Introduction to measurement-based quantum computation

Tzu-Chieh Wei (魏子傑)
C.N. Yang Institute for Theoretical Physics
Department of Physics & Astronomy
Stony Brook University

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Goals of this tutorial*

- Give some details to understand basic ingredients of measurement-based quantum computation (MBQC)

- Give pointers to related development/ application (fewer details)

- Will point out related talks in this conference

- Give some open problems

*This tutorial assumes little prior knowledge

Review papers

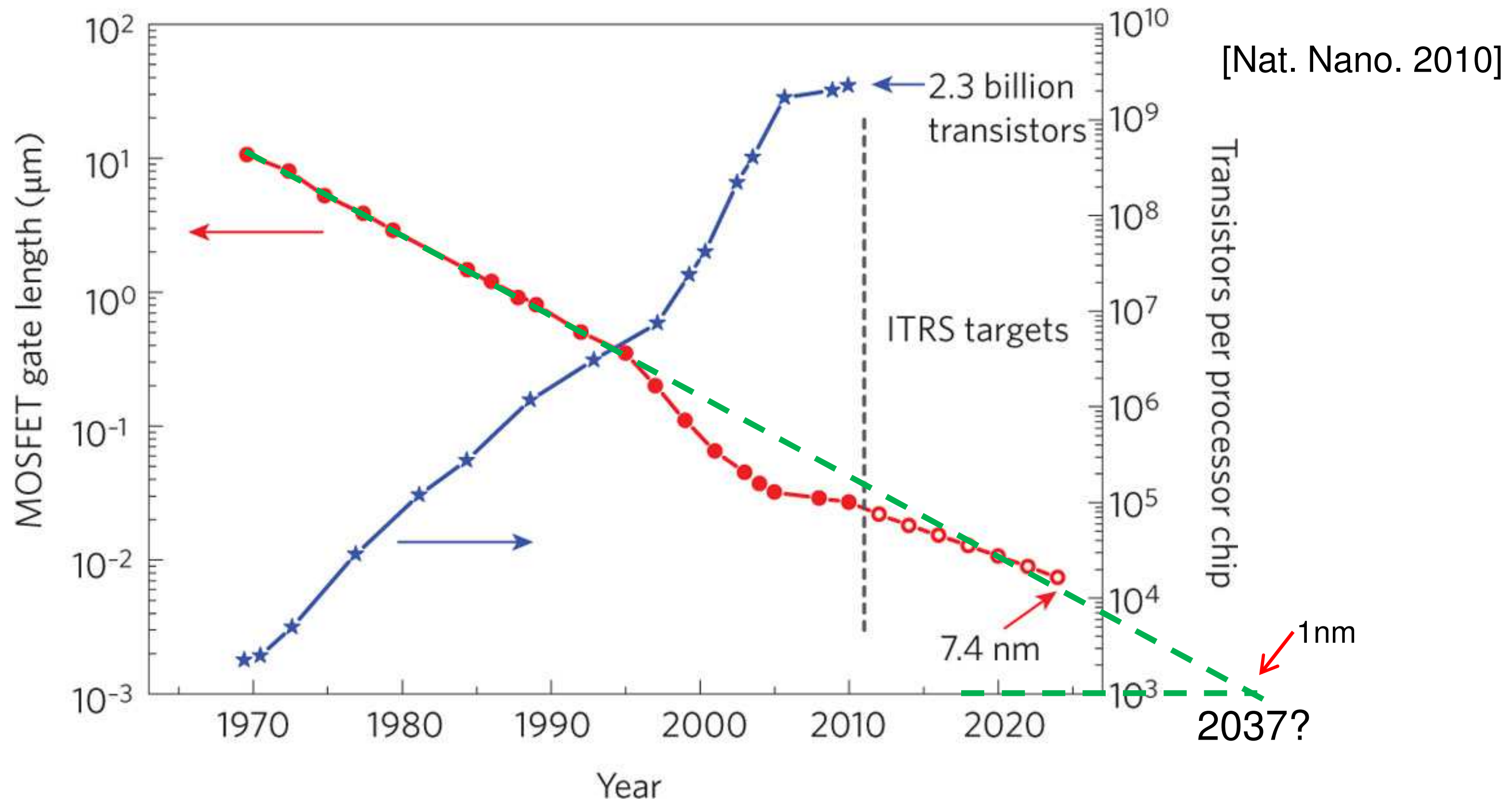
H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf & M. Van den Nest, "Measurement-based quantum computation" Nat. Phys. **5**, 19 (2009)

R. Raussendorf &T-C Wei, "Quantum computation by local measurement", Annual Review of Condensed Matter Physics, 3, 239 (2012)

L.C. Kwek, Z.H. Wei & Bei Zeng. "Measurement-Based Quantum Computing with Valence-Bond-Solids", Int. J. Mod. Phys. B 26, 123002 (2012)

Moore's Law:

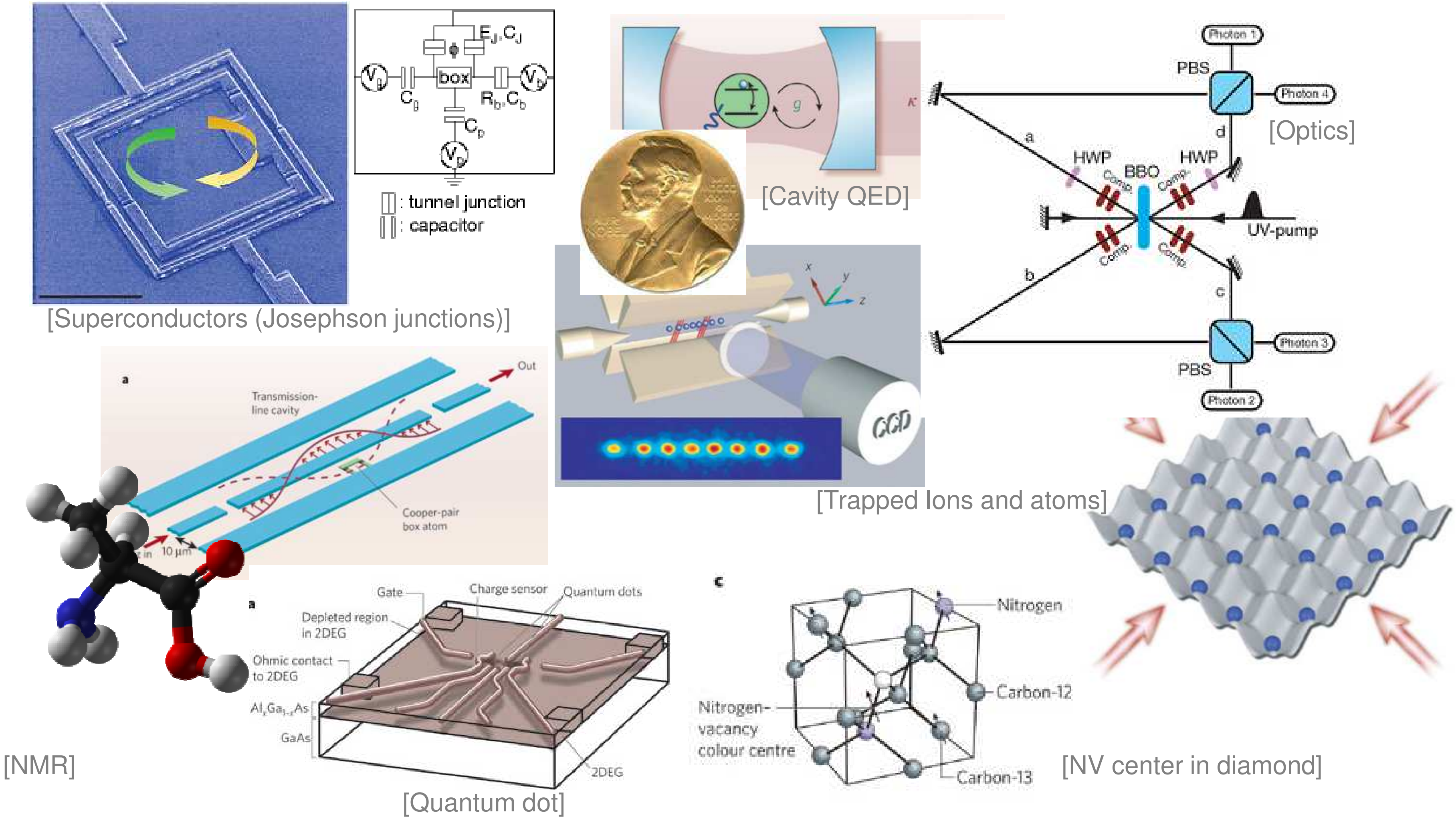
The number of transistors on a chip doubles ~every 2 years



➔ A transistor hits the size of a few atoms in about 20 years

➔ Quantum regime is inevitable

Candidate systems* for quantum computers



[Superconductors (Josephson junctions)]

[Cavity QED]

[Optics]

[Trapped Ions and atoms]

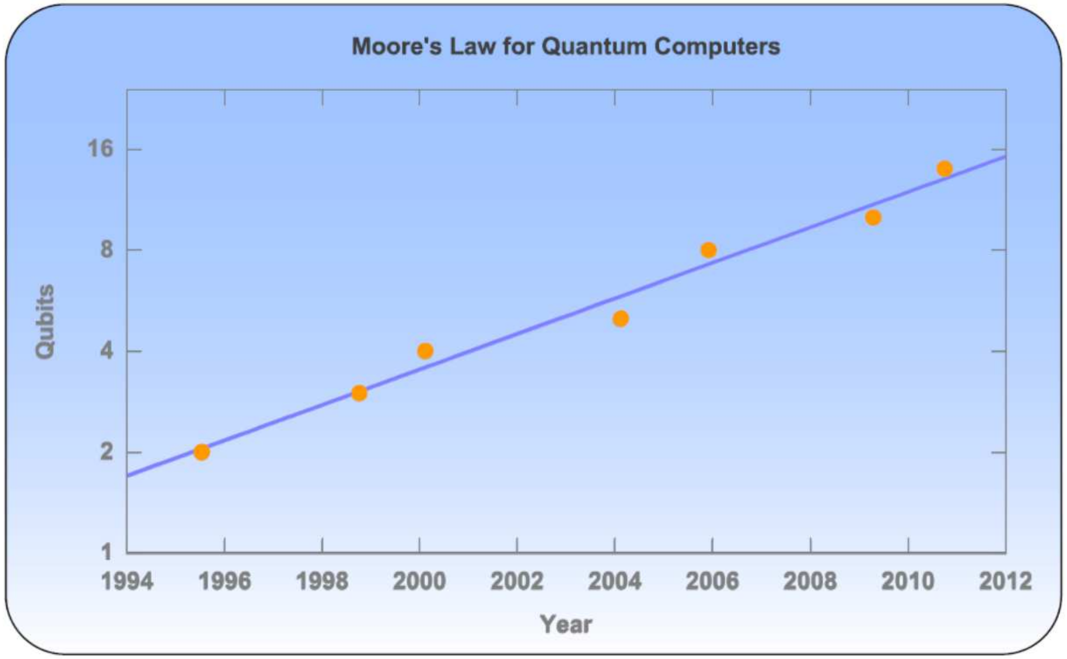
[NMR]

[Quantum dot]

[NV center in diamond]

*You may see many of these throughout this conference

New <u>quantum</u> Moore's Law?

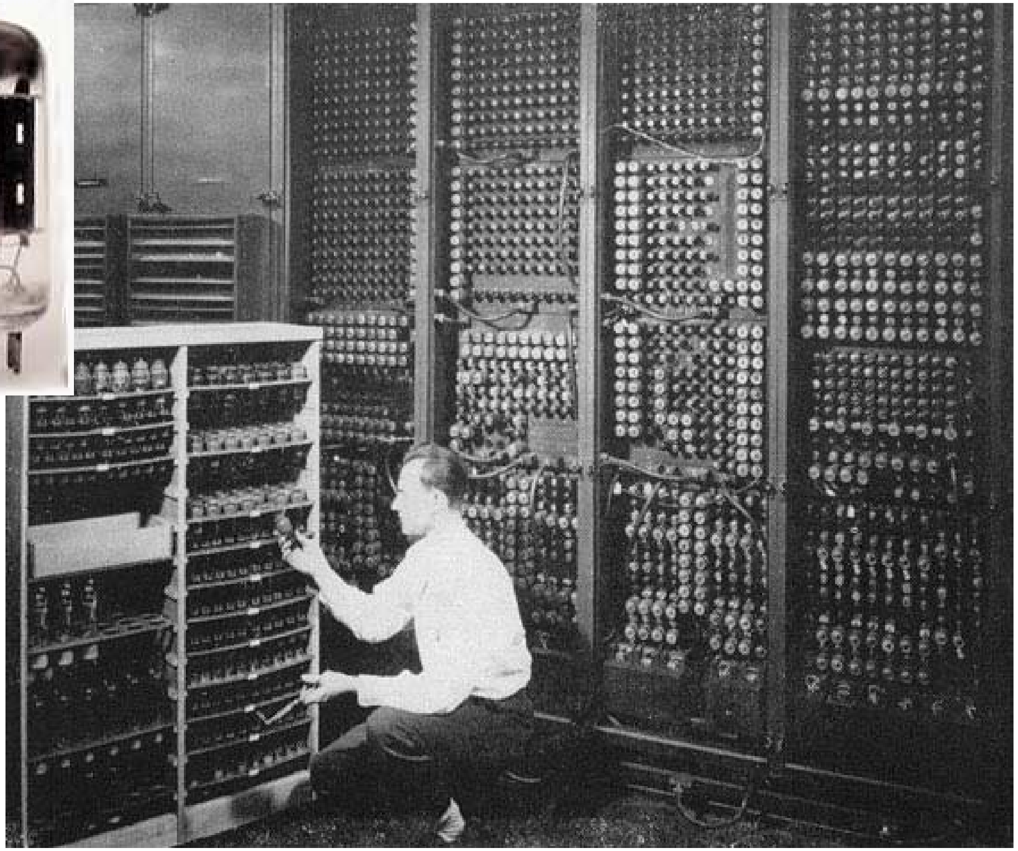


[H. Weimer]

- Number of qubits in ion trap

➔ Roughly doubles every 6 years!
(may depend on physical systems)
e.g. see Nathan Langford's tutorial on circuit QED

ENIAC – first generation computer



[1946]

Contained:
17,468 vacuum tubes,
7,200 crystal diodes,
1,500 relays,
70,000 resistors,
10,000 capacitors
5 million hand-soldered joints

Weighed 27 tons
About 8.5 by 3 by 100 feet
Took up 1800 square feet

20 ten-digit signed accumulators

When will the first-generation quantum computer appear?

Quantum computation in a nutshell

- Consider a function *f* and a corresponding unitary *U*:

$$U_f : |k\rangle \otimes |0\rangle \longrightarrow |k\rangle \otimes |f(k)\rangle$$

- Exploit quantum parallelism:

$$\left(\sum_{k=0}^{2^n-1} |k\rangle\right) \otimes |0\rangle \xrightarrow{U_f} \sum_{k=0}^{2^n-1} |k\rangle \otimes |f(k)\rangle$$

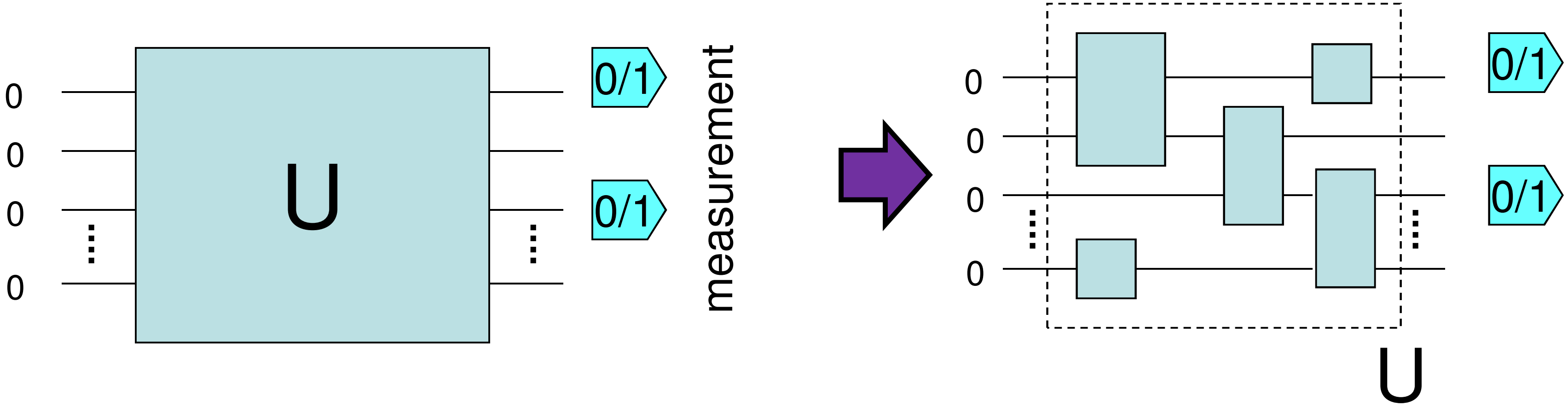
  - Naive measurement only gives one *f*(k) at a time
  - Good design of measurement may reveal properties of *f*
    ➔ *e.g. Shor's factoring algorithm*

- Factoring is hard:

1807082088687404805951656164405905566278102516769401349170127021450056662540244048387341127590812303371781887966563182013214880557 =(????....?) x (????...?)

