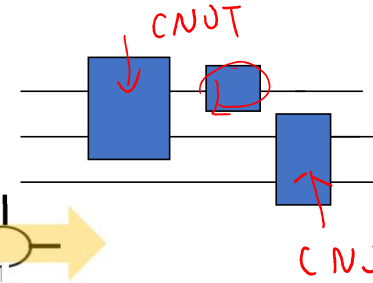
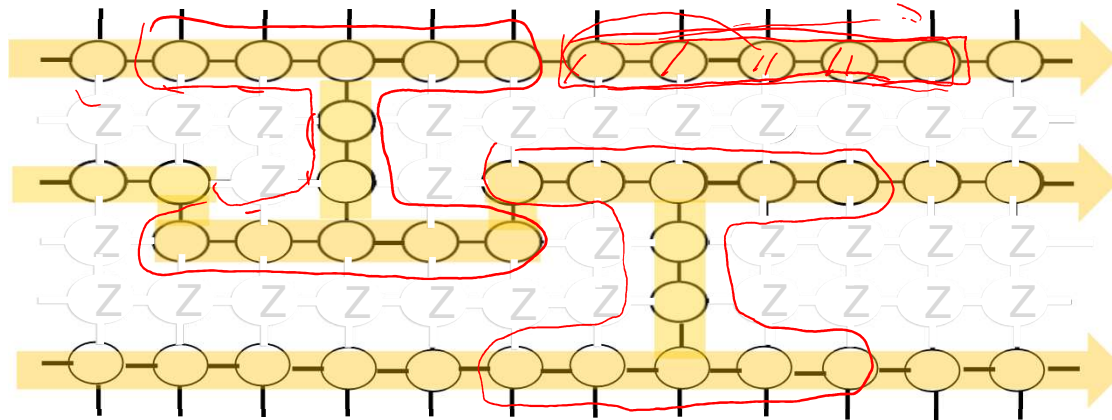
✓ If outcome = 1: $|1\rangle_a \prod_{b \in NB(a)} Z_b |-\rangle_1 |C\rangle_{234}$

Measurement-based QC: cluster state

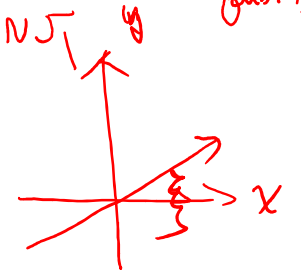
- Carve out entanglement structure by local Z measurement



[Raussendorf & Briegel PRL 01']



Z measurement
=> remove qubits



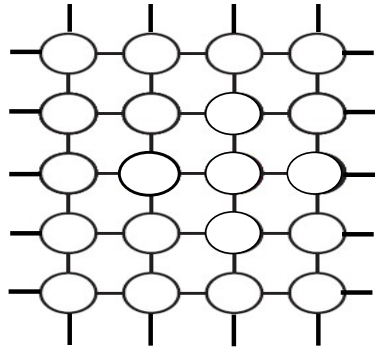
- (1) Each wire simulates one-qubit evolution (gates)
- (2) Each bridge simulates two-qubit gate (CNOT)



2D or higher dimension is needed for universal QC & Graph connectivity is essential (percolation)

2D cluster state is a resource for quantum computation

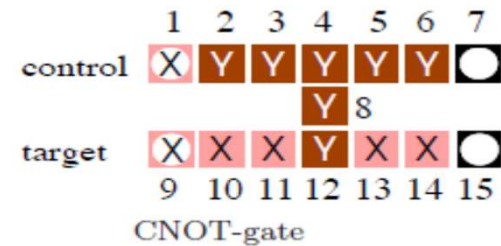
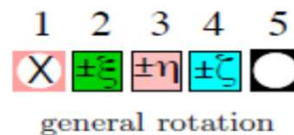
$$|C\rangle = \bigotimes_{\langle i,j \rangle} CZ_{ij} (|+\rangle|+\rangle \cdots |+\rangle)$$



- Whole entangled state is created first (by whatever means)
- Operations needed for *universal* QC are single-qubit measurements only

