Do poll 8/31-(1)

Today 8/31:

(1) Some review: Bloch sphere, mixed states, phase kickback, Deutsch algorithm
(2) Jupyter Notebook Demo
https://nbviewer.jupyter.org/url/insti.physics.sunysb.edu/~twei/Courses/Fall2020/PHY682/Demo-QubitSphere.ipynb
(3) More algorithms
(4) Week 2: From foundation to science-fiction teleportation

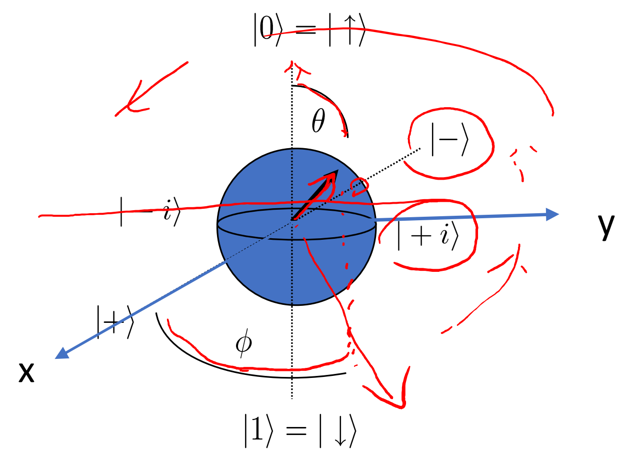
Review: Bloch sphere of a qubit

$$\boxed{|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle}$$ is the same as ← spinor

$$\rho_\psi = |\psi\rangle\langle\psi| = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$$ ← density

$$r_x = \sin\theta\cos\phi,\ r_y = \sin\theta\sin\phi,\ r_z = \cos\theta$$

$$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli



- ❑ Example: what is the state on the sphere along negative x axis (|->)? positive y axis (|i>)?

$|-\rangle$: $-x$ $\quad \theta = \frac{\pi}{2},\ \varphi = \pi \quad |-\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\pi}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|i\rangle$: $+y$: $\quad \theta = \frac{\pi}{2},\ \varphi = \frac{\pi}{2},\ |i\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\frac{\pi}{2}}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$|-i\rangle$: $-y$ $\quad \varphi = \frac{\pi}{2} + \pi$

$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- ❑ Example: How do we use quantum gates to obtain these states starting from |0> ?

$|0\rangle \xrightarrow{\text{Hadamard}} (|0\rangle + |1\rangle)/\sqrt{2} \qquad |1\rangle = X|0\rangle \xrightarrow{H} (|0\rangle - |1\rangle)/\sqrt{2}\ ; \quad |-i\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}|-\rangle$

Review: mixed state

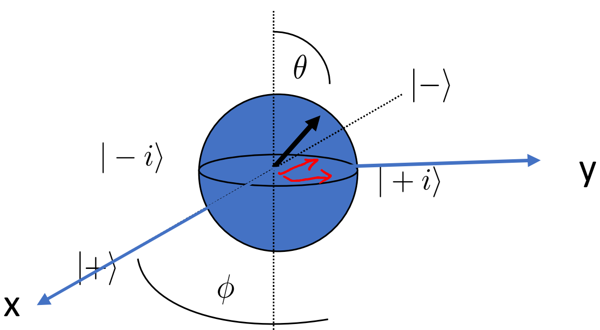
(mixture of pure state)

$$|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle$$ is the same as

$$\rho_\psi = |\psi\rangle\langle\psi| = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$$

$$r_x = \sin\theta\cos\phi,\ r_y = \sin\theta\sin\phi,\ r_z = \cos\theta$$

$$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$|-\rangle$ x-y plane mixture $|+i\rangle$

- Example: how do we represent equal weight mixture of |-> and |i> ?

mixture $(\alpha|-\rangle + \beta|i\rangle)$ not mixing

$$\rho = \frac{1}{2}|-\rangle\langle -| + \frac{1}{2}|i\rangle\langle i| \qquad \vec{r}_\rho = \frac{1}{2}\vec{r}_- + \frac{1}{2}\vec{r}_i$$

$tr(\hat{O}_s \rho)$

$= \langle\psi|\hat{O}_s|\psi\rangle$

- Example: a mixed state can arise from entanglement with other system

$$|\psi\rangle_{2\text{-qubit}} = \frac{1}{\sqrt{2}}\left(|-\rangle|0\rangle_{env} + |i\rangle|1\rangle_{env}\right) \Rightarrow$$

when I "ignore" (trace over) the env $\Rightarrow$ get mixed for the system.