=(39685999459597454290161126162883786067576449112810064832555157243)
x
(45534498646735972188403686897274408864356301263205069600999044599)

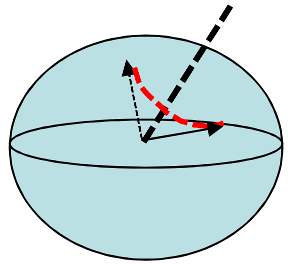
Quantum computation: Circuit model



- Building blocks

Universal gates

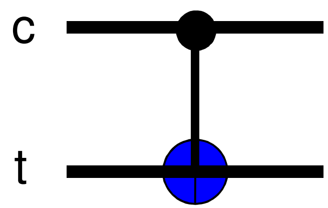
(1) One qubit gates: any rotation



(2) Two qubit gate: entangling

e.g., C-Z gate or
Controlled-NOT gate

CNOT:



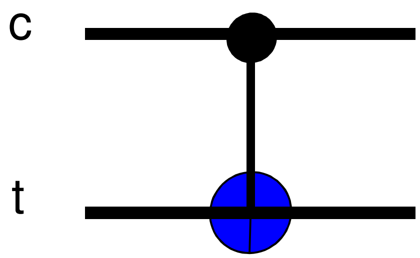
0 0 → 0 0
0 1 → 0 1
1 0 → 1 1
1 1 → 1 0

CNOT & CZ gates

CNOT:

$$\mathrm{CNOT} = |0\rangle_c\langle 0| \otimes I_t + |1\rangle_c\langle 1| \otimes X_t$$

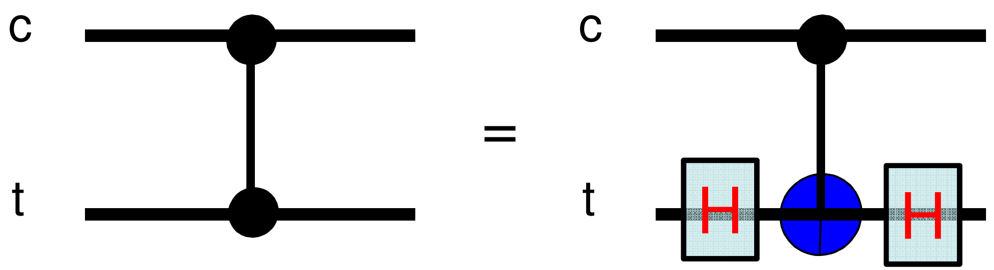
0 0 → 0 0
0 1 → 0 1
1 0 → 1 1
1 1 → 1 0



CZ:

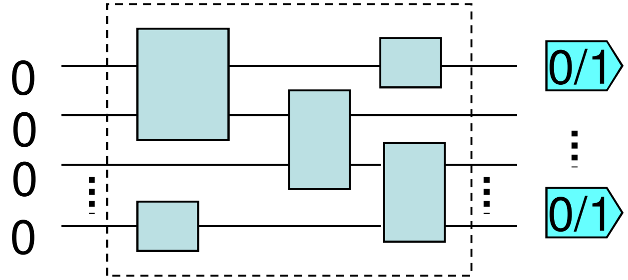
$$\mathrm{CZ} = |0\rangle_c\langle 0| \otimes I_t + |1\rangle_c\langle 1| \otimes Z_t$$

0 0 → 0 0
0 1 → 0 1
1 0 → 1 0
1 1 → - 1 1

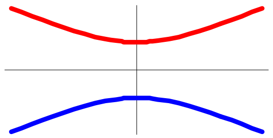


(Models of) Quantum Computation

- Circuit:

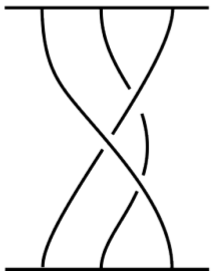


- Adiabatic:



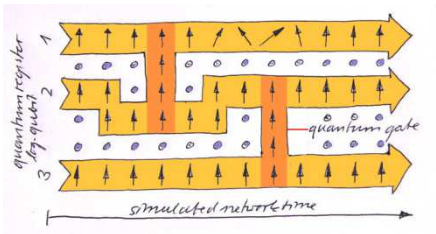
$$H(t) = \left(1 - \frac{t}{T}\right) H_{\text{initial}} + \frac{t}{T} H_{\text{final}}$$

- Topological:



using braiding of anyons to simulate quantum gates

- Measurement-based:



local measurement is the only operation needed

Outline

I. Introduction

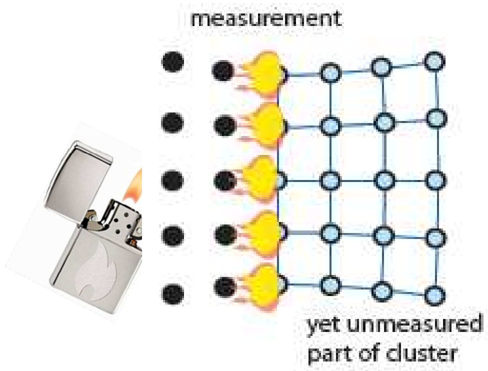
II. One-way cluster-state (or measurement-based) quantum computers

III. Other entangled resource states: Affleck-Kennedy -Lieb-Tasaki (AKLT) family

IV. Summary

Now focus on measurement-based
(or one-way) quantum computer:

which can "simulate" unitary evolution



Unitary operation by measurement?

- Intuition: entanglement as resource!

  - Controlled-Z (CZ) gate from Ising interaction

$$CZ_{12} = e^{-i\frac{\pi}{4}(1-\sigma_Z^{(1)})(1-\sigma_Z^{(2)})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

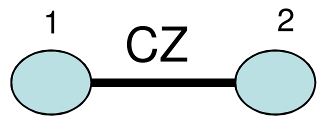
  - Entanglement is generated:

$$(a|0\rangle + b|1\rangle)\,|+\rangle \xrightarrow{CZ} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$

Unitary operation by measurement?

- Intuition: entanglement as resource!

$$(a|0\rangle + b|1\rangle)\,|+\rangle \xrightarrow{\text{CZ}} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$



$$|\mathrm{in}\rangle_1 \Longrightarrow U_1|\mathrm{in}\rangle_2$$

measurement

- Measurement on 1st qubit in basis

$$|\pm\xi\rangle = \left(e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle\right)/\sqrt{2}$$

with outcome denoted by $\pm = (-1)^s$

$$\cos(\xi)\sigma_x + \sin(\xi)\sigma_y$$

$\equiv$ $\equiv$

X *Y*

➔ Second qubit becomes

$${}_1\langle\pm\xi|\psi\rangle_{12} \sim a\,e^{i\xi/2}|+\rangle_2 \pm b\,e^{-i\xi/2}|-\rangle_2 \;= H\,e^{i\xi Z/2}Z^s(a|0\rangle_2 + b|1\rangle_2)$$

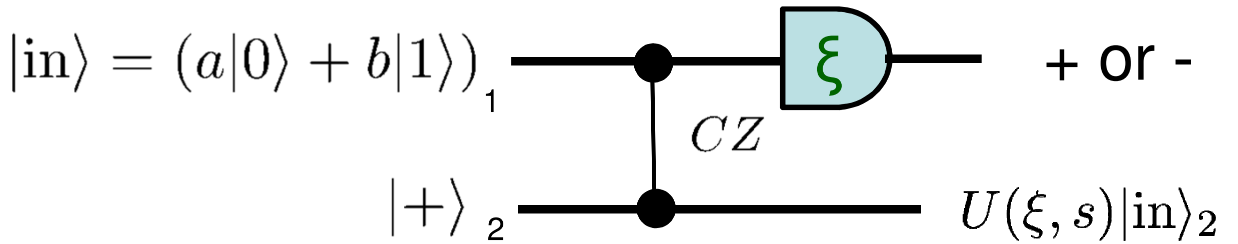
➔ A unitary gate is induced: $U(\xi, s) \equiv H\,e^{i\xi Z/2}Z^s$

$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Simulating arbitrary one-qubit gates

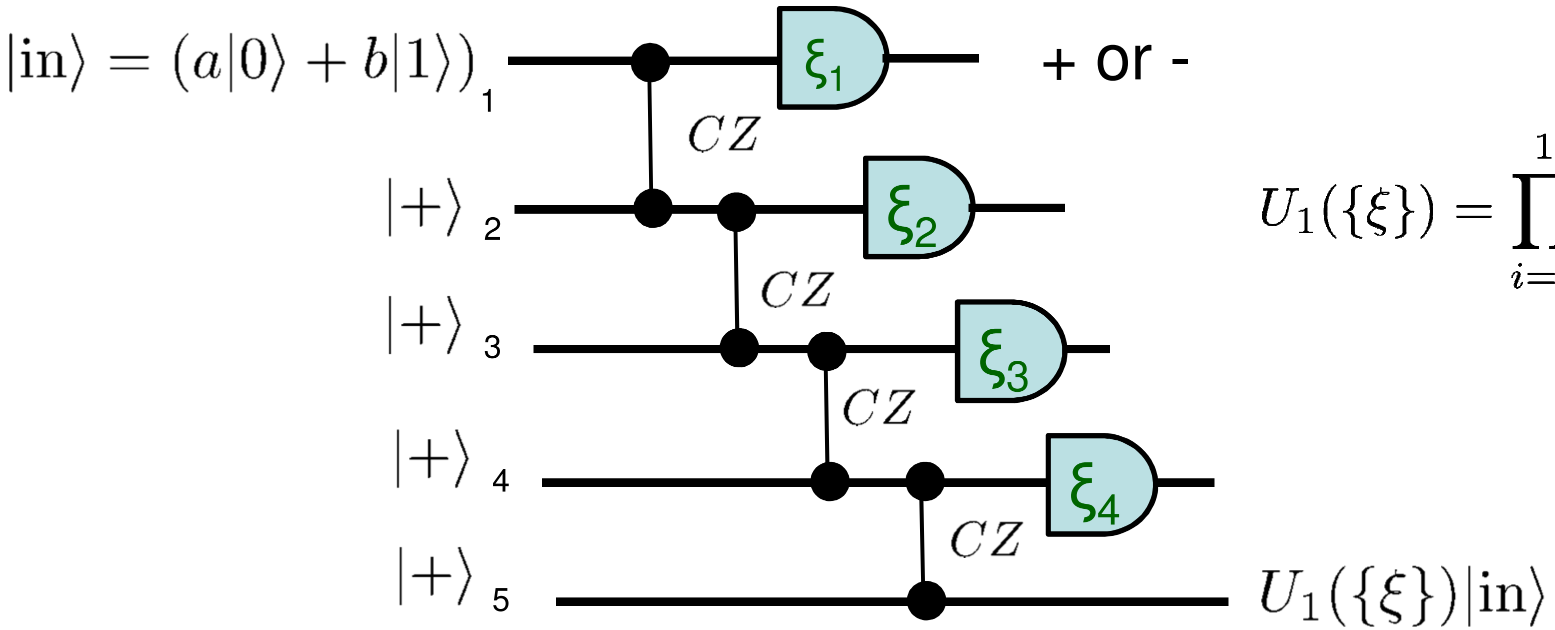
- In terms of circuit:

[*Raussendorf &* ***Wei***, Ann Rev Cond-Mat '12]



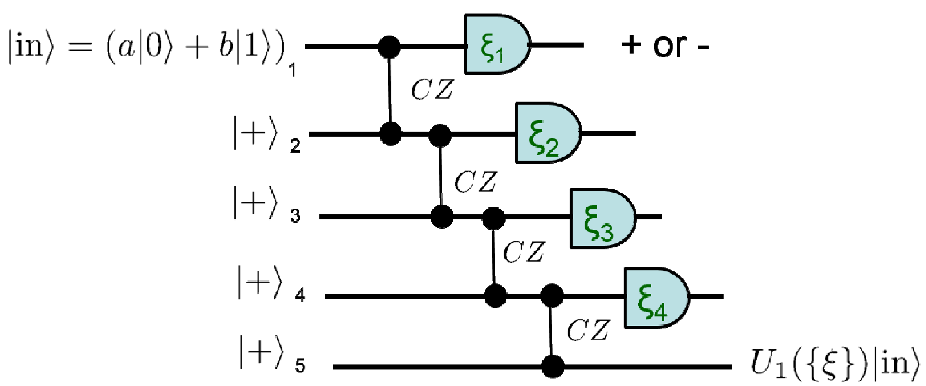
$$U(\xi, s) = H\, e^{i\xi Z/2} Z^s$$

- Can cascade this a few times:



$$U_1(\{\xi\}) = \prod_{i=4}^{1} U(\xi_i, s_i)$$

Example: arbitrary one-qubit gate



$$U_1(\{\xi\}) = \prod_{i=4}^{1} U(\xi_i, s_i) \qquad U(\xi, s) = H\, e^{i\xi Z/2} Z^s$$

- Consider: $\xi_1=0$ & construct arbitrary rotation

$$U_1(\{\xi\}) = \left(He^{i\xi_4 Z/2} Z^{s_4}\right)\left(He^{i\xi_3 Z/2} Z^{s_3}\right)\left(He^{i\xi_2 Z/2} Z^{s_2}\right)\left(HZ^{s_1}\right)$$

- Propagating Z's to left and use *HZH=X*:

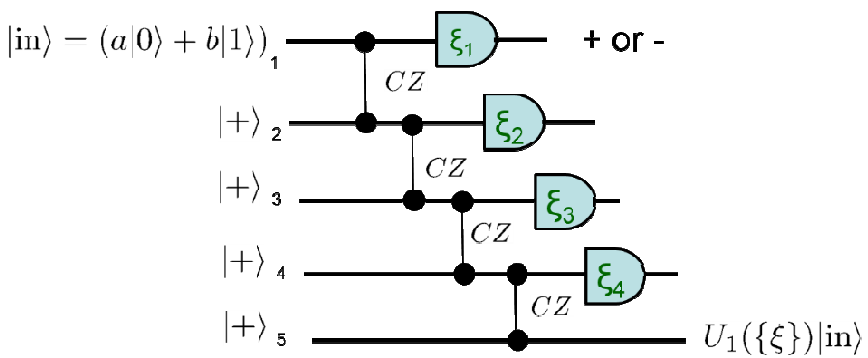
$$U_1(\{\xi\},\{s\}) = Z^{s_1+s_3} X^{s_2+s_4}\, e^{i(-1)^{s_1+s_3}\xi_4 X/2}\, e^{i(-1)^{s_2}\xi_3 Z/2}\, e^{i(-1)^{s_1}\xi_2 X/2}$$

- Take $\xi_2 = -(-1)^{s_1}\gamma,\ \xi_3 = -(-1)^{s_2}\beta,\ \xi_4 = -(-1)^{s_1+s_3}\alpha$

  we realize an Euler rotation, up to byproduct Z, X operators:

$$U_1(\{\xi\},\{s\}) = Z^{s_1+s_3} X^{s_2+s_4}\, e^{-i\alpha X/2}\, e^{-i\beta Z/2}\, e^{-i\gamma X/2}$$

Comments



$$U_1(\{\xi\}) = \prod_{i=4}^{1} U(\xi_i, s_i) \qquad U(\xi, s) = H\, e^{i\xi Z/2} Z^s$$

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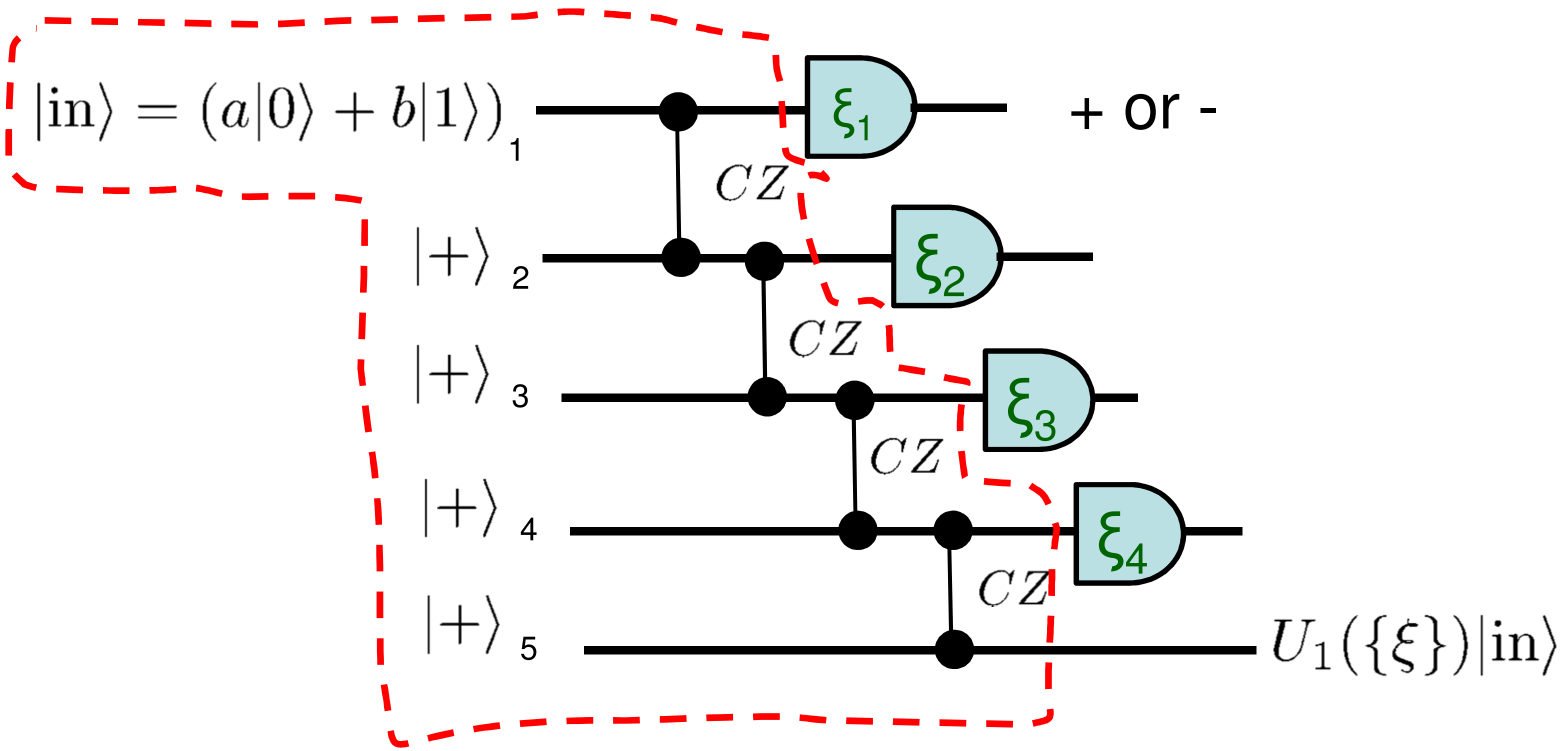
$$U_1(\{\xi\},\{s\}) = Z^{s_1+s_3} X^{s_2+s_4}\, e^{-i\alpha X/2}\, e^{-i\beta Z/2}\, e^{-i\gamma X/2}$$

➔ Note: measurement basis can depend on prior results

➔ Byproduct operators $Z^{s_1+s_3}X^{s_2+s_4}$ can be absorbed by modifying later measurement basis

➔ Byproduct operators on final measurement in Z basis (readout) can be easily taken into account (only X flips 0/1)

Linear cluster state: resource for simulating arbitrary one-qubit gates

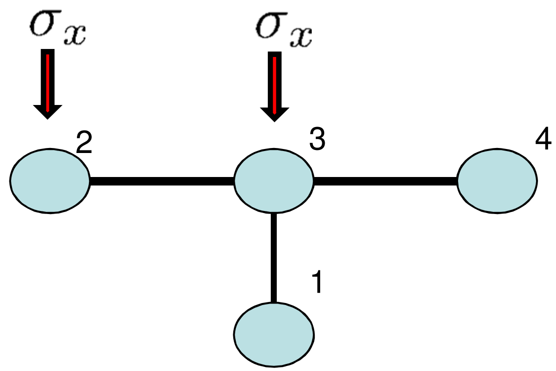


- May as well take $|\text{in}\rangle = |+\rangle$ the whole state before measurement ξ's is a highly entangled state ➔ 1D cluster state

Simulating CNOT by measurement

- Consider initial state

$$(a|0\rangle + b|1\rangle)_1\,(c|0\rangle + d|1\rangle)_2\,|+\rangle_3\,|+\rangle_4 \xrightarrow{CZ_{23}\,CZ_{13}\,CZ_{34}} |\psi\rangle_{1234}$$



$$|\psi\rangle_{1234} = |0\rangle_3\big(a|0\rangle_1 + b|1\rangle_1\big)\big(c|0\rangle_2 + d|1\rangle_2\big)|+\rangle_4 + |1\rangle_3\big(a|0\rangle_1 - b|1\rangle_1\big)\big(c|0\rangle_2 - d|1\rangle_2\big)|-\rangle_4$$

$$|\psi_{\mathrm{in}}\rangle_{12} \Rightarrow \mathrm{CNOT}|\psi_{\mathrm{in}}\rangle_{14}$$

- Measurement on 2nd and 3rd qubits in basis $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$

If outcome=++: an effective CNOT applied:

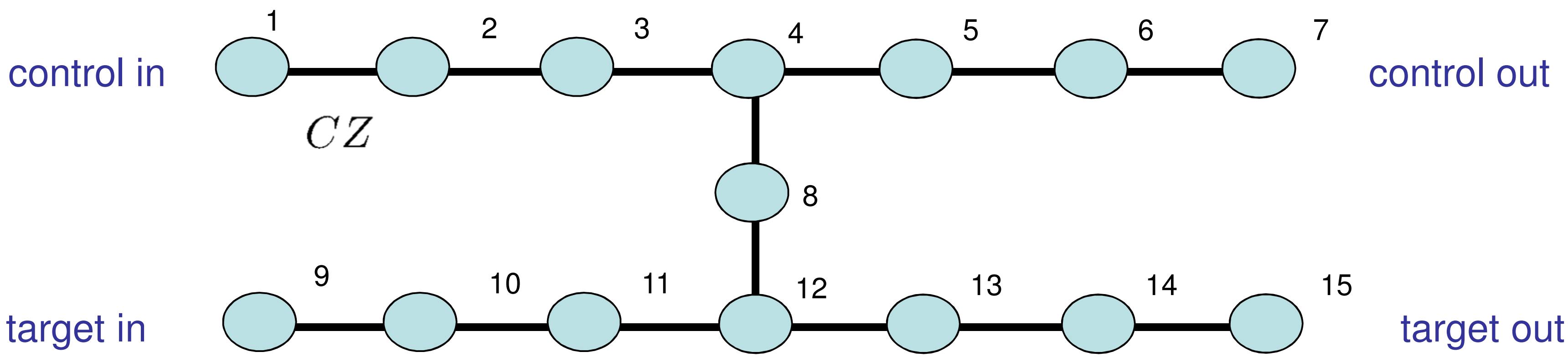
$$|\psi\rangle_{14} = {}_{23}\langle + + |\psi\rangle_{1234} \sim \mathrm{CNOT}_{14}(a|0\rangle_1 + b|1\rangle_1)(c|0\rangle_4 + d|1\rangle_4)$$

Can show: $|\psi_{\mathrm{out}}\rangle \sim Z_1^{s_2}\,X_4^{s_3}\,Z_4^{s_2}\,\mathrm{CNOT}_{14}|\mathrm{in}\rangle_{14}$

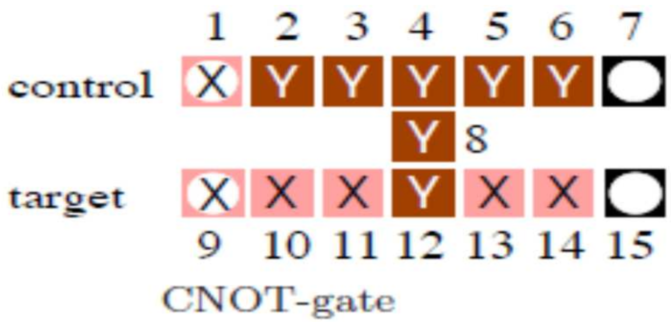
- Note the action of CZ gates can be pushed up front
(a 4-qubit "cluster" state can be used to simulating CNOT)

CNOT gate: symmetric design

[*Raussendorf &Briegel* PRL 01']



➔ The following measurement pattern simulates CNOT gate (via entanglement between wires)



Q: how do I know it implements CNOT?
byproduct operators=?