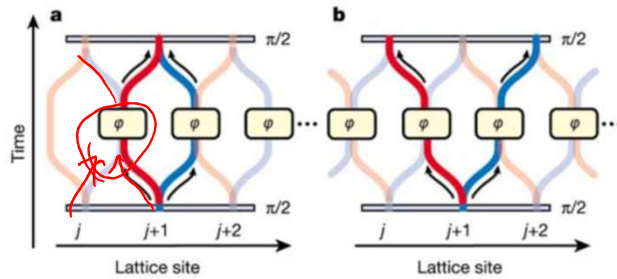
→ Pattern of measurement gives computation
(entanglement is being consumed → one-way)

→ Elementary “Lego pieces” for QC:



Realizations of cluster states

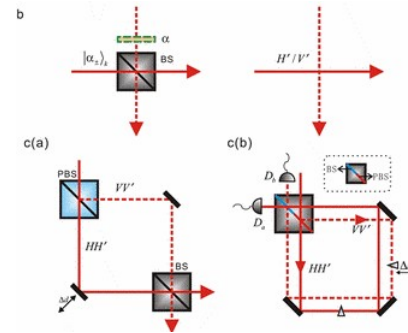
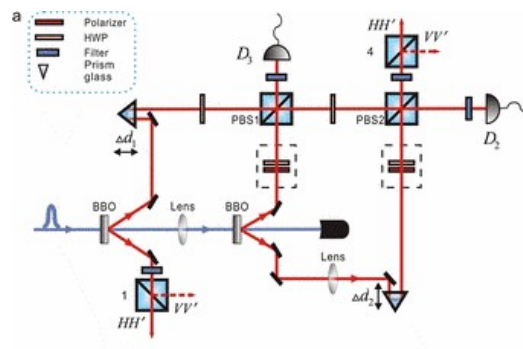
- Bloch's group: controlled collision in cold atoms (Nature 2003)



$$(1 - \sigma_z) (1 - \sigma_z)$$

Isz coupling

- J-W Pan's group: 4-photon 6 qubit and CNOT (PRL 2010)



Linear optic QC & cluster state

- Linear optic universal QC possible with single photon source, linear optic elements (beam splitters, mirrors, etc) & photon counting

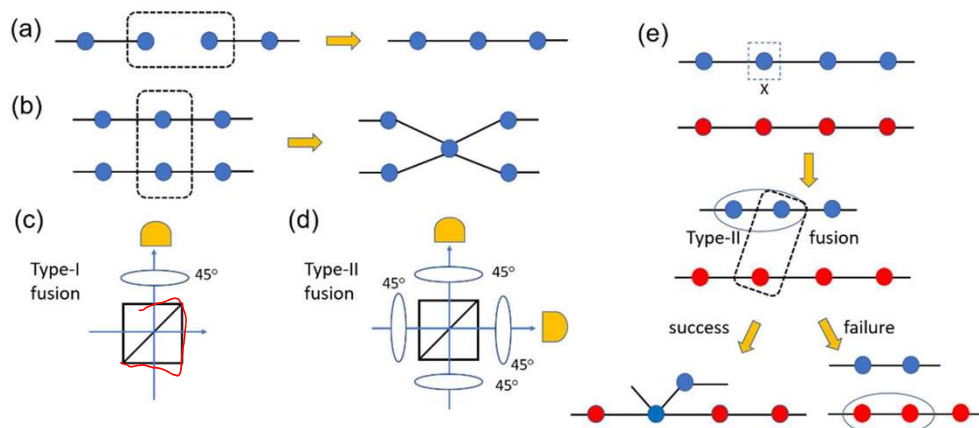
→ High overhead in entangling gates

[Knill, Laflamme & Milburn '01]

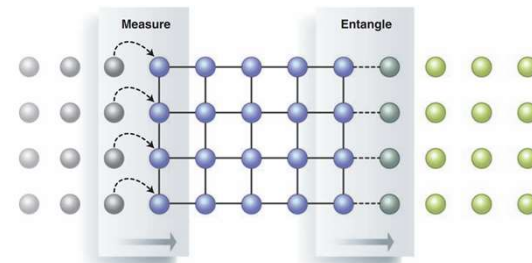
- Cluster state helps reduce this overhead

→ Grow cluster states efficiently

[Yoran & Reznik '03; Nielsen '04; Brown & Rudolph '05; Kieling, Rudolph & Eisert '07]



measurement of photon polarization can be very high

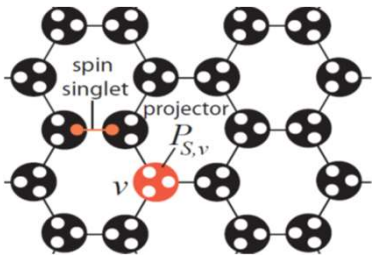


- Experimental prospect: see e.g. [O'Brien, Science '07]

Other universal states*

- ❑ So far no complete characterization for resource states
- ❑ Can they be unique ground state with 2-body Hamiltonians with a finite gap?