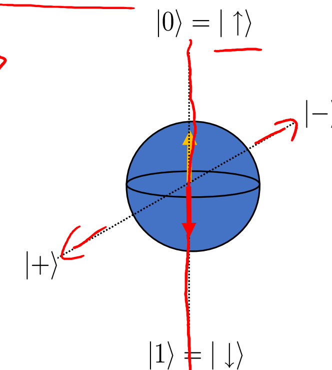
Review: observable and measurement

- Strong measurement projects wavefunction; outcome is often probabilistic

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \xrightarrow{\text{randomly}} |\uparrow\rangle \text{ or } |\downarrow\rangle$$

$$P_\uparrow = |\alpha|^2 = |\langle\uparrow|\psi\rangle|^2 \qquad \sigma_z = +1$$

$$P_\downarrow = |\beta|^2 = |\langle\downarrow|\psi\rangle|^2 \qquad \sigma_z = -1$$



'Observables' ⟷ 'basis' of measurement (and possible measured values)

$$Z = (+1)|\uparrow\rangle\langle\uparrow| + (-1)|\downarrow\rangle\langle\downarrow|$$

$$\text{Expectation value: } \langle\psi|Z|\psi\rangle = P_\uparrow\cdot(+1) + P_\downarrow\cdot(-1)$$

- Example: When the observable is X? What is the probability to measure +?

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (+1)|+\rangle\langle+| + (-1)|-\rangle\langle-|$$

$$P_+ = |\langle+|\psi\rangle|^2 \qquad P_- = |\langle-|\psi\rangle|^2$$

Suppose two-qubit eg.

$$|\psi\rangle_{sys,env} = \alpha|\uparrow\rangle|0\rangle_{env} + \beta|\downarrow\rangle|1\rangle_{env}$$

if measure the system

$$\begin{cases} \uparrow : |\alpha|^2 \quad env \Rightarrow |0\rangle \\ \downarrow : |\beta|^2, \ env \Rightarrow |1\rangle \end{cases}$$

Review: CNOT (controlled-NOT) gate and the balanced function

$$\mathrm{CNOT}_{12} = |\uparrow\rangle\langle\uparrow| \otimes I + |\downarrow\rangle\langle\downarrow| \otimes X = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$|0\rangle\langle 0|$

$00 \to 00$

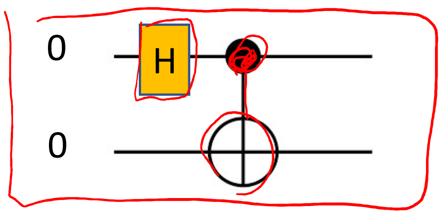
$01 \to 01$

$10 \to 11$

$11 \to 10$

$|x\rangle \otimes |b\rangle \longrightarrow |x\rangle \otimes |b + f(x)\rangle$

$$|\uparrow\uparrow\rangle \overset{H_1}{\Longrightarrow} \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |\uparrow\rangle \xrightarrow{\mathrm{CNOT}_{12}} \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

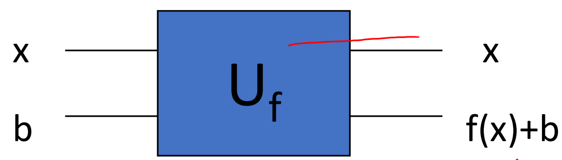


$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

where we use 0 and 1 instead of up and down arrows, and we have introduced the diagram for the CNOT gate

Review: `phase kickback'