Ans: see Theorem I in Raussendorf, Browne & Briegel PRA '03
(generalization to qudit: Zhou et al. PRA '03)

Theorem 1. Let $\mathcal{C}(g)=\mathcal{C}_I(g)\cup\mathcal{C}_M(g)\cup\mathcal{C}_O(g)$ with $\mathcal{C}_I(g)\cap\mathcal{C}_M(g)=\mathcal{C}_I(g)\cap\mathcal{C}_O(g)=\mathcal{C}_M(g)\cap\mathcal{C}_O(g)=\varnothing$ be a cluster for the simulation of a gate g, realizing the unitary transformation U, and $|\phi\rangle_{\mathcal{C}(g)}$ the cluster state on the cluster $\mathcal{C}(g)$.

Suppose the state $|\psi\rangle_{\mathcal{C}(g)}=P_{\{s\}}^{(\mathcal{C}_M(g))}(\mathcal{M})\,|\phi\rangle_{\mathcal{C}(g)}$ obeys the $2n$ eigenvalue equations

$$\sigma_x^{(\mathcal{C}_I(g),i)}(U\sigma_x^{(i)}U^\dagger)^{(\mathcal{C}_O(g))}|\psi\rangle_{\mathcal{C}(g)}=(-1)^{\lambda_{x,i}}|\psi\rangle_{\mathcal{C}(g)}, \tag{61}$$

$$\sigma_z^{(\mathcal{C}_I(g),i)}(U\sigma_z^{(i)}U^\dagger)^{(\mathcal{C}_O(g))}|\psi\rangle_{\mathcal{C}(g)}=(-1)^{\lambda_{z,i}}|\psi\rangle_{\mathcal{C}(g)},$$

with $\lambda_{x,i},\lambda_{z,i}\in\{0,1\}$ and $1\leqslant i\leqslant n$.

Then, on the cluster $\mathcal{C}(g)$ the gate g acting on an arbitrary quantum input state $|\psi_{\text{in}}\rangle$ can be realized according to Scheme 1 with the measurement directions in $\mathcal{C}_M(g)$ described by $\mathcal{M}^{(\mathcal{C}_M(g))}$ and the measurements of the qubits in $\mathcal{C}_I(g)$ being σ_x measurements. Thereby, the input and output state in the simulation of g are related via

$$|\psi_{\text{out}}\rangle=UU_\Sigma|\psi_{\text{in}}\rangle, \tag{62}$$

where U_Σ is a byproduct operator given by

$$U_\Sigma=\bigotimes_{(\mathcal{C}_I(g)\ni i)=1}^{n}(\sigma_z^{[i]})^{s_i+\lambda_{x,i}}(\sigma_x^{[i]})^{\lambda_{z,i}}. \tag{63}$$

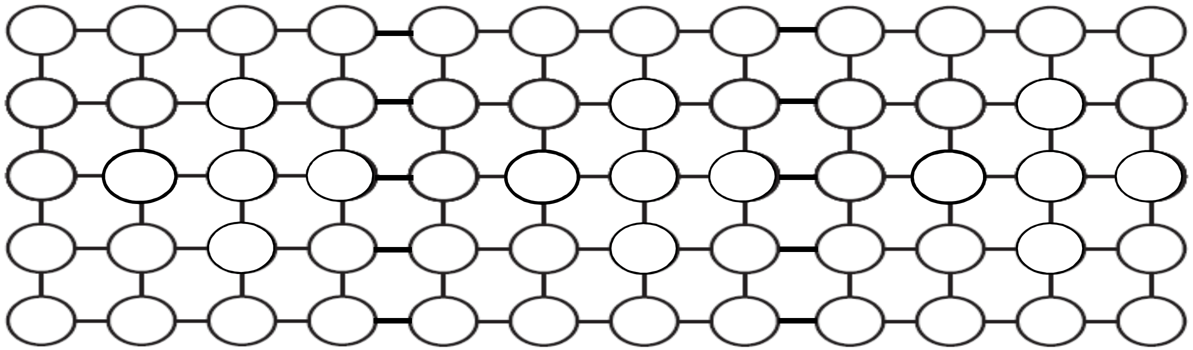
[Raussendorf, Browne & Briegel PRA '03]

2D cluster state and graph states

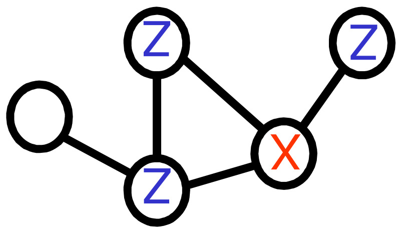
- Can be created by applying CZ gates to each pair with edge

$$|G\rangle = \bigotimes_{\langle i,i\rangle} CZ_{ij} \left(|+\rangle|+\rangle\cdots|+\rangle\right)$$

[Raussendorf&Briegel '01]



- Cluster state: special case of general "graph" states



$$K_v|G\rangle = |G\rangle, \;\forall\, \text{vertex}\, v$$

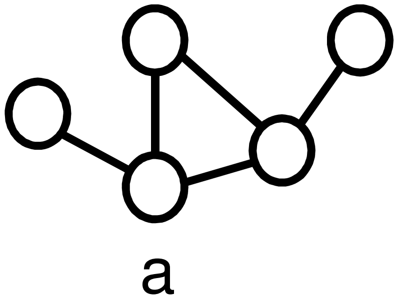
$$K_v = X_v \bigotimes_{u\in \mathrm{Nb}(v)} Z_u$$

(can show this, using above def. of G)

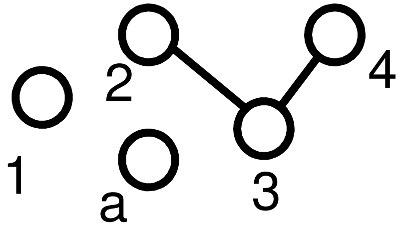
➔ Uniquely define the state G, also via Hamiltonian $H = -\sum_v K_v$

Z measurement on graph state

- The effect is just to remove the measured qubit, keeping the remaining entanglement structure



$$|\Psi_G = |0\rangle_a|\Psi_{G\setminus a}\rangle + |1\rangle_a \left(\prod_{b\in NB(a)} Z_b \right) |\Psi_{G\setminus a}\rangle$$



➔ Graph after Z measurement on a:

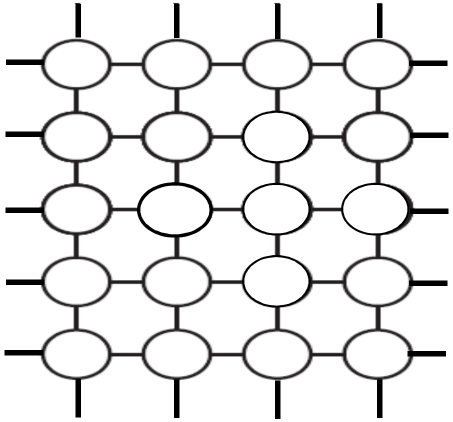
✓ If outcome =0: $|0\rangle_a|+\rangle_1|C\rangle_{234}$ $\quad |C\rangle_{234}$: linear cluster state

✓ If outcome =1: $|0\rangle_a\,|-\rangle_1\,Z_2Z_3|C\rangle_{234}$

- For X & Y measurements, see [Hein, Eisert, Briegel '04, Hein et al. '06]

2D cluster state is a resource for quantum computation

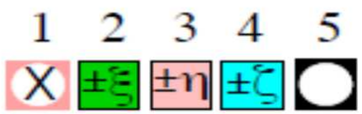
$$|C\rangle = \bigotimes_{\langle i,i\rangle} CZ_{ij}\left(|+\rangle|+\rangle\cdots|+\rangle\right)$$



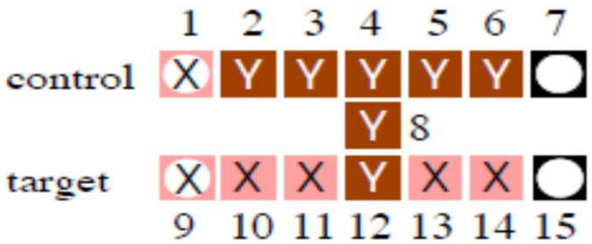
- Whole entangled state is created first (by whatever means)
- Operations needed for *universal* QC are single-qubit measurements only

➔ Pattern of measurement gives computation
(entanglement is being consumed → one-way)

➔ Elementary "Lego pieces" for QC:

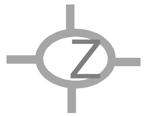


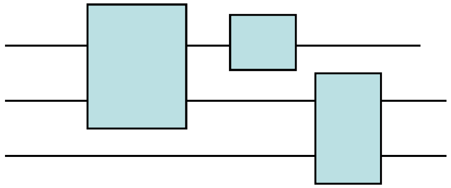
general rotation

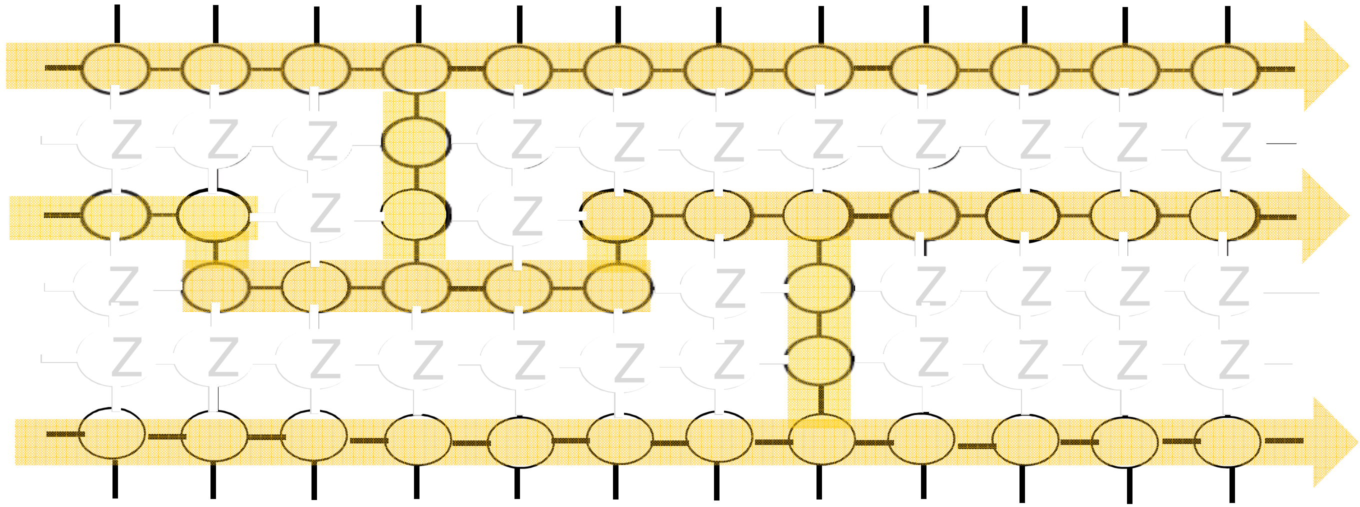


CNOT-gate

Cluster state for universal computation

- Carve out entanglement structure by local Z measurement 





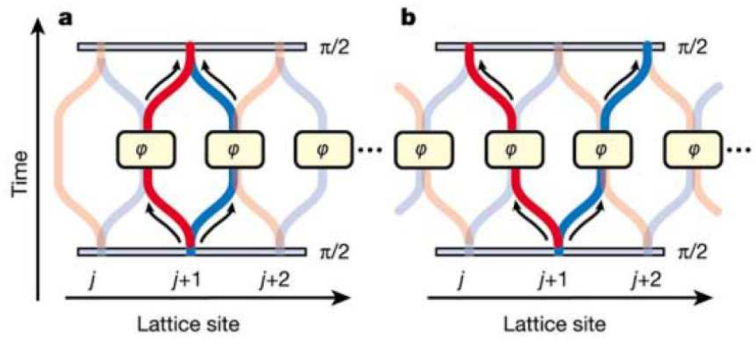
(1) Each wire simulates one-qubit evolution (gates)

(2) Each bridge simulates two-qubit gate (CNOT)

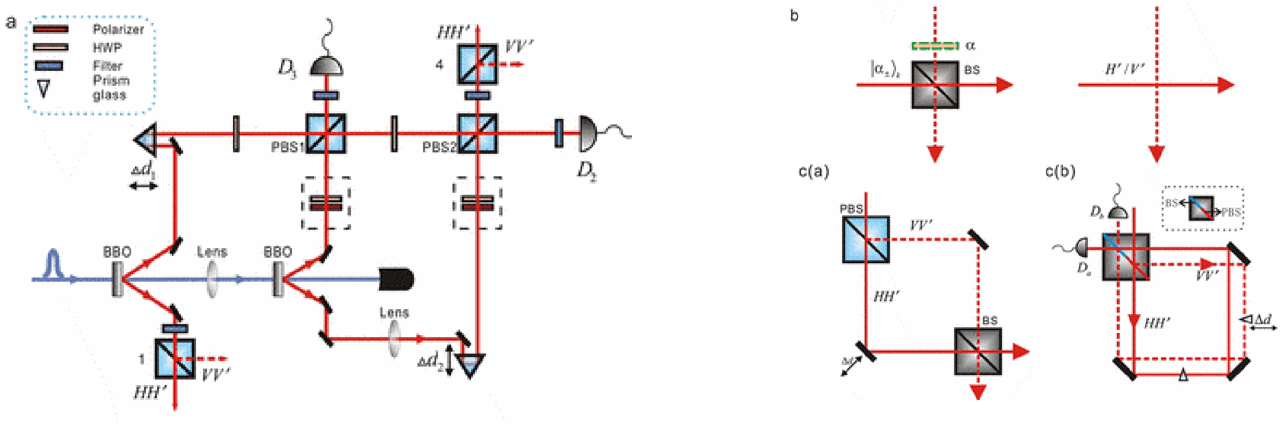
 2D or higher dimension is needed for universal QC & Graph connectivity is essential (percolation)

Realizations of cluster states

- Bloch's group: controlled collision in cold atoms (Nature 2003)

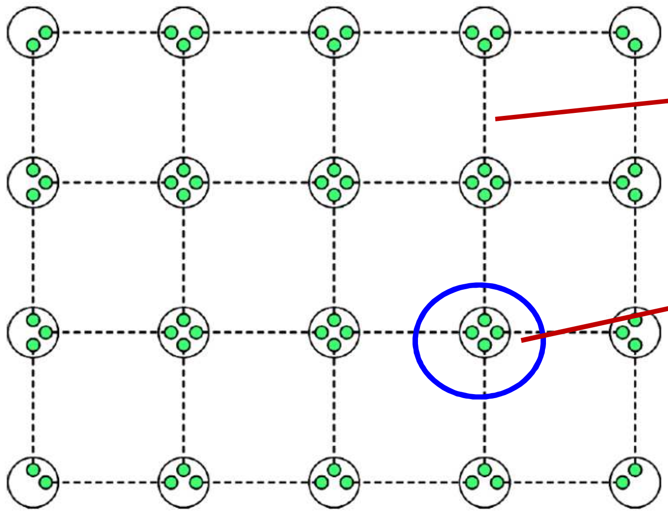


- J-W Pan's group: 4-photon 6 qubit and CNOT (PRL 2010)



Cluster state: a valence-bond picture

- Cluster state = a valence-bond state [Verstraete & Cirac ’04]
  = a projected entangled pair state (PEPS)



- ➢ Bond of two virtual qubits =

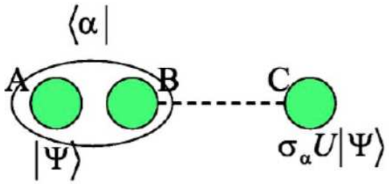
$$\mathrm{CZ}|++\rangle = |0\rangle|+\rangle + |1\rangle|-\rangle$$

- ➢ Projection of several virtual qubits to physical qubit =

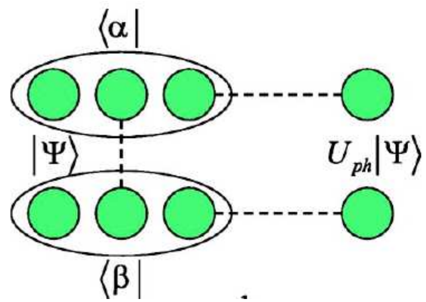
$$P = |0\rangle\langle 0000| + |1\rangle\langle 1111|$$

- Quantum computation via teleportation [see also Gottesman & Chuang ‘99]

➢ 1-qubit gate:



➢ 2-qubit gate:

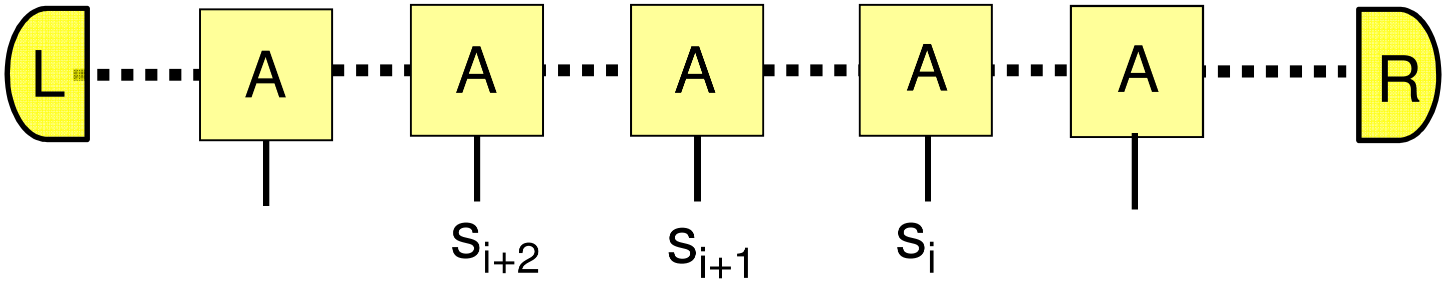


QC in correlation space

- Previous picture of valence bond was generalized by Gross and Eisert using matrix product states (MPS) and PEPS