➔ If so, create resources by cooling!



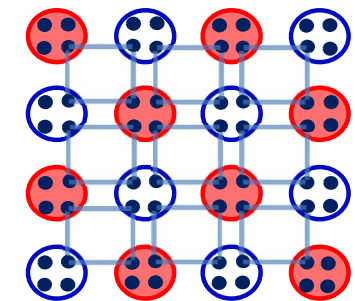
- ❖ Affleck-Kennedy-Lieb-Tasaki (AKLT) family of states [AKLT '87, '88]

➔ Unique ground states of two-body interacting Hamiltonians

- 1D (not universal): [Gross et al. '07, '10] [Brennen & Miyake '08?]
- 2D (universal): [Wei, Affleck & Raussendorf '11] [Miyake '11] [Wei et al. '13-'15]

- ❖ Symmetry-protected topological states

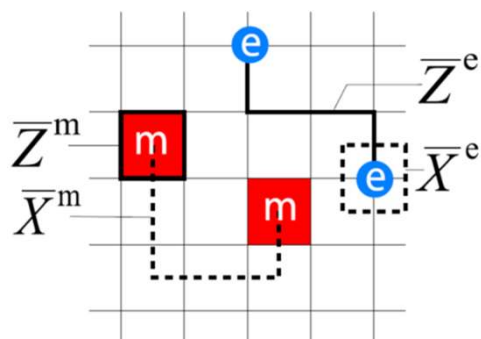
- 1D (not universal): [Else, Doherty & Bartlett '12] [Miller & Miyake '15] [Prakash & Wei '15] [Stephen et al. '17]
- 2D (universal, *not much explored*): [Poulsen Nautrup & Wei '15, Miller & Miyake '15, Chen, Prakash & Wei '18, Raussendorf et al. '18]



How do we make MBQC fault tolerant?

- Key idea: use 3-dimensional cluster state and measurement pattern simulates braiding

- 2d Surface code:



1. Physical qubit on edge

2. Most plaquettes and vertices (stars)

$$\begin{array}{c} \sigma_z \\ \square \\ \sigma_z \end{array} = 1 = \begin{array}{c} \sigma_x \\ \sigma_x \\ \sigma_x \end{array}$$

is enforced in code space

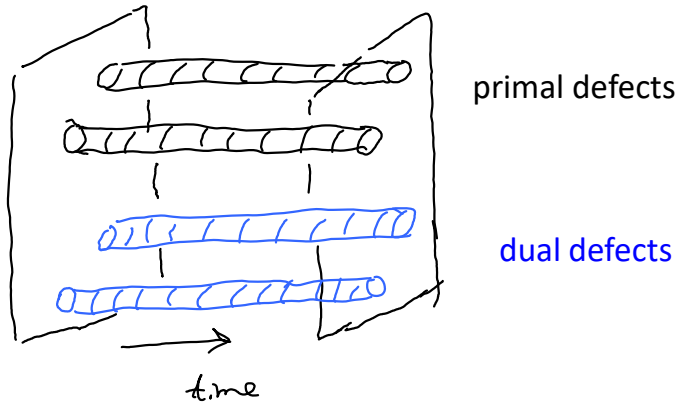
3. When a pair of plaquettes (or stars) not enforced, it gives rise to a **logical** qubit (see above and logical X and Z operators)

- Braiding gives topologically protected logical operations

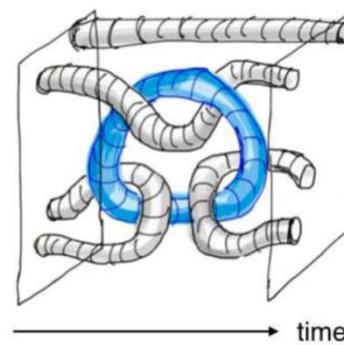
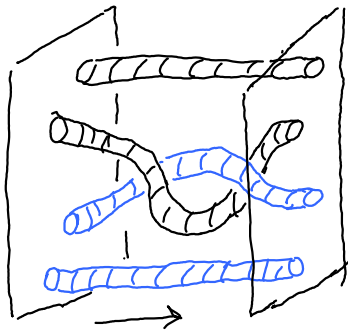
☹ But not universal, requires **magic state injection to logical** qubit (see above and logical X and Z operators)