- Suppose the effect of the circuit is to compute f(x) and add it to second register:



$$|x\rangle \otimes |b\rangle \rightarrow |x\rangle \otimes |f(x)+b\rangle$$

"−1" eigen state of X

$0 \oplus 1 = 1$
$1 \oplus 1 = 0$
$0 \oplus 0 = 0$

- If we send in $|x\rangle \otimes (|0\rangle - |1\rangle)$, ignoring normalization

$$|x\rangle \otimes (|0\rangle \cancel{-} |1\rangle) \rightarrow |x\rangle \otimes (|f(x)\rangle - |f(x)+1\rangle)$$

$$= |x\rangle \otimes \boxed{(-1)^{f(x)}}(|0\rangle \cancel{-} |1\rangle)$$

0 0

forget this

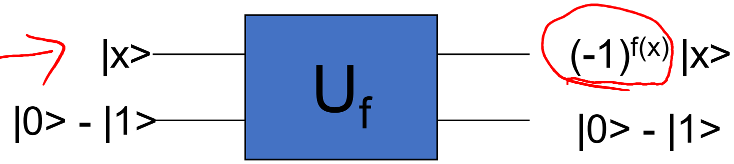
By linearity & superposition

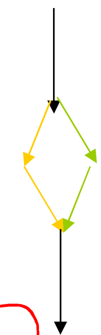
verify
$f(0)=0$
$f(0)=1$

- Phase kickback:

$$|x\rangle \otimes (|0\rangle - |1\rangle) \rightarrow \boxed{(-1)^{f(x)}}|x\rangle \otimes (|0\rangle - |1\rangle)$$

Deutsch algorithm: one function call



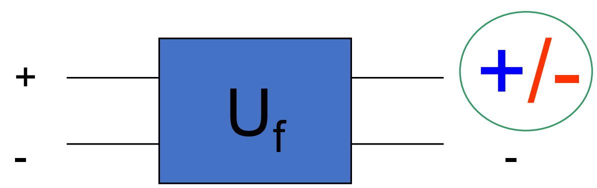


Quantum explores Parallel universes?

- Consider sending |0>+|1> in first register (and |0>-|1> in the second):

$$|0\rangle + |1\rangle \to (-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle \to \begin{cases} (-1)^{f(0)}(|0\rangle + |1\rangle) & \text{constant} \\ (-1)^{f(0)}(|0\rangle - |1\rangle) & \text{balanced} \end{cases}$$

- Quantum computers: need one evaluation only and measure in +/- basis $|\pm\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$



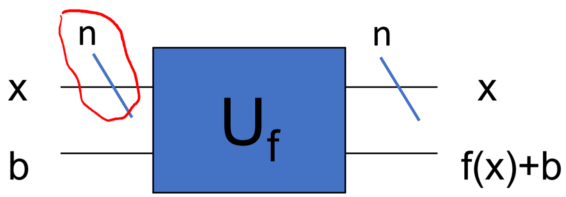
➔ Measure: + constant
- balance

- First hint that quantum computer can be powerful!

Exercise: Deutsch-Josza Algorithm

Here we consider unknown function *f* that maps from n-bits to 1-bit. We are promised that f is either constant (f= the same value) or balanced (the latter means exactly half of inputs f(x)=1, and other half f(x)=0). This generalizes Deutsch's problem from one bit to n bits.

n lines



$$|x\rangle \otimes |b\rangle \rightarrow |x\rangle \otimes |f(x)+b\rangle$$

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|x\rangle = [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]^{\otimes n} = \underbrace{H \otimes H \otimes \dots H}_{n} \underbrace{|0\rangle \otimes |0\rangle \otimes \dots |0\rangle}_{n}$$

$$|b\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H|s_1\rangle = \frac{1}{\sqrt{2}} \sum_{z_1=0,1} (-1)^{s_1 z_1} |z_1\rangle \qquad s_1 = 0 \text{ or } 1$$

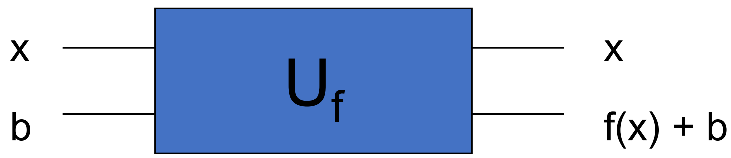
$$H^{\otimes n} = \underbrace{H \otimes H \otimes \cdots \otimes H}$$

$$H^{\otimes n}|s_1 s_2 \cdots s_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{z_1 \cdots z_n} (-1)^{\sum s_i z_i} |z_1 z_2 \cdots z_n\rangle$$

(1) Show the quantum state after the circuit.
(2) Show that if f is constant, the first register is always +…+
(3) Show that if f is balanced, the first register is always orthogonal to +…+