[Gross & Eisert '07, Gross et al. '07]

- Illustrate with 1D cluster state:

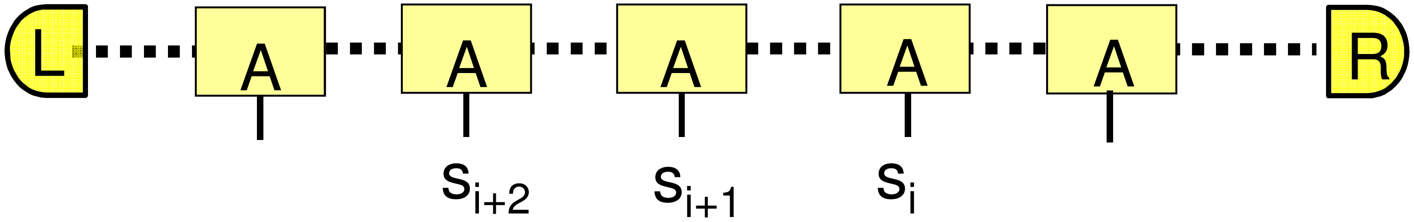


$$|\Psi\rangle = \sum_{\{s\}'s} \vec{L} \cdot A_{s_n} \cdots A_{s_{i+1}} \cdot A_{s_i} \cdots A_{s_1} \cdot \vec{R}\, |s_n, ..., s_i, ...s_1\rangle$$

- Measurement outcome φ_i at site i: $$A(\phi_i) \equiv \sum_{s_i} \langle \phi_i | s_i \rangle A_{s_i}$$

$$\langle \phi_n, ...\phi_i, ..., \phi_1 | \Psi \rangle = \vec{L} \cdot A(\phi_n) \cdots A(\phi_i) \cdots A(\phi_1) \cdot \vec{R}$$

Cluster state QC: in correlation space



- Measurement outcome φ_i at site i: $A(\phi_i) \equiv \sum_{s_i} \langle \phi_i | s_i \rangle A_{s_i}$

$$\langle \phi_n, ...\phi_i, ..., \phi_1 | \Psi \rangle = \vec{L} \cdot A(\phi_n) \cdots A(\phi_i) \cdots A(\phi_1) \cdot \vec{R}$$

- As spins are measured, the boundary vector R is operated by gates

$$|R\rangle \rightarrow A_1(\phi_1)|R\rangle \rightarrow A_1(\phi_2)A_1(\phi_1)|R\rangle \rightarrow \cdots$$

- For 1D cluster state: $A(0) = |+\rangle\langle 0|,\ A(1) = |-\rangle\langle 1|$
  - ➔ measure in basis $|\pm\xi\rangle = \left(e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle\right)/\sqrt{2}$
  - ➔ obtain same 1-qubit gate as before:

$$A(\xi, s) = e^{i\xi/2}|+\rangle\langle 0| + (-1)^s e^{-i\xi/2}|-\rangle\langle 1| = He^{i\xi Z/2}Z^s$$

- 2-qubit gates use 2D PEPS ➔ see Gross & Eisert ‘07

Comment: deriving MPS for cluster state

$$|0+\rangle + |1-\rangle = \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} = \begin{pmatrix} |+0\rangle & |+1\rangle \\ |-0\rangle & |-1\rangle \end{pmatrix}$$

$$P_v = |0\rangle\langle 00| + |1\rangle\langle 11|$$

- MPS form:

$$P_v \begin{pmatrix} |+0\rangle & |+1\rangle \\ |-0\rangle & |-1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |0\rangle & |1\rangle \\ |0\rangle & -|1\rangle \end{pmatrix} = |0\rangle \left(|+\rangle\langle 0|\right) + |1\rangle \left(|-\rangle\langle 1|\right)$$

$$A(0) = |+\rangle\langle 0|, \;\; A(1) = |-\rangle\langle 1|$$

Related talk:

Monday Session A: 4. [3:00-3:20] **Anurag Anshu, Itai Arad and Aditya Jain.** *How local is the information in MPS/PEPS tensor networks?*

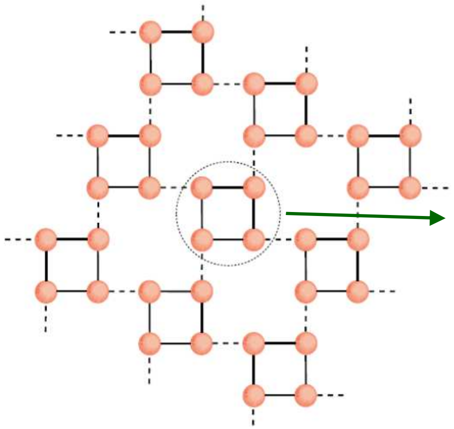
Cluster states: not unique ground state of 2-body Hamiltonians

- First proved by Nielsen [Haselgrov, Nielsen & Osborne '03, Nielsen '04]
- Van den Nest et al. proved for general (connected) graph states G:

  ➔ For approximation: ground-state of 2-body Hamiltonian can be ϵ-close to G, but the gap is proportional to ϵ [Van den Nest et al. '08]

- Bartlett & Rudolph constructed a two-body Hamiltonian such that the ground state is approximately an encoded cluster state

[Bartlett & Rudolph '06]



$$H_S = -\sum_{\mu \in S} \sum_{i \sim i'} \sigma^z_{(\mu,i)} \otimes \sigma^z_{(\mu,i')}$$

$$V = -\sum_{(\mu,i)\sim(\nu,j)} (\sigma^z_{(\mu,i)} \otimes \sigma^x_{(\nu,j)} + \sigma^x_{(\mu,i)} \otimes \sigma^z_{(\nu,j)})$$

- Darmawan & Bartlett constructed encoded cluster state by deforming the AKLT Hamiltonian [Darmawan & Bartlett '14]

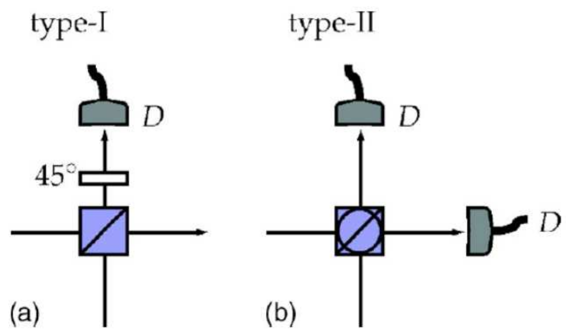
Linear optic QC & cluster state

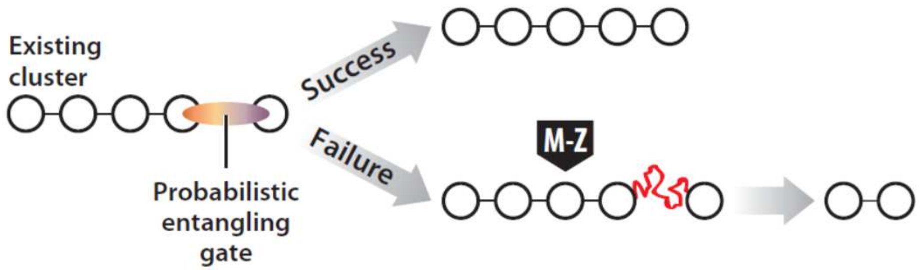
- Linear optic universal QC possible with single photon source, linear optic elements (beam splitters, mirrors, etc) & photon counting

  ➔ High overhead in entangling gates [Knill, Laflamme & Milburn '01]

- Cluster state helps reduce this overhead [Yoran & Reznik '03; Nielsen '04; Browen & Ruldoph '05; Kieling, Rudolph &Eisert '07]

  ➔ Grow cluster states efficiently



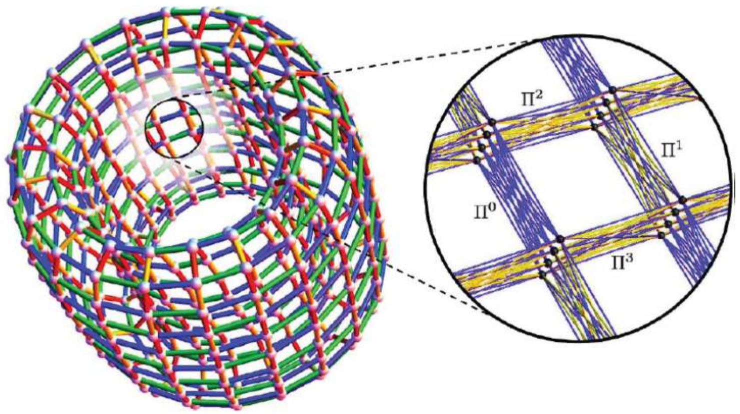


- Experiments: see e.g. [O'Brien Science '07]

Create continuous-variable cluster states

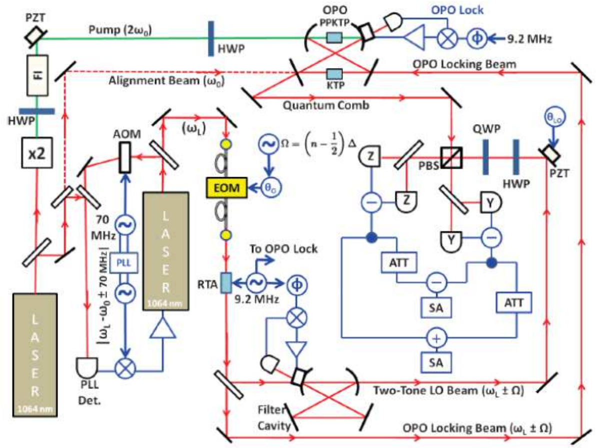
- ❑ Use frequency comb and parametric amplifier in cavity

  - ➢ Theory: [Menicucci et al ‘06, ’08]

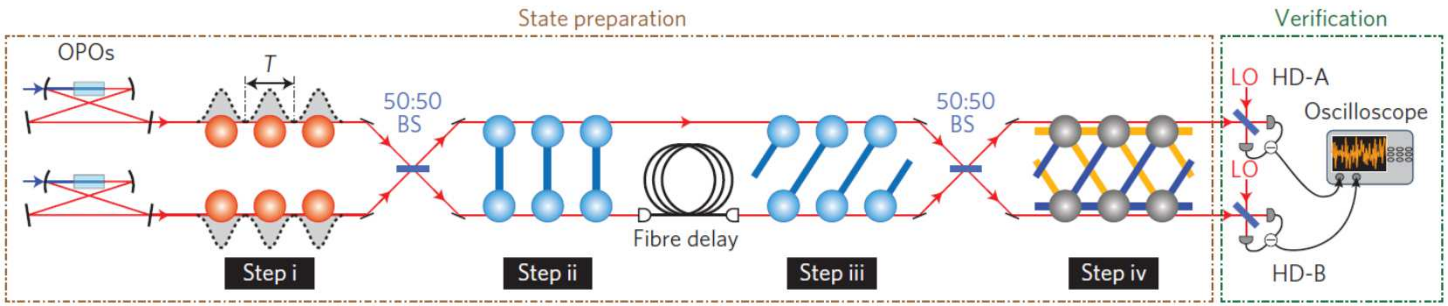


$X(s) = e^{-is\hat{p}}$ and $Z(t) = e^{it\hat{q}}$ $\qquad C_Z = \exp(i\hat{q} \otimes \hat{q})$

  - ➢ Experiment I: [60 modes in Pfister group ’11]



- ➢ Experiment II: [> 10,000 modes in Furusawa group ‘12]



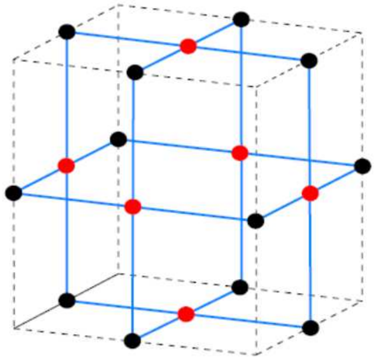
Related talks:

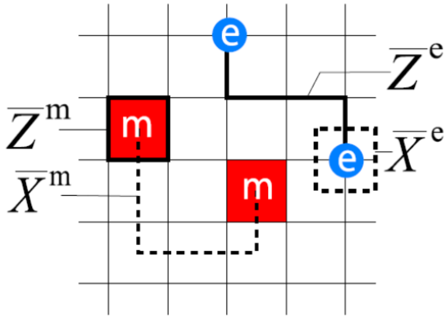
Thursday Session A:
5. [3:20-3:40] **Hoi-Kwan Lau and Martin Plenio.** *Universal Quantum Computing with Arbitrary Continuous-Variable Encoding*

6. [3:40-4:00] **Alessandro Ferraro, Oussama Houhou, Darren Moore, Mauro Paternostro and Tommaso Tufarelli.** *Measurement-based quantum computation with mechanical oscillators*

Fault tolerant cluster-state QC

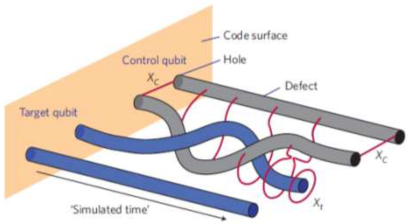
- Uses a 3d cluster state and implements surface codes in each 2d layer

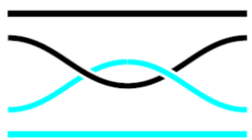


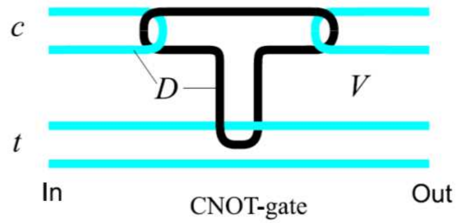


[Raussendorf, Harrington & Goyal '07]

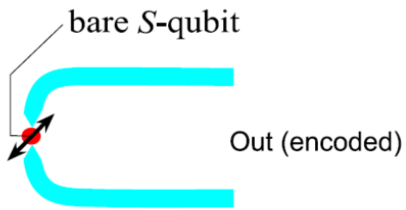
- CNOT is achievable







- Uses magic-state distillation to achieve non-Clifford gate



➔ Error threshold 0.75%, qubit loss threshold 24.9%

[Barrett & Stace '10]

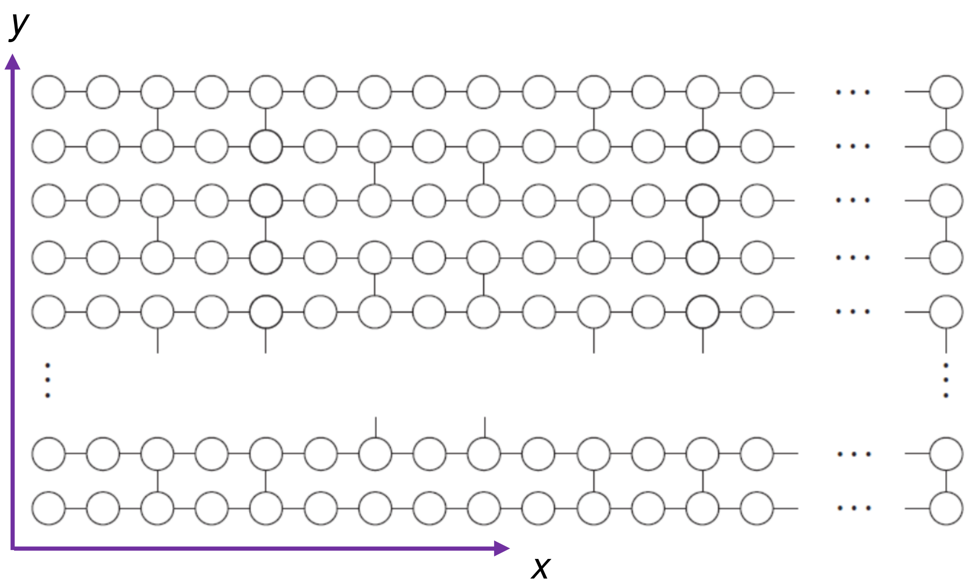
Related talk:

Friday 10:30-11:00 [Long] **Guillaume Dauphinais and David Poulin.** *Fault Tolerant Quantum Memory for non-Abelian Anyons*

Universal blind quantum computation

[Broadbent, Fitzsimons & Kashefi '09]

- Using the following cluster state (called brickwork state)



- Alice prepares

$$|\Psi\rangle = \bigotimes_{x,y} \left(|0\rangle_{x,y} + e^{i\theta_{x,y}}|1\rangle_{x,y}\right)$$

with random

$$\theta_{x,y} = 0, \pi/4, \ldots 7\pi/4$$

- Bob entangles all qubits according to the brickwork graph via CZ gates

- Alice tells Bob what measurement basis for Bob to perform and he returns the outcome (compute like one-way computer)

➔ Alice can achieve her quantum computation without Bob knowing what she computed!!

1 Alice computes $\phi'_{x,y}$ where $s^X_{0,y} = s^Z_{0,y} = 0$. $\qquad \phi'_{x,y} = (-1)^{s^X_{x,y}}\phi_{x,y} + s^Z_{x,y}\pi$

2 Alice chooses $r_{x,y} \in_R \{0,1\}$ and computes $\delta_{x,y} = \phi'_{x,y} + \theta_{x,y} + \pi r_{x,y}$.

3 Alice transmits $\delta_{x,y}$ to Bob. Bob measures in the basis $\{|+_{\delta_{x,y}}\rangle, |-_{\delta_{x,y}}\rangle\}$.

4 Bob transmits the result $s_{x,y} \in \{0,1\}$ to Alice.

5 If $r_{x,y} = 1$ above, Alice flips $s_{x,y}$; otherwise she does nothing.

➔ Realized in an exp. Barz et al. 2012

We have seen the cluster states on the square lattice and the brickwork lattice for universal for quantum computation

Q: How much do we know about the general cluster/graph states?