Surface code QC

- Example: 2 logical qubits (one primal and one dual)

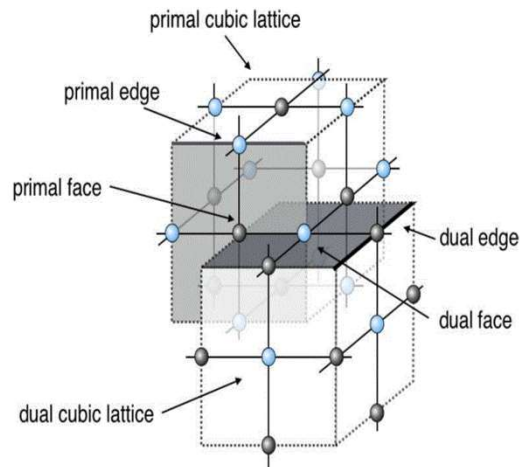


- CNOT gates (between primal and dual; between two primal logical qubits)
→ simplest to implement

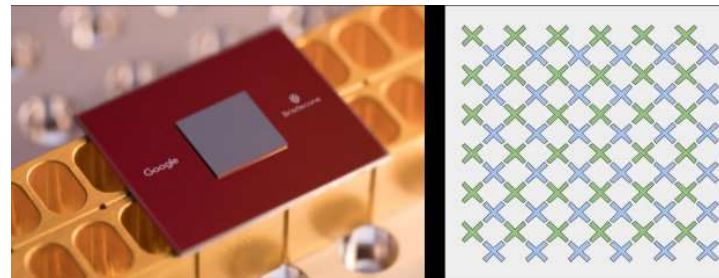


Fault-tolerant MBQC

- Key idea: use 3-dimensional cluster state and measurement pattern simulates braiding
- The diagrams literally translate to measurement pattern (replace time direction by the third dimension)
- The 3d cluster state is given by a lattice with the following unit cell:

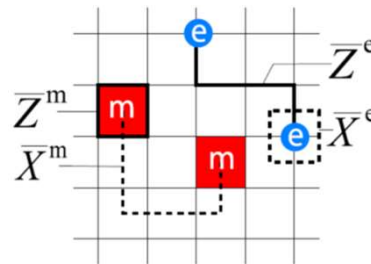
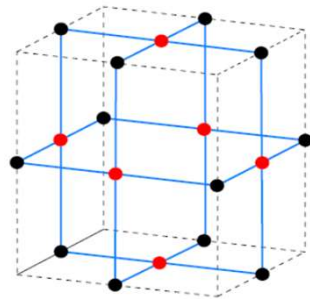


- Gives a high error threshold: 0.75%
- 2d circuit version is what Google plans to use



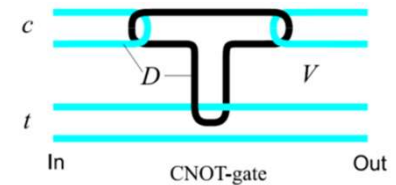
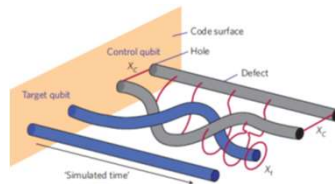
Fault tolerant cluster-state QC

- ❑ Uses a 3d cluster state and implements surface codes in each 2d layer

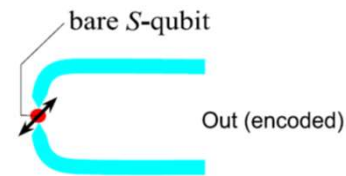


[Raussendorf, Harrington & Goyal '07]

- ❑ CNOT is achievable by measurement



- ❑ Uses magic-state distillation to achieve non-Clifford gate (by measurement)



➔ Error threshold 0.75%, qubit loss threshold 24.9%

[Barrett & Stace '10]

One application of measurement-based QC

Suppose we have a cloud quantum computer server.

Q: Is it possible to run on this cloud quantum computer without the server figuring out what the client is actually running?

A: Blind quantum computation