Quantum Parallelism

- Consider the unitary evolution that evaluates f(x)



x=00..0, 00..1,…, 11..1
binary rep. of 0,1,2,..

$$|x\rangle \otimes |b\rangle \rightarrow |x\rangle \otimes |f(x) + b\rangle$$

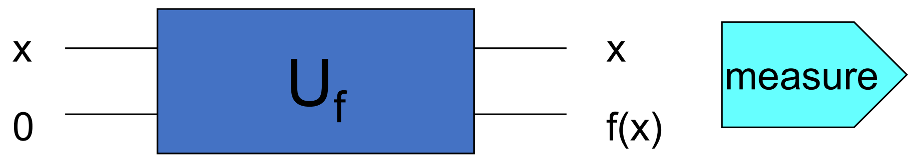
- Use superposition inputs:

$$(|0\rangle + |1\rangle + |2\rangle + \ldots) \otimes |0\rangle$$

$$\rightarrow (|0\rangle \otimes |f(0)\rangle + |1\rangle \otimes |f(1)\rangle + |2\rangle \otimes |f(2)\rangle + \ldots)$$

- Parallelism → superposition of (argument, fcn value)
  → potential power of quantum computers!

Measurement causes complication



$$\sum_{k=0}^{2^n-1} |k\rangle \otimes |0\rangle \rightarrow \sum_{k=0}^{2^n-1} |k\rangle \otimes |f(k)\rangle$$

- To obtain answer: Need to measure!

  - e.g. measure first register: *k* ➔ second register: *f*(k)
    only one answer at a time ☹ (and k is random)

  - But can measure in different basis or/and second register

    e.g. measure second register, obtain f_0 ,
    ➔ first register in superposition of x such that $f(x) = f_0$

➔ QC useful only for determining symmetry properties of f

More quantum algorithms

- Quantum Algorithm Zoo http://math.nist.gov/quantum/zoo/

  This is a comprehensive catalog of quantum algorithms. If you notice any errors or omissions, please email me at **stephen.jordan@nist.gov**. Your help is appreciated and will be acknowledged.

  Algebraic and Number Theoretic Algorithms

  Algorithm: Factoring
  Speedup: Superpolynomial
  Description: Given an *n*-bit integer, find the prime factorization. The quantum algorithm of Peter Shor solves this in $\widetilde{O}(n^3)$ time [82,125]. The fastest known classical algorithm for integer factorization is

- Notable ones:
  - Shor's factoring [~ exponential speedup]
  - Grover's searching [~ quadratic speedup]
  - Quantum Algorithm for Linear System: $A\vec{x} = b$
    [~ can be exponential speedup]
    aka HHL (Harrow-Hassidim-Lloyd) algorithm

Shor

Grover

We will first see a Jupyter Notebook demo

(Just to give you an idea of the programming as an alternative way to learn quantum computing. You don't need to understand all the details; as long as you can modify and have fun with the codes.)

https://nbviewer.jupyter.org/url/insti.physics.sunysb.edu/~twei/Courses/Fall2020/PHY682/Demo-QubitSphere.ipynb

We will then look at two other simple algorithms.

Berstein-Vazirani algorithm

f: 1 bit to 1 bit

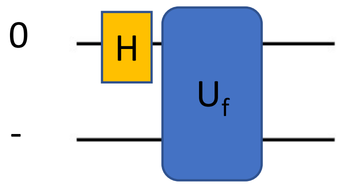
- Simplest case: one qubit and the linear function is f(x)= a.x

$f(x) = a\,x$

a's 0 or 1

$a=0 \quad f(x)=0$

$a=1 \quad f(x)=x$



$$|0\rangle - |1\rangle$$

$$|x\rangle \otimes |b\rangle \longrightarrow |x\rangle \otimes |f(x)+b\rangle$$

$f(x)=x$

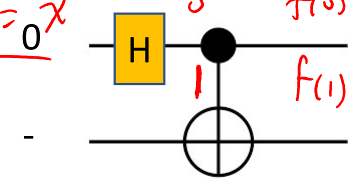
$$\text{first register}: |0\rangle \overset{H}{\rightarrow} |0\rangle + |1\rangle \overset{U_f}{\rightarrow} (-1)^{\overset{0}{f(0)}}|0\rangle + \underline{(-1)^{\overset{1}{f(1)}}}|1\rangle$$