Universality in graph/cluster states

- Beyond square & brickwork: other 2D *graph/cluster* states on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal

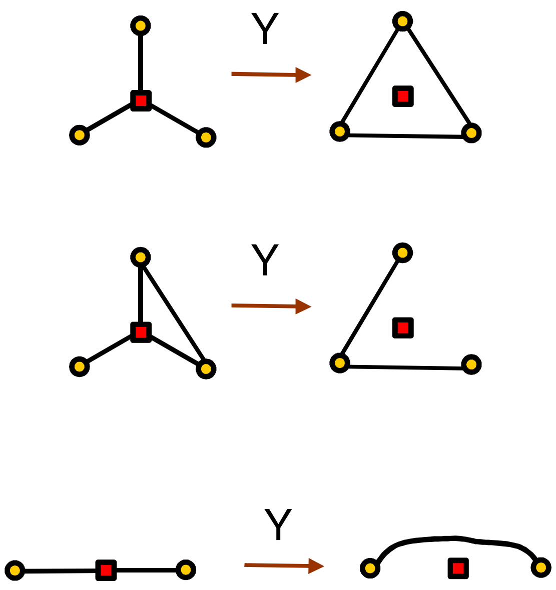
  → Can use local measurement to convert one to the other (with fewer qubits, but still macroscopic) [*Van den Nest et al.* '06]

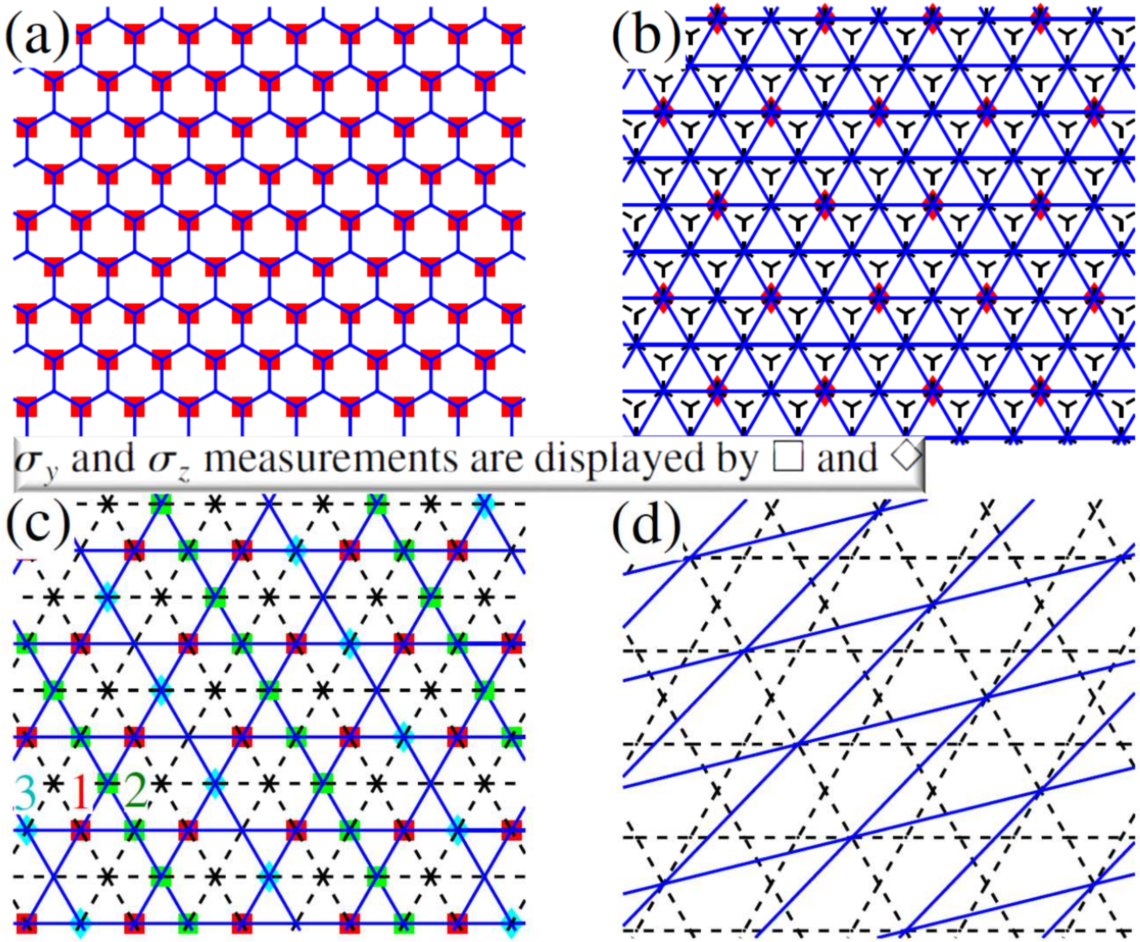
Graph states on regular lattices

❖ Beyond square & brickwork: other 2D *graph/cluster* states on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal

[*Van den Nest et al.* '06]

➔ local measurement converts one to another





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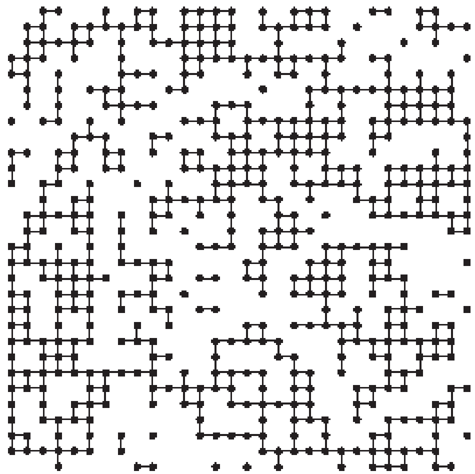
  [*Van den Nest et al.* '06]

  ➔ Can use local measurement to convert one to the other (with fewer qubits, but still macroscopic)

- Faulty square lattice (degree ≤ 4)

  [Browne et al. '08]

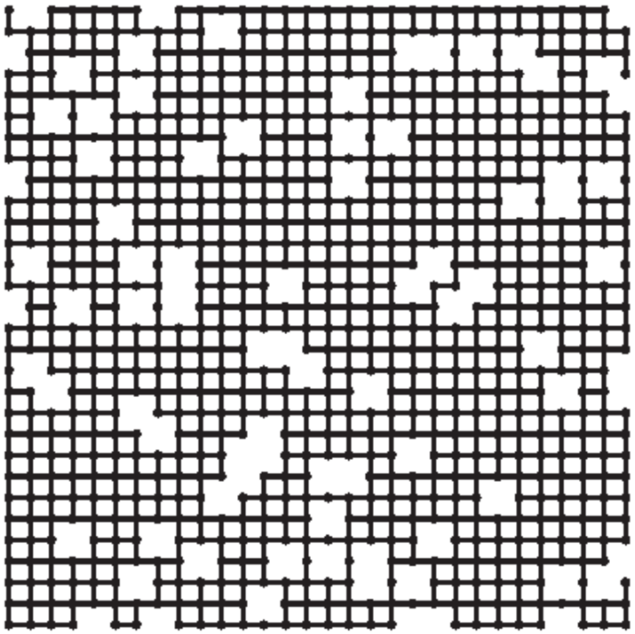
  ➔ As long as it is sufficiently connected (a la percolation), can find sub-graph ~ honeycomb



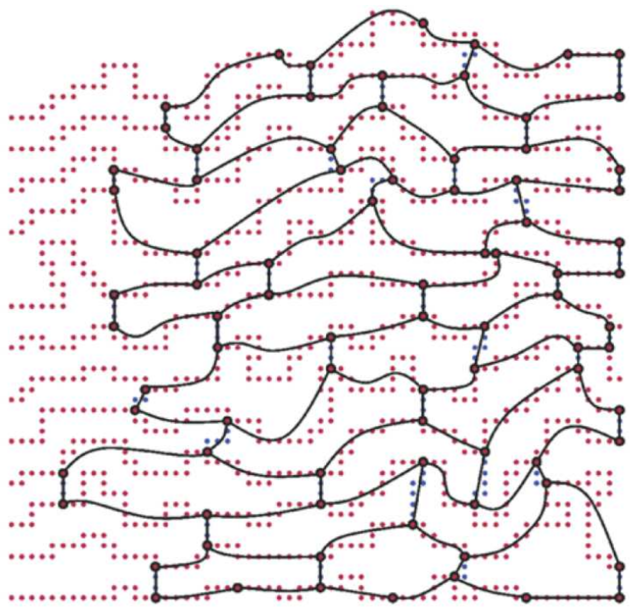
Cluster state on faulty lattice

[Browne et al. '08]

- No qubits on empty sites (degree ≤ 4) ←→ site percolation
- But assume perfect CZ gates $|G\rangle = \bigotimes_{\langle i,i\rangle} CZ_{ij} \left(|+\rangle|+\rangle\cdots|+\rangle\right)$
- As long as probability of occupied sites > site percolation threshold → still universal for MBQC



Local measurement



Universality in graph/cluster states

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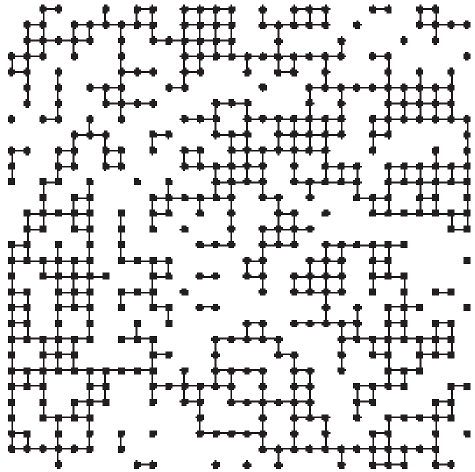


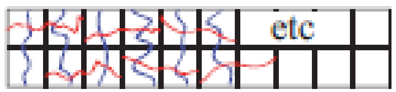
- Faulty square lattice (degree ≤ 4)

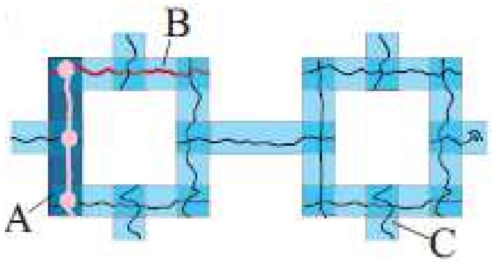
  [Browne et al. '08]

  ➔ As long as it is sufficiently connected (a la percolation), can find sub-graph ~ honeycomb

- Any 2D planar random graphs in supercritical phase of percolation are universal

  [Wei, Affleck & Raussendorf.'12]





Other universal states

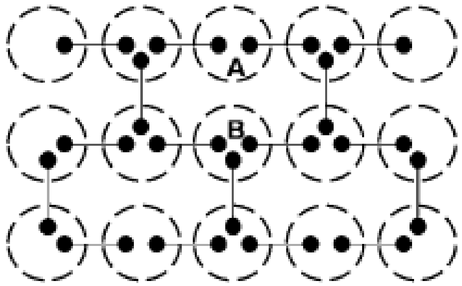
- So far no complete characterization for resource states
- Can they be unique ground state with 2-body Hamiltonians with a finite gap?

  ➔ If so, create resources by cooling!

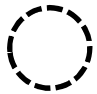
- TriCluster state [*Chen et al.* '09]
- Affleck-Kennedy-Lieb-Tasaki (AKLT) family of states [*AKLT* '87, '88]
  - 1D (not universal): [*Gross & Eisert* '07, '10] [*Brennen & Miyake* '08?]
  - 2D (universal): [*Wei, Affleck & Raussendorf* '11] [*Miyake* '11] [*Wei et al.* '13-'15]
- Symmetry-protected topological states
  - 1D (not universal): [*Else, Doherty & Bartlett '12*] [*Miller & Miyake* '15] [*Prakash & Wei* '15]
  - 2D (universal, *but not much explored*): [*Poulsen Nautrup & Wei* '15] [*Miller & Miyake* '15]

Example ground state of two-body Hamiltonian as computational resource

- TriCluster state (6-level) [*Chen, Zeng, Gu, Yoshida & Chuang*, PRL'09]



$$= |00\rangle + |01\rangle + |10\rangle - |11\rangle$$



$$P_{\mathrm{Tric}} = |\tilde{0}\rangle\langle 000| + |\tilde{1}\rangle\langle 111| + |\tilde{2}\rangle\langle 100| + |\tilde{3}\rangle\langle 011| + |\tilde{4}\rangle\langle 010| + |\tilde{5}\rangle\langle 101|$$

$$H^{\star}_{triC} = \sum_{a} \left(h_{ab} + h_{ba} + h_{\frac{b}{a}} \right)$$

$$\begin{aligned}
h_{ab} \;=\; & \\
& 2(2S_{a_z}-5)(2S_{a_z}-3)(2S_{a_z}-1)(2S_{a_z}+1)(4S_{a_z}+11) \\
& (2S_{b_z}+5)(2S_{b_z}+3)(2S_{b_z}-1)(2S_{b_z}+1)(4S_{b_z}-11) \\
- \; & 75\sqrt{2}S_{a_+}(2S_{a_z}-5)(2S_{a_z}+3)(2S_{a_z}-1)(2S_{a_z}+1) \\
& (48S_{b_z}^4+64S_{b_z}^3-280S_{b_z}^2-272S_{b_z}+67) \\
+ \; & 75\sqrt{2}(48S_{a_z}^4-64S_{a_z}^3-280S_{a_z}^2+272S_{a_z}+67) \\
& S_{b_+}(2S_{b_z}-5)(2S_{b_z}-3)(2S_{b_z}-1)(2S_{b_z}+3) \\
+ \; & 4\sqrt{10}S_{a_+}^3(2S_{a_z}-1)(2S_{a_z}-3)\times \\
& (128S_{b_z}^5+560S_{b_z}^4-2840S_{b_z}^2-3848S_{b_z}+675) \\
+ \; & 4\sqrt{10}(128S_{a_z}^5-560S_{a_z}^4+2840S_{a_z}^2-3848S_{a_z}-675) \\
& S_{b_+}^3(2S_{b_z}-5)(2S_{b_z}-3)+h.c.
\end{aligned}$$

$$\begin{aligned}
h_{\frac{b}{a}} \;=\; & \\
& -25(2S_{a_z}-5)(2S_{a_z}-3)(2S_{a_z}+3)(2S_{a_z}+5) \\
+ \; & 25S_{a_+}^3(2S_{a_z}-5)(2S_{a_z}-1) \\
& (224S_{b_z}^5-16S_{b_z}^4-1968S_{b_z}^3+40S_{b_z}^2+3550S_{b_z}-9) \\
- \; & 12S_{a_+}^5 \\
& (416S_{b_z}^5-80S_{b_z}^4-3600S_{b_z}^3+520S_{b_z}^2+5994S_{b_z}-125) \\
+ \; & h.c.+(a \Leftrightarrow b)\,,
\end{aligned}$$

Too much entanglement is useless

[Gross, Flammia & Eisert '09; Bremner, Mora & Winter '09]

- States (*n*-qubit) possessing too much geometric entanglement E_g are not universal for QC (i.e if $E_g > n - \delta$)

$$E_g\left(|\Psi\rangle\right) = -\log_2 \max_{\phi\in\mathcal{P}} |\langle\phi|\Psi\rangle|^2 \qquad \mathcal{P} = \text{set of product states}$$

- Intuition: if state is very high in geometric entanglement, every local measurement outcome has low probability

  ➔ whatever local measurement strategy, the distribution of outcomes is so random that one can simulate it with a random coin (thus not more powerful than classical random string)

- Moreover, states with high entanglement are typical:

  those with $E_g < n - 2\log_2(n) - 3$ is rare, i.e. with fraction $< e^{-n^2}$

➔ Universal resource states are rare!!

Outline

I. Introduction

II. One-way (measurement-based) quantum computers

III. Other entangled resource states: AKLT family

IV. Summary

A new direction: valence-bond ground states of isotropic antiferromagnet

- AKLT (Affleck-Kennedy-Lieb-Tasaki) states/models
  - Importance: provide strong support for Haldane's conjecture on spectral properties of spin chains [*AKLT* '87,88]
  - Provide concrete example for symmetry-protected topological order [*Gu & Wen* '09, '11]

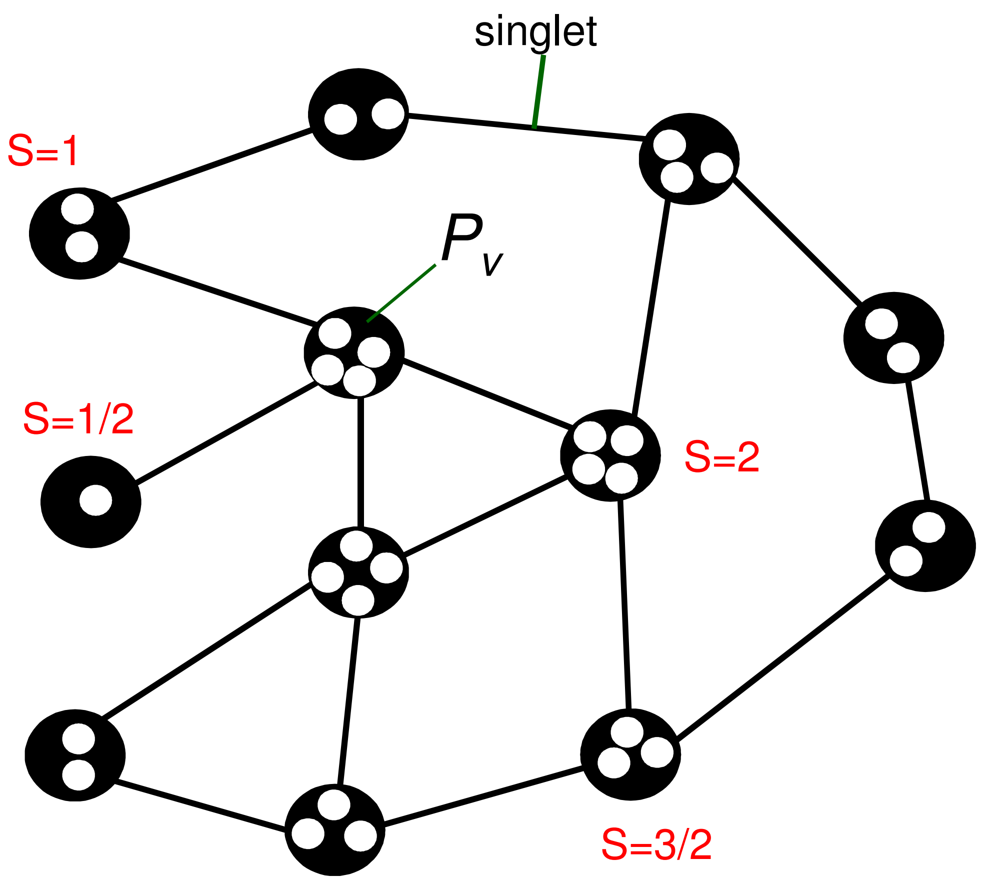
- States of spin S=1,3/2, 2,.. (defined on any lattice/graph)

  ➔ Unique* ground states of gapped# two-body isotropic Hamiltonians

$$H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j)$$ *f(x)* is a polynomial

*w/ appropriate boundary conditions; #gap proved in 1D; evidence in 2D: *Garcia-Saez,Murg,**Wei*** ' 12

(hybrid) AKLT state defined on any graph



- # virtual qubits = # neighbors
- S= # neighbors / 2
- Physical spin Hilbert space = symmetric subspace of qubits

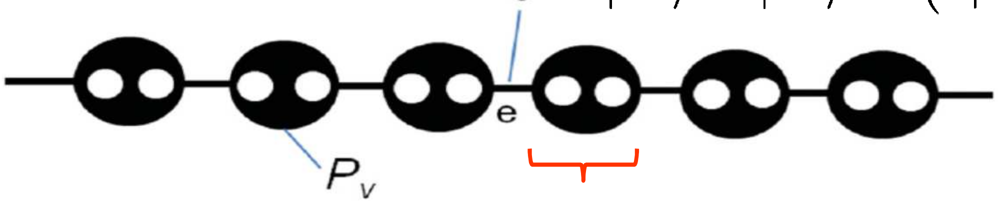
P_v = projection to symmetric subspace of n qubit ≡ spin n/2

1D AKLT state for simulating 1-qubit gates

- Easy to see from its matrix product state (MPS)

[*Gross & Eisert, PRL* '07] [*Brennen & Miyake, PRL* '09]

$$|01\rangle - |10\rangle = \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} |1\rangle \\ -|0\rangle \end{pmatrix}$$



$$\begin{pmatrix} |1\rangle \\ -|0\rangle \end{pmatrix} \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} = \begin{pmatrix} |10\rangle & |11\rangle \\ -|00\rangle & -|01\rangle \end{pmatrix}$$

$$P_v = |+1\rangle\langle 00| + |0\rangle(\langle 01| + \langle 10|)/\sqrt{2} + |-1\rangle\langle 11|$$

- MPS form: $|0\rangle \equiv |z\rangle,\ |+1\rangle \equiv -(|x\rangle + i|y\rangle)/\sqrt{2},\ \ |-1\rangle \equiv (|x\rangle - i|y\rangle)/\sqrt{2}$

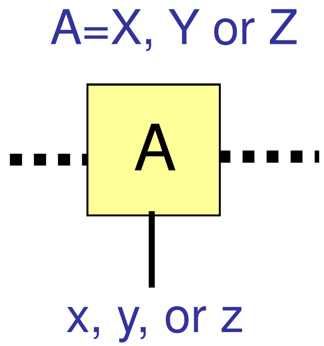
$$P_v \begin{pmatrix} |10\rangle & |11\rangle \\ -|00\rangle & -|01\rangle \end{pmatrix} = \begin{pmatrix} |0\rangle/\sqrt{2} & |-1\rangle \\ -|+1\rangle & -|0\rangle\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(|x\rangle X + |y\rangle Y + |z\rangle Z)$$

➔ Gates with superposition of X, Y, Z are achievable

➔ Arbitrary 1-qubit gates possible (but universal QC requires 2-qubit gates) ➔ any 2D AKLT states universal?

Hamiltonian & SPT order

- 1D spin-1 AKLT state $|x\rangle X + |y\rangle Y + |z\rangle Z$
  is ground state of the gapped 2-body Hamiltonian



$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2$$

- AKLT is a symmetry-protected topological (SPT) state,
  e.g. by $Z_2 x Z_2$ symmetry (rotation around x or z by 180^o)

- Under transformation on physical spins:

$$|0\rangle \equiv |z\rangle,\ |+1\rangle \equiv -(|x\rangle + i|y\rangle)/\sqrt{2},\ \ |-1\rangle \equiv (|x\rangle - i|y\rangle)/\sqrt{2}$$

$$U_z(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad |z\rangle \to |z\rangle,\ |x\rangle \to -|x\rangle,\ |y\rangle \to -|y\rangle \quad A \to Z \cdot A \cdot Z$$

$$U_x(\pi) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad |z\rangle \to -|z\rangle,\ \ |x\rangle \to |x\rangle,\ |y\rangle \to -|y\rangle \quad A \to X \cdot A \cdot X$$

➔ Projective representation (e.g. Z & X) of symmetry implies SPT order

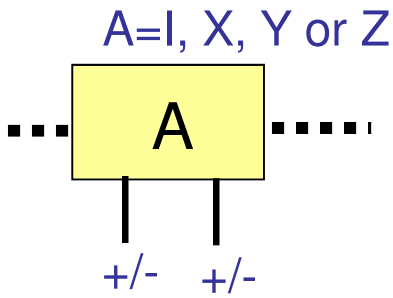
SPT order of cluster state

- MPS for cluster state (single site):

$$A(0) = |+\rangle\langle 0|,\ A(1) = |-\rangle\langle 1|$$

➔ +/- basis: $A(+) \sim A(0) + A(1) = H,\ A(-) = HZ$

- Two sites:



$$A(++) = H^2 = 1,\ A(+-) = H(HZ) = Z$$

$$A(-+) = (HZ)H = X,\ A(--) = (HZ)^2 = XZ$$

- Under XIXI… on physical spins:

$$\left.\begin{array}{l} A(++) \to A(++),\ A(+-) \to A(+-) \\ A(-+) \to -A(-+),\ A(--) \to -A(--) \end{array}\right\} A(\alpha,\beta) \to Z \cdot A(\alpha,\beta) \cdot Z$$

- Similarly for IXIX… : $A(\alpha,\beta) \to X \cdot A(\alpha,\beta) \cdot X$

➔ projective representation ➔ SPT order

SPT order & gates

- AKLT is a symmetry-protected topological (SPT) state, e.g. by $Z_2 x Z_2$ symmetry (rotation around x or z by 180°) with Hamiltonian

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2$$

- 1D cluster state is also a SPT state, e.g. by $Z_2 x Z_2$ symmetry (*XIXI… or IXIX..*) with Hamiltonian

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

- Generic states in such 1D SPT phase

$$A_\alpha = \sigma_\alpha \otimes B_\alpha$$

logical subspace — junk subspace

[*Else et al.* '12]

[*Prakash & Wei* '15]

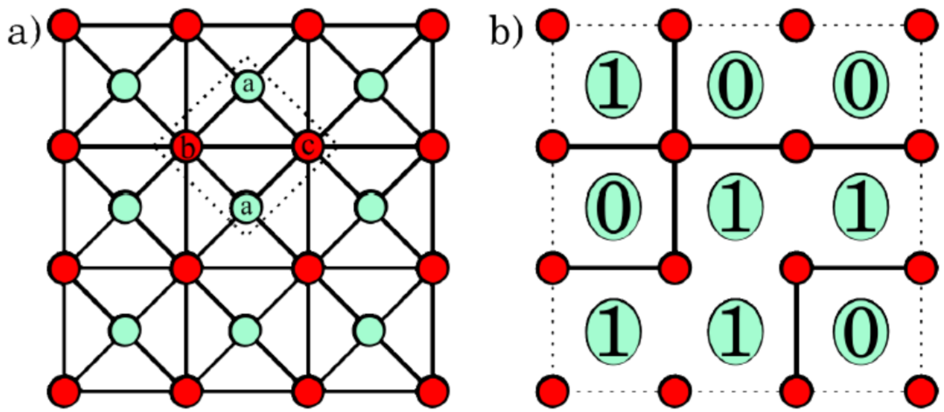
➔ Only identity gate (up to Pauli) is protected

➔ But arbitrary 1-qubit gate is possible, e.g. with S_4 symmetry [*Miller & Miyake* '15]

2D SPT states for universal QC

- A “Control-control-Z state”: [Miller & Miyake ‘15]

ψ= CCZ (Control-Control-Z) gates applied to all triangles with |+++ ..++>



(with symmetry $Z_2xZ_2xZ_2$)

- Fixed-point wavefunctions of generic SPT states (with any nontrivial SPT order) are universal resource; see

  Thursday Session A: 4. [3:00-3:20] **Hendrik Poulsen Nautrup and Tzu-Chieh Wei.** *Symmetry-protected topologically ordered states for universal quantum computation*

In the remaining, we will focus on AKLT family of states for universal quantum computation

Converting 1D AKLT state to cluster state

singlet $|01\rangle - |10\rangle$

P_v

$$P_v = |+1\rangle\langle 00| + |0\rangle(\langle 01| + \langle 10|)/\sqrt{2} + |-1\rangle\langle 11|$$

- Via adaptive local measurement (i.e. state reduction)

  [*Chen, Duan, Ji & Zeng '10*]

- Via fixed POVM [*Wei, Affleck & Raussendorf '11*]

  ➔ *generalizable to 2D AKLT:* $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$

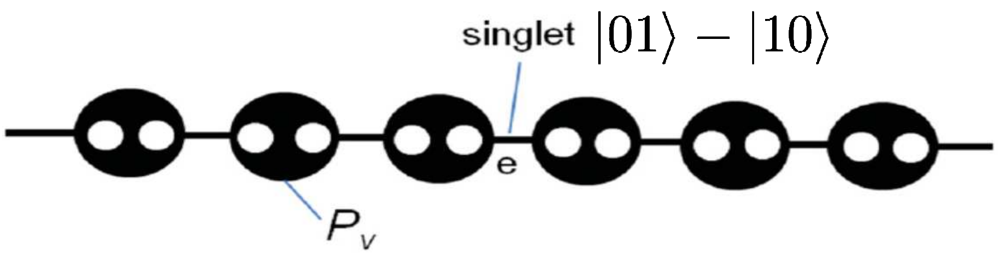
$$F_x \sim |S_x = 1\rangle\langle S_x = 1| + |S_x = -1\rangle\langle S_x = -1| \sim |++\rangle\langle ++| + |--\rangle\langle --|$$

$$F_y \sim |S_y = 1\rangle\langle S_y = 1| + |S_y = -1\rangle\langle S_y = -1| \sim |i,i\rangle\langle i,i| + |-i,-i\rangle\langle -i,-i|$$

$$F_z \sim |S_z = 1\rangle\langle S_z = 1| + |S_z = -1\rangle\langle S_z = -1| \sim |00\rangle\langle 00| + |11\rangle\langle 11|$$

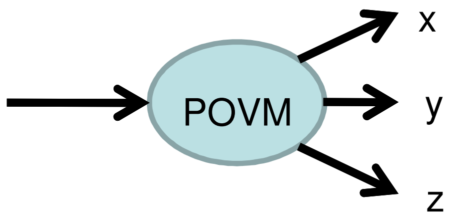
➔ *Outcome labeled by x,y, z:* $|\psi\rangle \rightarrow F_\alpha|\psi\rangle$

POVM: 1D AKLT state ➔ cluster state

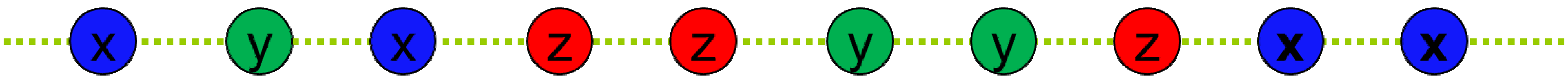


[*Wei, Affleck & Raussendorf ’11, `12*]

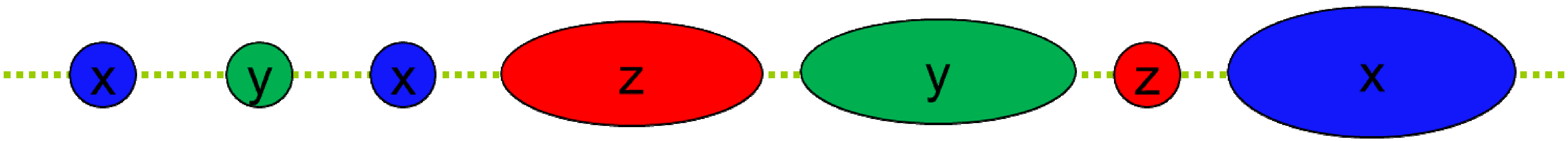
- ❑ POVM: $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$



e.g. for the outcome (labeled x, y, z)



➔ the post-measurement state is an encoded 1D cluster state with graph:

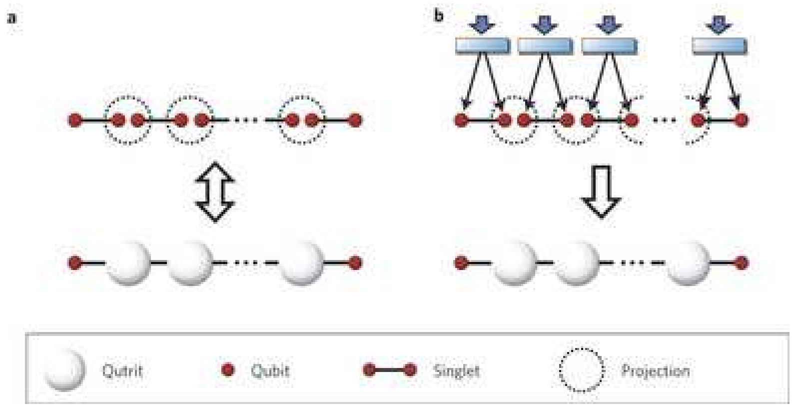


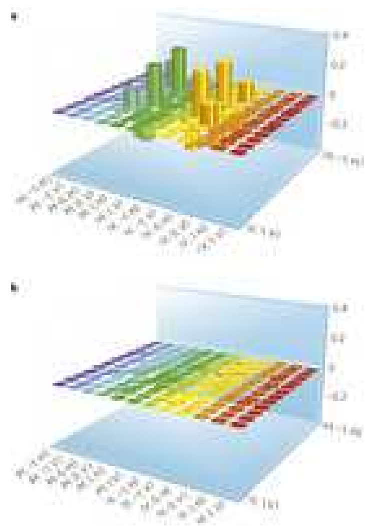
➔ 1 logical qubit = 1 domain = consecutive sites with same outcome

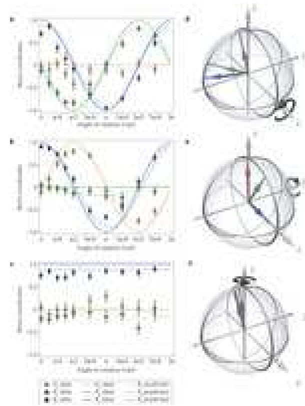
➔ This generalizes to some 2D AKLT states (with S ≤ 2)

Realizations of 1D AKLT state

- Resch's group: photonic implementation (Nature Phys 2011)







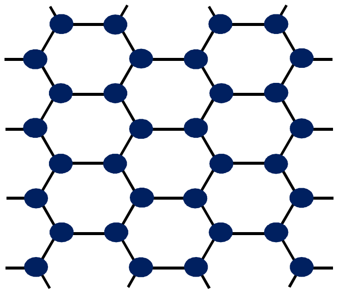
2D AKLT states for quantum computation?

- On various lattices

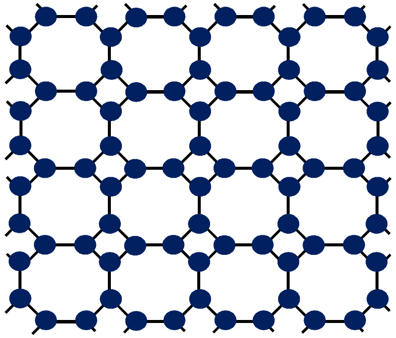
Wei, *Affleck & Raussendorf*, PRL '11; Miyake '11;
Wei, PRA '13, ***Wei***, *Haghnegahdar& Raussendorf, PRA '14*
Wei *& Raussendorf '15*

spin-3/2:

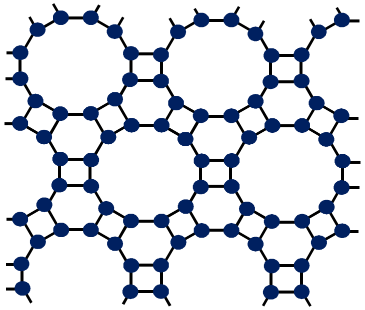
honeycomb



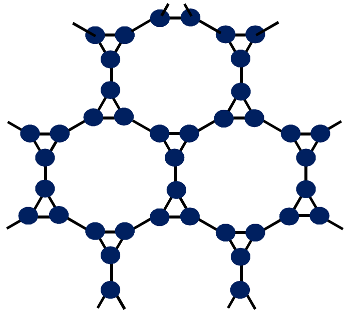
square-octagon



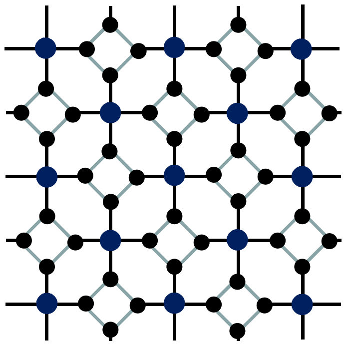
'cross'



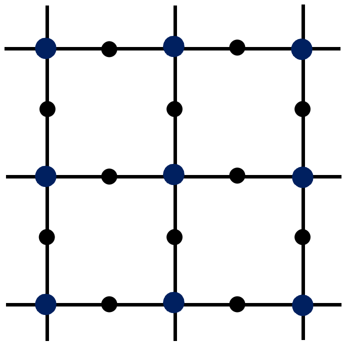
star



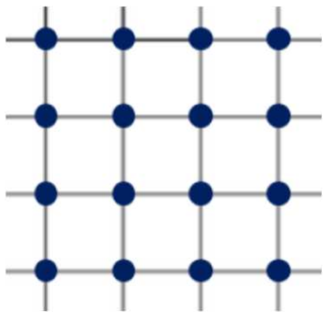
square-hexagon
(spin-2 spin-3/2 mixture)



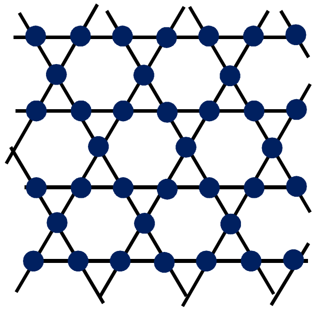
decorated-square
(spin-2 spin-1 mixture)



square
(spin-2)

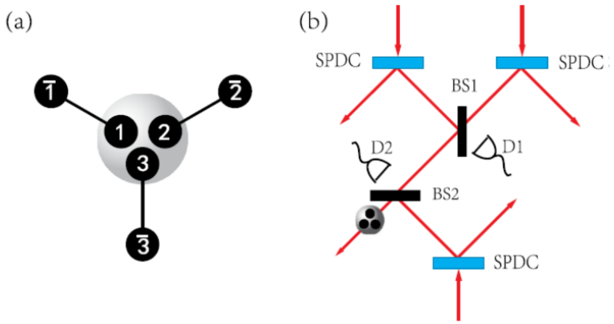


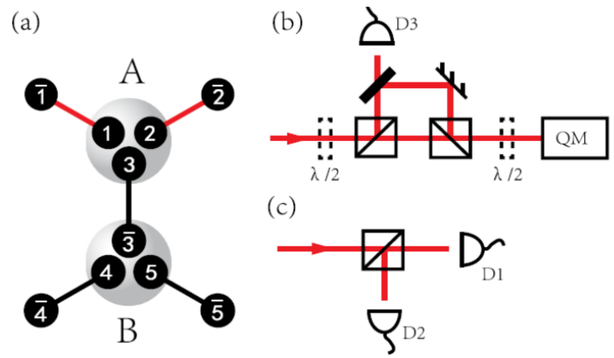
Kagome
(spin-2)



Proposal for 2D AKLT states

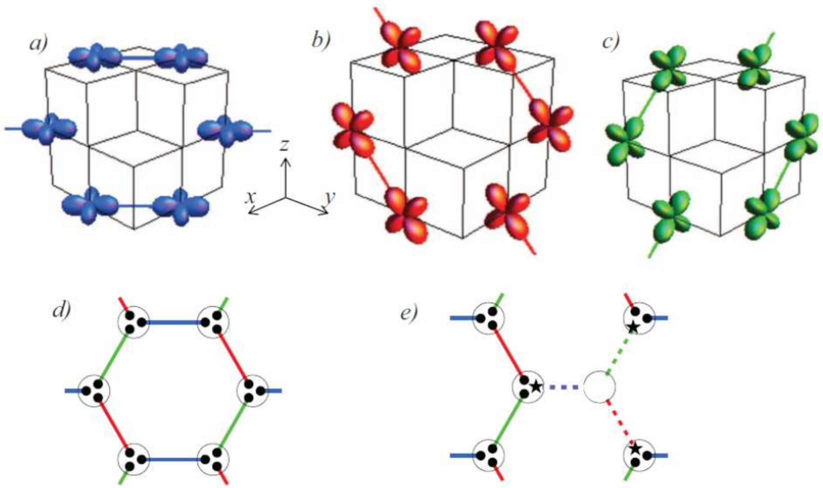
- Liu, Li and Gu [JOSA B 31, 2689 (2014)]



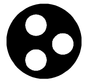


- Koch-Janusz, Khomskii & Sela [PRL 114, 247204 (2015)]

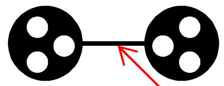
  t_{2g} electrons in Mott insulator



AKLT states on trivalent lattices

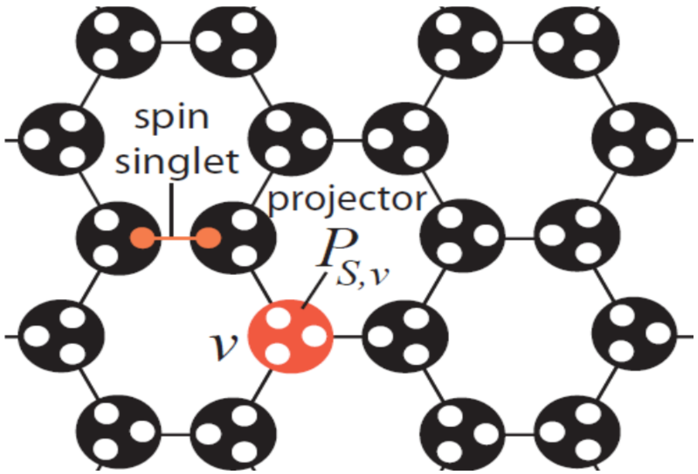
- Each site: three virtual qubits  ≡ spin 3/2 (in general: S= #nbr /2)

  ➔ physical spin = symmetric subspace of qubits

- Two virtual qubits on an edge form a singlet 

  $|01\rangle - |10\rangle$

$$P = |3/2\rangle\langle 000| + |-3/2\rangle\langle 111| + |1/2\rangle\langle W| + |-1/2\rangle\langle \overline{W}|$$



$$|000\rangle \leftrightarrow \left|S = \frac{3}{2}, S_z = \frac{3}{2}\right\rangle$$

$$|111\rangle \leftrightarrow \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

→ Effective qubit

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$

Use generalized measurement (POVM)

$$F_z = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_z + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_z\right)$$

$$F_x = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_x + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_x\right)$$