For a=1, f(x)=x, and thus it is a CNOT

$$\rightarrow |0\rangle - |1\rangle$$

presence of a=1 can be detected in +/- basis

f(x) = x



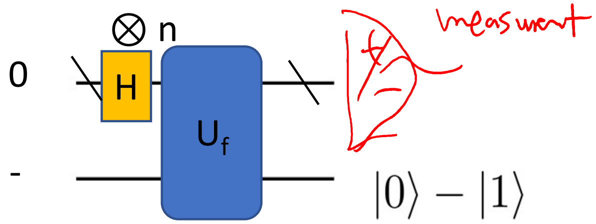
f(0) = 0 → add 0 to b (no action)

f(1) = 1 → add 1 to b (to flip)

n-qubit Berstein-Vazirani algorithm

- For n qubits: the linear function is **f(x) = a.x**, where **a** & **x** are both n-component binary vectors

measure "−" ⇒ bit is 1



$$a \cdot x = \sum_{i=1}^{n} a_i x_i = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

$$|x\rangle \otimes |b\rangle \rightarrow |x\rangle \otimes |f(x) + b\rangle$$

$$\text{first register} : |0^{\otimes n}\rangle \xrightarrow{H^{\otimes n}} \sum_{x_i' s} |x_1\rangle \otimes |x_2\rangle \otimes \cdots |x_n\rangle$$

$$\xrightarrow{U_f} \sum_{x_i' s} (-1)^{\sum_i a_i x_i} |x_1\rangle \otimes |x_2\rangle \otimes \cdots |x_n\rangle = \otimes_i (|0\rangle + (-1)^{a_i} |1\rangle)_i$$

✓ Presence of a_i=1 can be detected in +/- basis

Simon's algorithm*

- Consider a function f:{0,1}^{n} → finite set X.
  We are promised that there is some "hidden" string $\boldsymbol{s}=s_1s_2..s_n$ such that f($\boldsymbol{x}$)=f($\boldsymbol{y}$) if and only if $\boldsymbol{x}=\boldsymbol{y}$ or $\boldsymbol{x}=\boldsymbol{y}\oplus\boldsymbol{s}$ (bitwise XOR)
  ➔ Find string ***s***

x = 000
s = 010
y = 010
evaluate to same value

- Observation: n-qubit Hadamard

$$|\mathbf{0} \equiv 0^{\otimes n}\rangle \xrightarrow{H^{\otimes n}} \frac{1}{2^{n/2}} \sum_{z_i' s} |z_1\rangle \otimes |z_2\rangle \otimes \cdots |z_n\rangle = \frac{1}{2^{n/2}} \sum_{\mathbf{z}} |\mathbf{z}\rangle$$

$$|\mathbf{s} \equiv s_1 \ldots s_n\rangle \xrightarrow{H^{\otimes n}} \frac{1}{2^{n/2}} \sum_{\mathbf{z}} (-1)^{\mathbf{s}\cdot\mathbf{z}} |\mathbf{z}\rangle$$

$s \cdot z = \sum_i s_i z_i$ (binary addition)

➢ If we have a superposition:

$$\frac{1}{\sqrt{2}}(|\mathbf{0}\rangle + |\mathbf{s}\rangle) \xrightarrow{H^{\otimes n}} \frac{1}{2^{(n+1)/2}} \sum_{\mathbf{z}} (1 + (-1)^{\mathbf{s}\cdot\mathbf{z}})|\mathbf{z}\rangle$$

$$= \frac{1}{2^{(n-1)/2}} \sum_{\mathbf{z}\in\{\mathbf{s}\}^{\perp}} |\mathbf{z}\rangle$$

➔ no amplitude for $\boldsymbol{s}\cdot\boldsymbol{z}$=1 (mod 2)
i.e. only get $\boldsymbol{z}$ orthogonal to $\boldsymbol{s}$

$s \cdot z = 0$

Simon's algorithm (cont'd)*

$$\frac{1}{\sqrt{2}}(|\mathbf{0}\rangle + |\mathbf{s}\rangle) \xrightarrow{H^{\otimes n}} \frac{1}{2^{(n-1)/2}} \sum_{\mathbf{z}\in\{\mathbf{s}\}^{\perp}} |\mathbf{z}\rangle$$