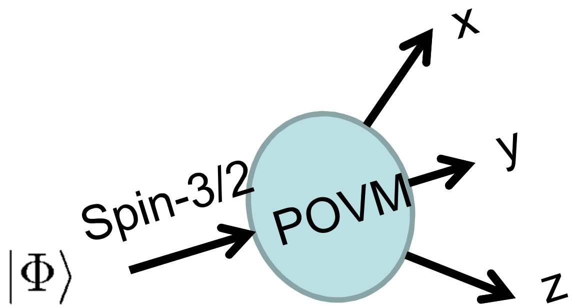
$$F_y = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_y + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_y\right)$$

[***Wei***, *Affleck & Raussendorf* '11
Miyake '11]

Completeness:

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$$

- POVM gives random outcome x, y and z at each site



$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z}|\Phi\rangle$$

➔ Can show POVM on all sites converts AKLT to a graph state
(graph depends on random x, y and z outcomes)

Proving graph state

Let us first explain the notation. Consider a central vertex $C \in V(G_0(\{F\}))$ and all its neighboring vertices $C_\mu \in V(G_0)$. Denote the POVM outcome for all $\mathcal{L}$ sites $v \in C, C_\mu$ by a_c and a_μ, respectively. Denote by E_μ the set of $\mathcal{L}$ edges that run between C and C_μ. Denote by E_c the set of $\mathcal{L}$ edges internal to C. Denote by V_c the set of all qubits in C, and by V_μ the set of all qubits in C_μ. (Recall that there are four qubit locations per $\mathcal{L}$ vertex $v \in C, C_\mu$.) Extending Eq. (33) of Ref. [17] to the spin-2 case, we have

$$\mathcal{K}_C = \bigotimes_\mu \bigotimes_{e\in E_\mu} (-1)\sigma_{a_\mu}^{(u(e))}\sigma_{a_\mu}^{(v(e))} \bigotimes_{e'\in E_c}(-1)\sigma_b^{(v_1(e'))}\sigma_b^{(v_2(e'))}$$
$$= (-1)^{|E_c|+\sum_\mu |E_\mu|} \bigotimes_\mu \bigotimes_{e\in E_\mu} \sigma_{a_\mu}^{(u(e))}\sigma_{a_\mu}^{(v(e))}$$
$$\times \bigotimes_{e'\in E_c} \sigma_b^{(v_1(e'))}\sigma_b^{(v_2(e'))}.$$

We take the following convention for b as reported in Table II. For POVM outcome $a_c = z$, we take $b = x$; for $a_c = x$, we take $b = z$; for $a_c = y$, we take $b = z$. With this choice we have

$$\mathcal{K}_C = (-1)^{|E_c|+\sum_\mu |E_\mu|} \bigotimes_\mu (\otimes_{e\in E_\mu}\lambda_{u(e)}) Z_\mu^{|E_\mu|}$$
$$\times \bigotimes_{e\in E_\mu} \sigma_{a_\mu}^{v(e)}\sigma_b^{v(e)} X_c.$$

$$\mathcal{K}_C = (-1)^{|E_c|+\sum_\mu |E_\mu|} \bigotimes_\mu (\otimes_{e\in E_\mu}\lambda_{u(e)}) Z_\mu^{|E_\mu|}$$
$$\times \left(\bigotimes_{a_\mu \neq b} \otimes_{e\in E_\mu} \lambda_{v(e)} \right) Q_c,$$

$$Q_c = \begin{cases} i^{n_{\neq b}} X_c & \text{if } n_{\neq b} \text{ is even} \\ -i^{1+n_{\neq b}}(-1)^{\delta_{a_c,x}} Y_c & \text{if } n_{\neq b} \text{ is odd} \end{cases}$$

$$n_{\neq b} \equiv \sum_{\mu, a_\mu \neq b} |E_\mu|$$

TABLE II. The choice of b and $a_{\mu\neq b}$.

a_c	z	x	y
b	x	z	z
$a_{\mu\neq b}$	y	y	x

POVM outcome	z	x	y
Stabilizer generator	$\lambda_i\lambda_j\sigma_z^{[i]}\sigma_z^{[j]}$	$\lambda_i\lambda_j\sigma_x^{[i]}\sigma_x^{[j]}$	$\lambda_i\lambda_j\sigma_y^{[i]}\sigma_y^{[j]}$
Logical $\overline{X}$ operator	$\bigotimes_{j=1}^{4\lvert C\rvert}\sigma_x^{[j]}$	$\bigotimes_{j=1}^{4\lvert C\rvert}\sigma_z^{[j]}$	$\bigotimes_{j=1}^{4\lvert C\rvert}\sigma_z^{[j]}$
Logical $\overline{Z}$ operator	$\lambda_i\sigma_z^{[i]}$	$\lambda_i\sigma_x^{[i]}$	$\lambda_i\sigma_y^{[i]}$

Probability of POVM outcomes

- Measurement gives random outcomes, but what is the probability of a given set of outcomes?

$$P(\{\alpha(v)\}) \sim \langle\psi_{\mathrm{AKLT}}|\bigotimes_v F^\dagger_{\alpha(v)}F_{\alpha(v)}|\psi_{\mathrm{AKLT}}\rangle$$

- Can evaluate this using coherent states; alternatively use tensor product states

- Turns out to be a geometric object

[***Wei***,*Affleck & Raussendorf*, PRL '11 & PRA '12]

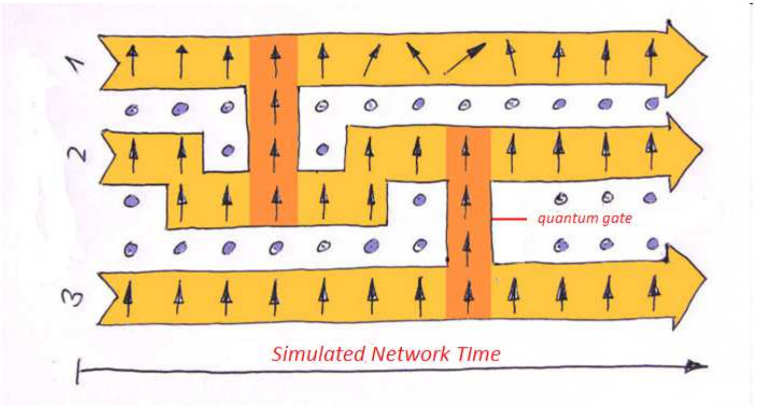
$$P(\{\alpha(v)\}) \sim 2^{|V|-|\mathcal{E}|}$$

Difference from 1D case: graph & percolation

[***Wei***,*Affleck & Raussendorf* PRL'11]

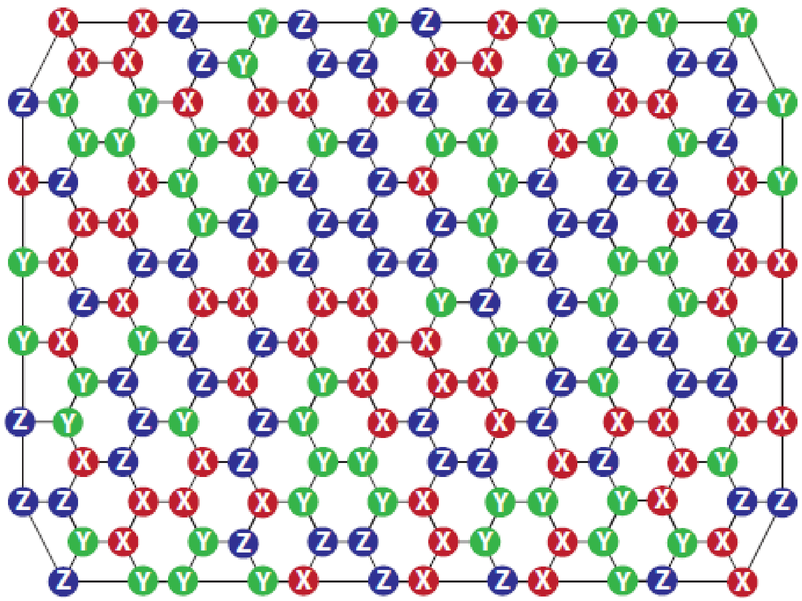
1. What is the graph? which determines the graph state
➔ How to identify the graphs ?

2. Are they percolated? (if so, universal resource)

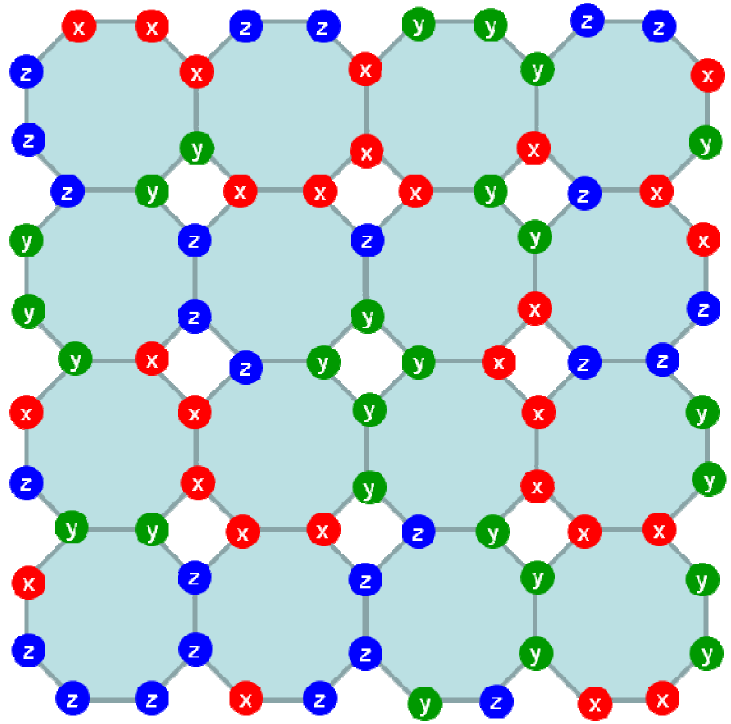


Recipe: construct graph for ‘the graph state’

- Examples: random POVM outcomes x, y, z



honeycomb

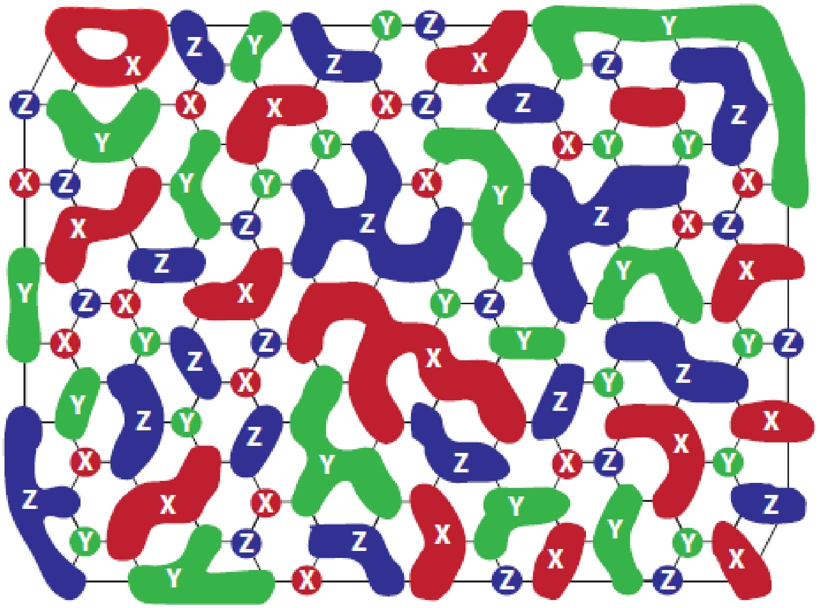


square octagon

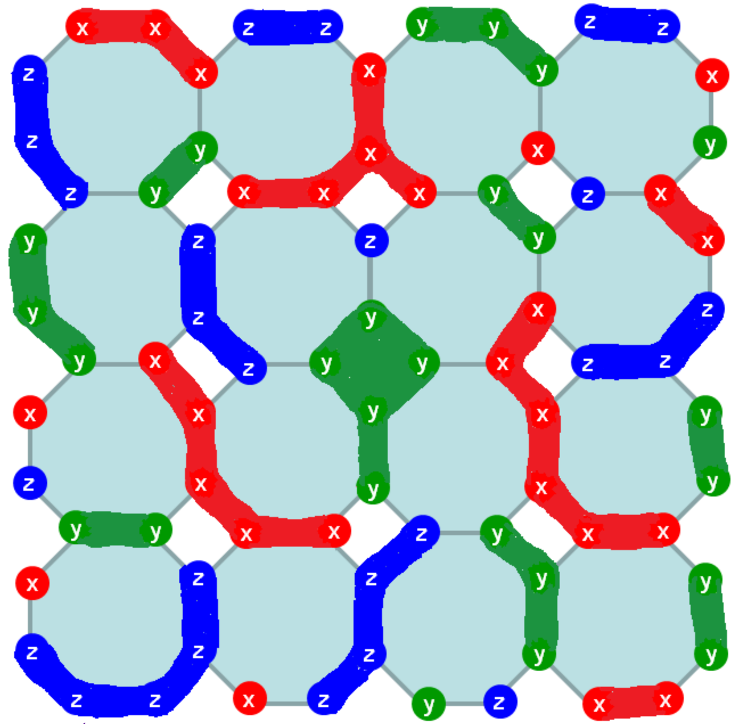
$$P(\{\alpha(v\}) \sim 2^{|V|-|\mathcal{E}|}$$

Step 1: Merge sites to "domains" ➔ vertices

- 1 domain = 1 logical qubit



honeycomb

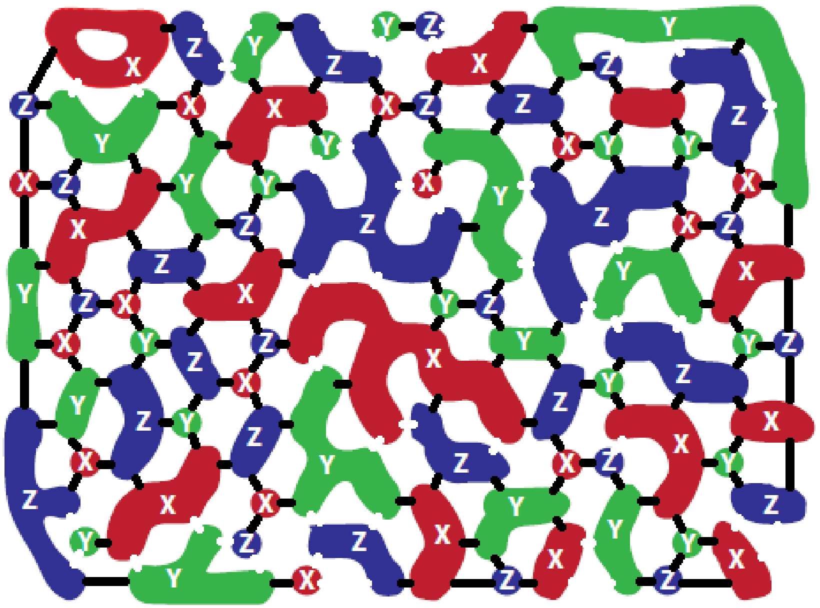


square octagon

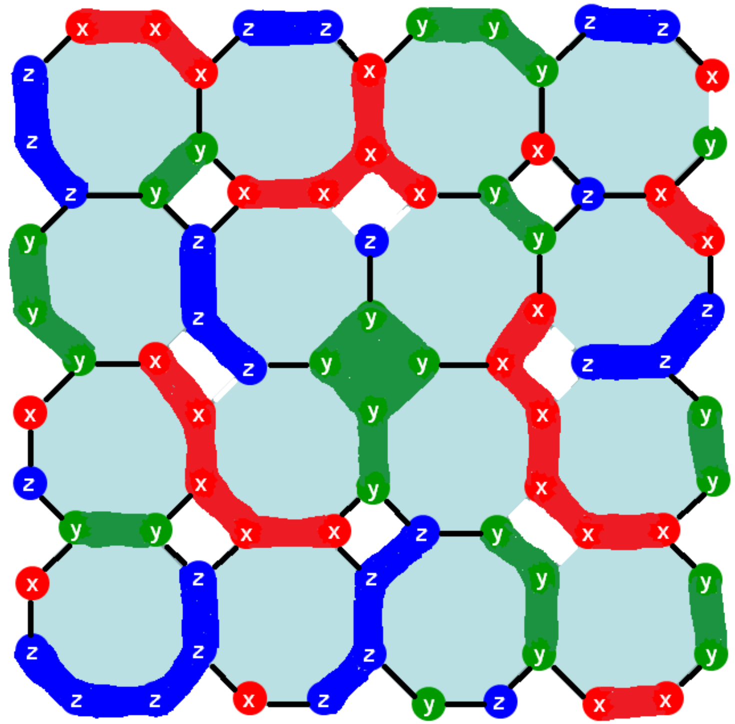
$|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$ $|\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$: encoding of a logical qubit

Step 2: edge correction between domains

- Even # edges = 0 edge, Odd # edges = 1 edge
  (due to $\sigma_z^2 = I$ in the C-Z gate)



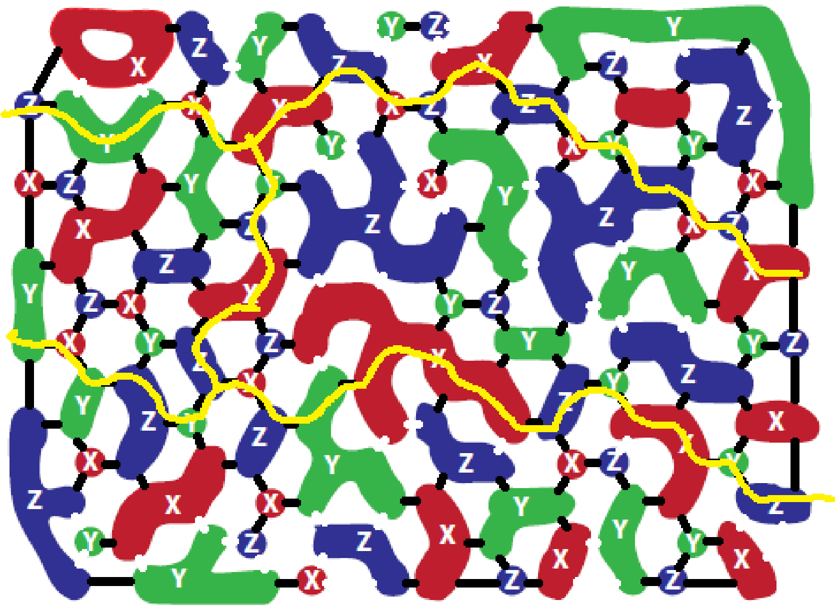
honeycomb

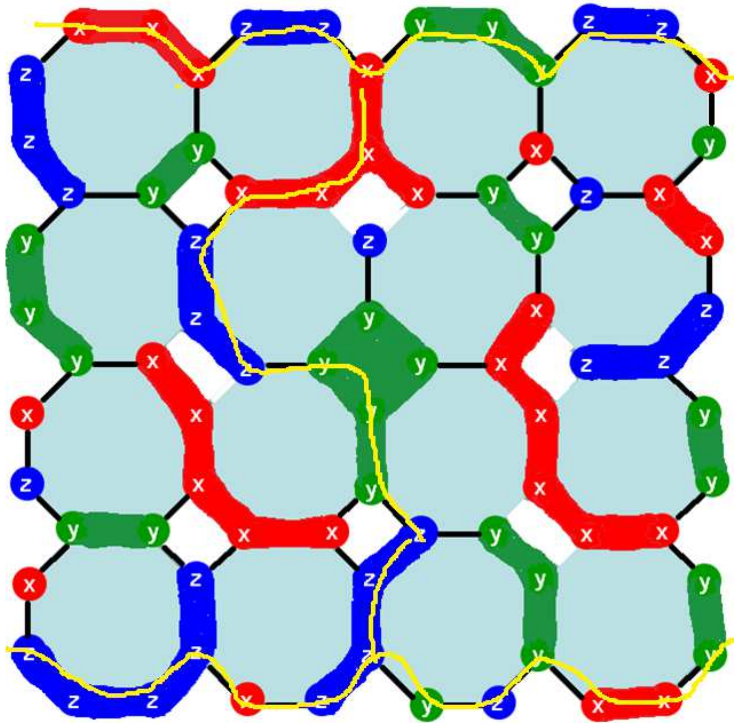


square octagon

Step 3: Check connections (percolation)