- More generally $$\frac{1}{\sqrt{2}}(|\mathbf{x}\rangle + |\mathbf{x}\oplus\mathbf{s}\rangle) \xrightarrow{H^{\otimes n}} \frac{1}{2^{(n-1)/2}} \sum_{\mathbf{z}\in\{\mathbf{s}\}^{\perp}} (-1)^{\mathbf{x}\cdot\mathbf{z}}|\mathbf{z}\rangle$$

Algorithm for Simon's Problem

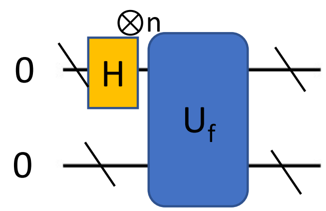
1. Set a counter $i = 1$.
2. Prepare $\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle|\mathbf{0}\rangle$.
3. Apply U_f, to produce the state

$$\sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle|f(\mathbf{x})\rangle.$$

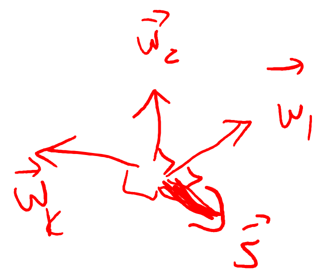
4. (optional[2]) Measure the second register.
5. Apply $H^{\otimes n}$ to the first register.
6. Measure the first register and record the value $\mathbf{w}_i$.
7. If the dimension of the span of $\{\mathbf{w}_i\}$ equals $n-1$, then go to Step 8, otherwise increment i and go to Step 2.
8. Solve the linear equation $\mathbf{W}\mathbf{s}^T = \mathbf{0}^T$ and let $\mathbf{s}$ be the unique non-zero solution.
9. Output $\mathbf{s}$.

$(|x_0\rangle + |x_0 \oplus s\rangle)|f_0\rangle$

$$U_f : |\mathbf{x}\rangle \otimes |\mathbf{b}\rangle \to |\mathbf{x}\rangle \otimes |f(\mathbf{x}) \oplus \mathbf{b}\rangle$$



$w_i \perp s$



Week 2: From foundation to science-fiction teleportation: Bell inequality, teleportation of states and gates, entanglement swapping, remote state preparation, superdense coding, and superdense teleportation

Quantum entangled states have correlations stronger than classical states

are useful as well

A simple equality and an inequality

We have seen measurement of observables X, Y, Z or any one-qubit operator

$$\vec{r}\cdot\vec{\sigma}, \quad \text{where}\, \vec{\sigma} \equiv (X, Y, Z), \quad |\vec{r}| = 1$$

read out ± 1 $(\vec{r}\cdot\vec{\sigma})$ $\boxed{|\vec{r}|=1}$

gives an eigenvalue randomly, which is ±1 in this case.

- It is interesting that for four variables a, a’, b, b’ which can be ±1, we have:

$$ab + ab' + a'b - a'b' = a(b+b') + a'(b-b') = \boxed{\pm 2}$$

$b+b' = 0$ or $+2$ -2

$b-b' = 2$ or -2 0

- Thus, for any probability distribution p(a,a’,b,b’) we have [using **E** to denote expectation]

$$-2 \le \mathbf{E}(ab + ab' + a'b - a'b') \equiv \sum_{a,a',b,b} p(a,a',b,b')(ab + ab' + a'b - a'b') \le 2$$

In the context of measuring **two choices of observables** at **two** locations A: a & a’, B: b & b’, we have the so-called Clauser-Horne-Shimony-Holt (CHSH) inequality:

$$-2 \le \mathbf{E}(a,b) + \mathbf{E}(a,b') + \mathbf{E}(a',b) - \mathbf{E}(a',b') \le 2$$

CHSH-Bell inequality (I_{2222})

CHSH generalized John Bell's idea (his original Bell inequality). The assumption is that a source emits e.g. a pair of photons

The choice of measurement axis (a or a') at A or (b or b') at B **cannot affect the outcome of the other side**. Nevertheless, **outcomes can be correlated** and described by some unknown-to-us distribution (depending on some hidden variable λ). This is also called the "Local hidden variable" theory

$$E_L(a,b) \equiv \int d\lambda\, \rho(\lambda) A(a,\lambda) B(b,\lambda)$$

where $A(a,\lambda) = \pm 1$ and $B(b,\lambda) = \pm 1$ are predetermined results for the measurement settings a for A and b for B depending on the local hidden variable λ; ρ(λ) is its distribution. Locality requires that the outcome $A(a,\lambda)$ does not depend on setting b and that of $B(b, \lambda)$ does not depend on setting a.

Violation of CHSH-Bell inequality

By averaging over the local hidden variable, we still have

$$|E_L(a,b) + E_L(a,b') + E_L(a',b) - E_L(a',b')| \leq 2. \tag{1}$$

Quantum mechanics can violate this inequality. To be specific, the operators to be measured are the Pauli operators $\vec{\sigma}$. Let $E_Q(a,b) \equiv \langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle$ denote expectation of repeated measurement with along axes of unit vectors $\vec{a}$ and $\vec{b}$, respectively. Define

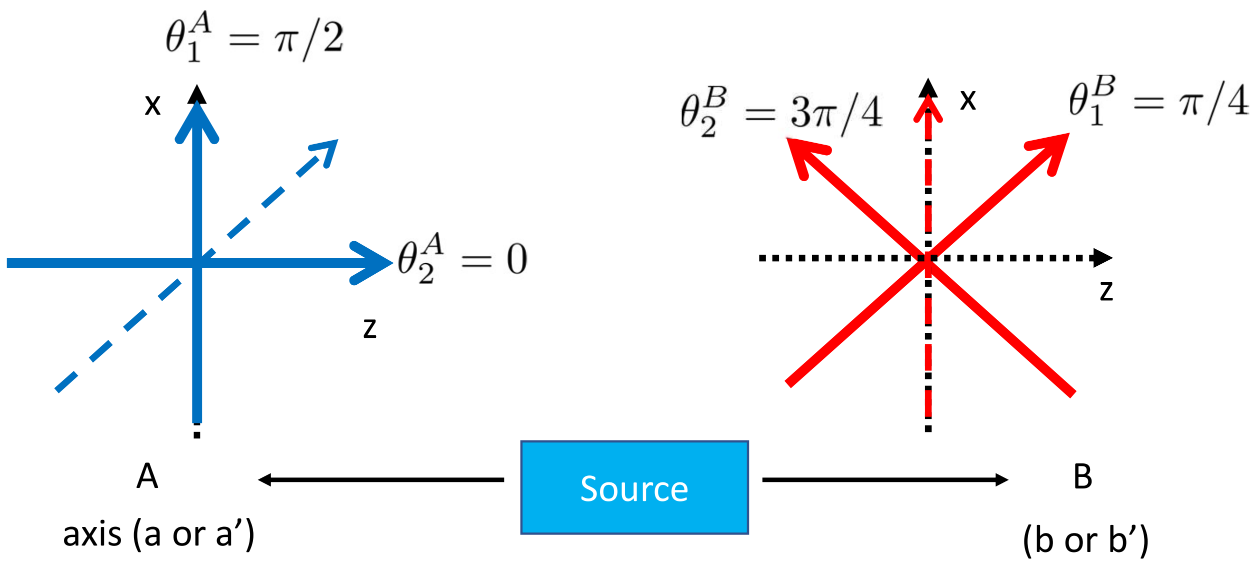
$$2B \equiv \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b} + \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b'} + \vec{\sigma}\cdot\vec{a'}\otimes\vec{\sigma}\cdot\vec{b} - \vec{\sigma}\cdot\vec{a'}\otimes\vec{\sigma}\cdot\vec{b'}.$$

For a singlet state $|\psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, $\max_{a,a',b,b'} |\langle\psi|2B|\psi\rangle| = 2\sqrt{2}$, which can be achieved for the settings $\theta_a = \pi/2$, $\theta_a' = 0$, $\theta_b = \pi/4$, and $\theta_b' = 3\pi/4$, where the angles are measured from the z-axis in the $z - x$ plane.

Violation of Bell inequality

- ❑ Measurement along axes 1 and 2 of A & B are used to check violation of Bell inequality

note: $\langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle = -\vec{a}\cdot\vec{b}$



- ➢ The bound 2√2 is the Tsirelson bound. Deriving maximal violation and measurement settings for an arbitrary state is a math problem; see Horodecki et al.

Phys. Lett. A 200, 340 (1995) and Phys. Lett. A 210, 223 (1996).