- Sufficient number of wires if graph is in supercritical phase (percolation)

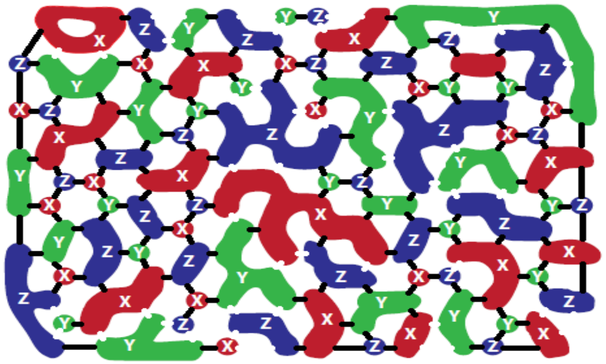


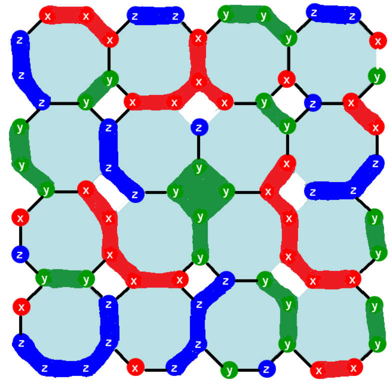


✓ Verified this for honeycomb, square octagon and cross lattices
➔ AKLT states on these are universal resources

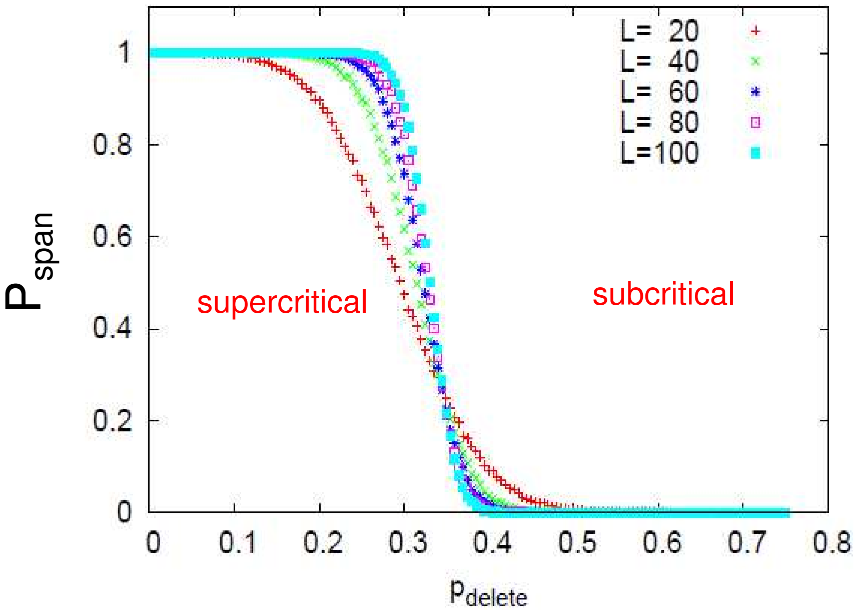
How robust is connectivity?

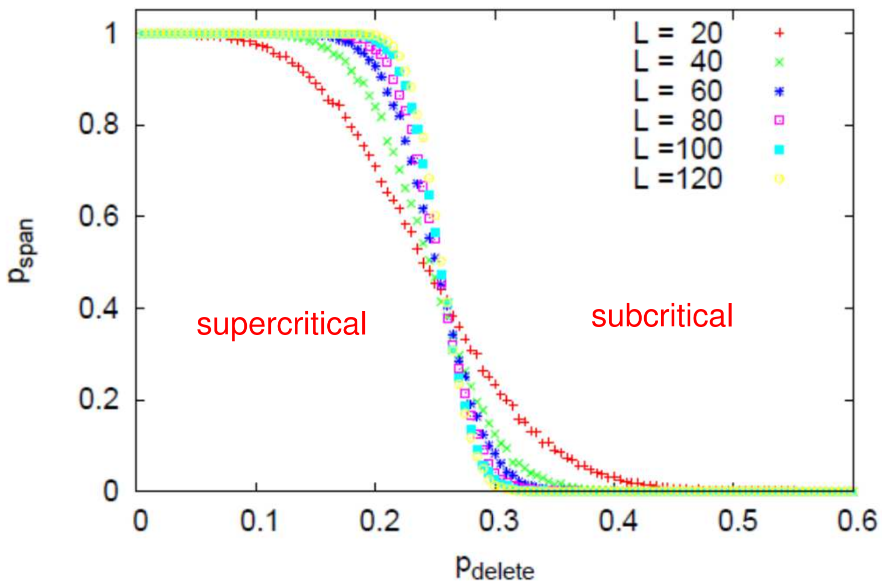
- Characterized by artificially removing domains to see when connectivity collapses (phase transition)



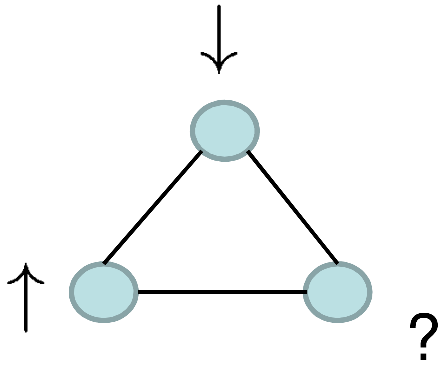


[*Wei* '13]

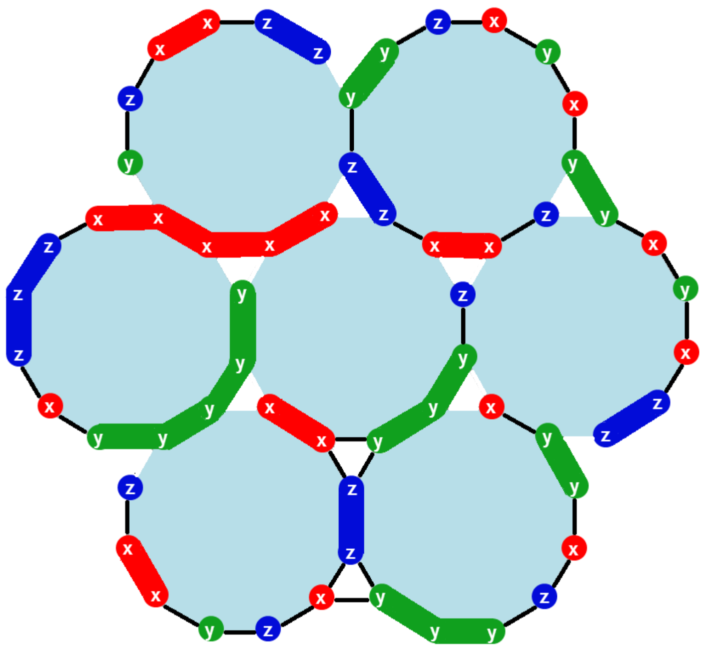




Frustration on star lattice

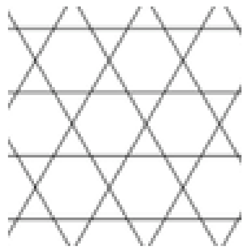


➔ Cannot have POVM outcome xxx, yyy or zzz on a triangle



➔ Consequences:

(1) Only 50% edges on triangles occupied
< p_{th} ≈0.5244 of Kagome
→ disconnected graph



(2) Simulations confirmed: graphs not percolated

➔ AKLT on star likely NOT universal

Difficulty for spin-2

- Technical problem: trivial extension of POVM does NOT work!

$$
\begin{aligned}
F_z &= |2\rangle\langle 2|_z + |-2\rangle\langle -2|_z \\
F_x &= |2\rangle\langle 2|_x + |-2\rangle\langle -2|_x \\
F_y &= |2\rangle\langle 2|_y + |-2\rangle\langle -2|_y
\end{aligned}
$$

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z \neq c \cdot I$$

➔ Leakage out of logical subspace (error)

- Fortunately, can add elements K's to complete the identity

$$
\left\{
\begin{aligned}
& F_\alpha = \sqrt{\frac{2}{3}}\big(|S_\alpha = +2\rangle\langle S_\alpha = +2| + |S_\alpha = -2\rangle\langle S_\alpha = -2|\big) \\
& K_\alpha = \sqrt{\frac{1}{3}}\big(|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\big) \qquad |\phi_\alpha^-\rangle \equiv \sqrt{\frac{1}{2}}\big(|S_\alpha = 2\rangle - |S_\alpha = -2\rangle\big) \\
& \alpha = x, y, z
\end{aligned}
\right.
$$

[*Wei, Haghnegahdar, Raussendorf* '14]

Completeness: $\displaystyle\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I$

Another difficulty: sample POVM outcomes

$$p(\{F,K\}) = \langle \mathrm{AKLT}| \bigotimes_u F^\dagger_{\alpha(u)} F_{\alpha(u)} \bigotimes_v K^\dagger_{\beta(v)} K_{\beta(v)} |\mathrm{AKLT}\rangle = ?$$ [*Wei*, *Raussendorf* '15]

- How to calculate such an *N*-body correlation function?

Lemma. If there exists a set Q (subset of D_K) such that $-\otimes_{\mu\in Q}(-1)^{|V_\mu|}X_\mu$ is in the stablizer group $\mathcal{S}(|G_0\rangle)$ of the state $|G_0\rangle$, then $p(\{F,K\})=0$. Otherwise,

$$p(\{F,K\}) = c\left(\frac{1}{2}\right)^{|\mathcal{E}|-|V|+2|J_K|-\dim\big(\ker(H)\big)},$$

where c is a constant.

$$\begin{cases} |G_0\rangle \sim \bigotimes_v F_{\alpha(v)}|\mathrm{AKLT}\rangle \\ D_K\text{: set of domains having all sites POVM } K \\ (H)_{\mu\nu} = 1 \text{ if } \{\mathcal{K}_\mu, X_\nu\} = 0, \text{ and } (H)_{\mu\nu} = 0 \text{ otherwise} \end{cases}$$

➔ Bottom line: can use Monte Carlo sampling

Local POVM: 5-level to (2 or 1)-level

$$F_\alpha = \sqrt{\frac{2}{3}}\left(|S_\alpha = +2\rangle\langle S_\alpha = +2| + |S_\alpha = -2\rangle\langle S_\alpha = -2|\right)$$

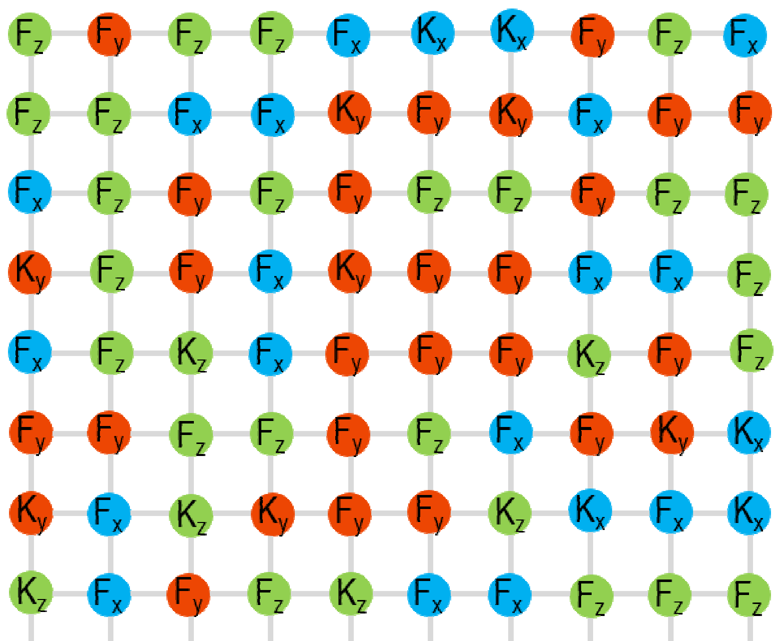
[Wei, Haghnegahdar, Raussendorf '14]

$$K_\alpha = \sqrt{\frac{1}{3}}\left(|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\right) = \frac{1}{\sqrt{2}}|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\, F_\alpha \qquad |\phi_\alpha^\pm\rangle \equiv \sqrt{\frac{1}{2}}\left(|S_\alpha = 2\rangle \pm |S_\alpha = -2\rangle\right)$$

$$\alpha = x, y, z$$

Completeness: $\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I$

- POVM gives random outcome F_x, F_y, F_z, K_x, K_y, K_z at each site



➔ Local action (depends on outcome):

$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z}|\Phi\rangle$$

or

$$|\Phi\rangle \longrightarrow K_{\alpha=x,y,\text{ or z}}|\Phi\rangle$$

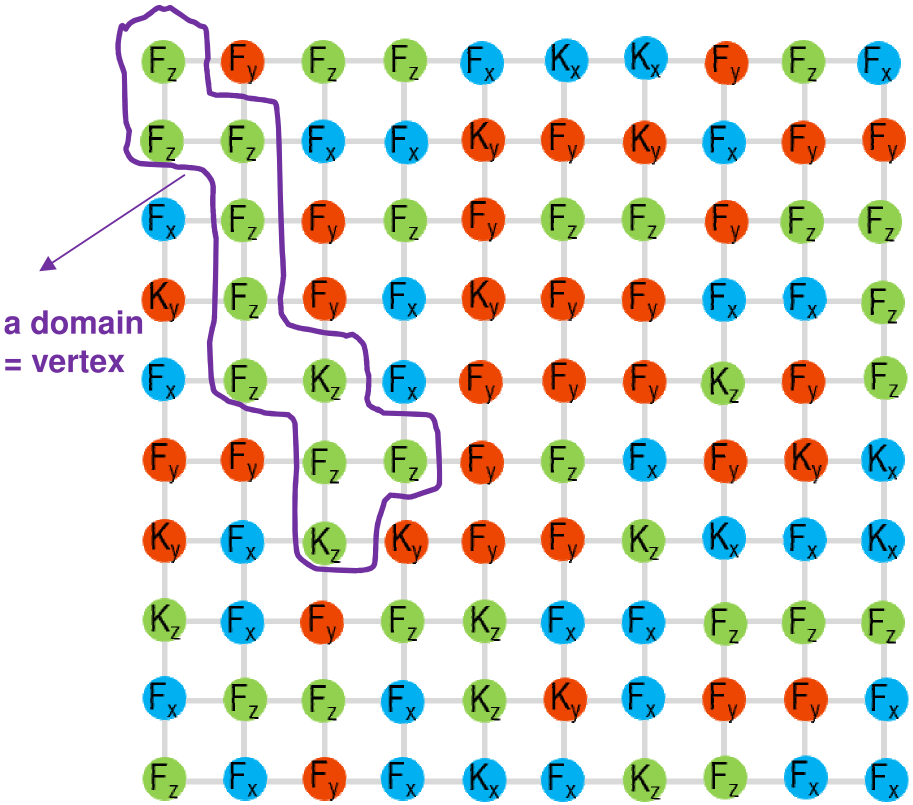
Post-POVM state: graph state

$$F_\alpha = \sqrt{\frac{2}{3}}\left(|S_\alpha = +2\rangle\langle S_\alpha = +2| + |S_\alpha = -2\rangle\langle S_\alpha = -2|\right)$$

[Wei, Haghnegahdar, Raussendorf '14]

$$K_\alpha = \sqrt{\frac{1}{3}}\left(|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\right) = \frac{1}{\sqrt{2}}|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\,F_\alpha \qquad |\phi_\alpha^\pm\rangle \equiv \sqrt{\frac{1}{2}}\left(|S_\alpha = 2\rangle \pm |S_\alpha = -2\rangle\right)$$

$$\alpha = x, y, z$$



- If *F* outcome on **all** sites
  ➔ a *planar* graph state

$$|G_0\rangle = \bigotimes_v F_{\alpha_v}^{(v)}|\mathrm{AKLT}\rangle$$

✓ **Vertex = a domain of sites with same color (x, y or z)**

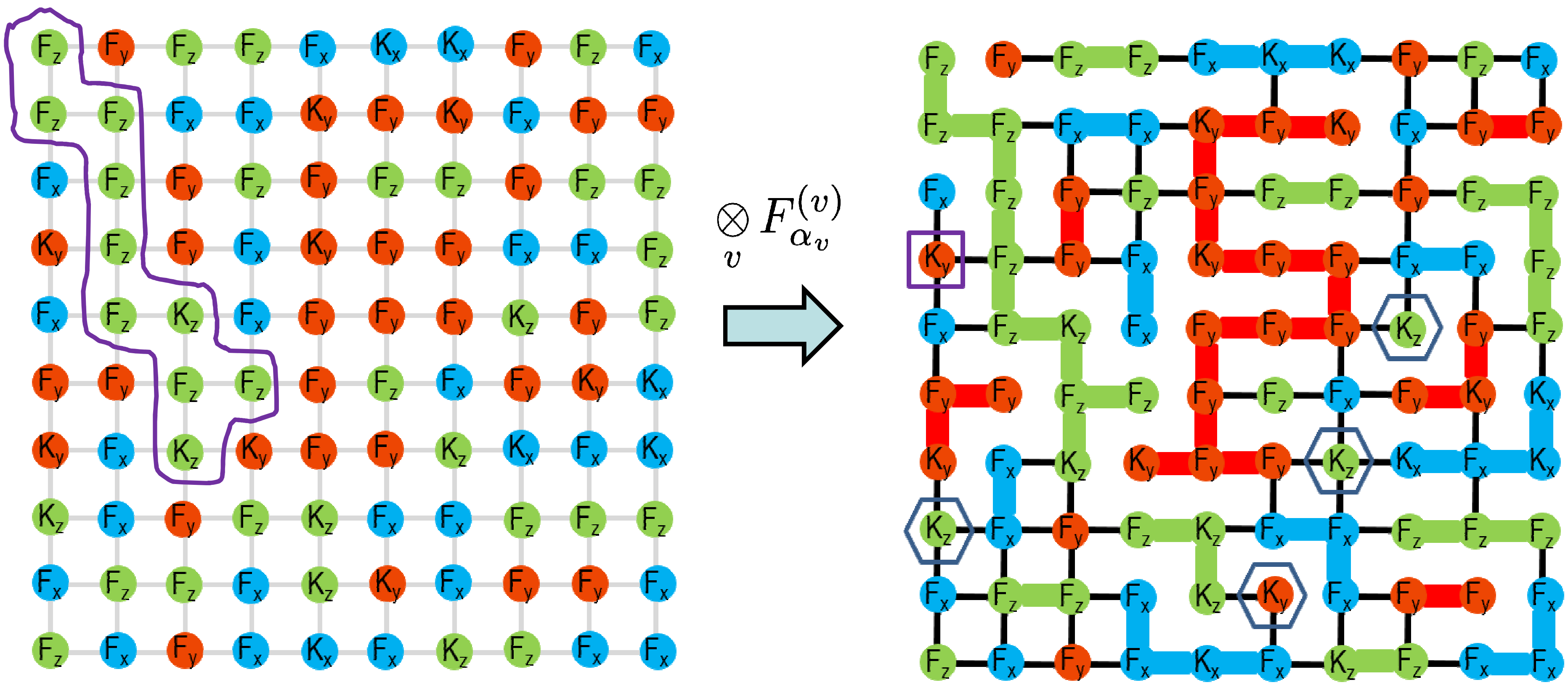
- *K* outcome = *F* followed by $\phi^\pm$ measurement (then *post-selecting* '-' result)

➔ Either
(1) shrinks domain size [trivial] or
(2) logical X or Y measurement [nontrivial]

POVM ➔ Graph of the graph state

Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)

$$|G_0\rangle = \bigotimes_v F^{(v)}_{\alpha_v} |\mathrm{AKLT}\rangle$$



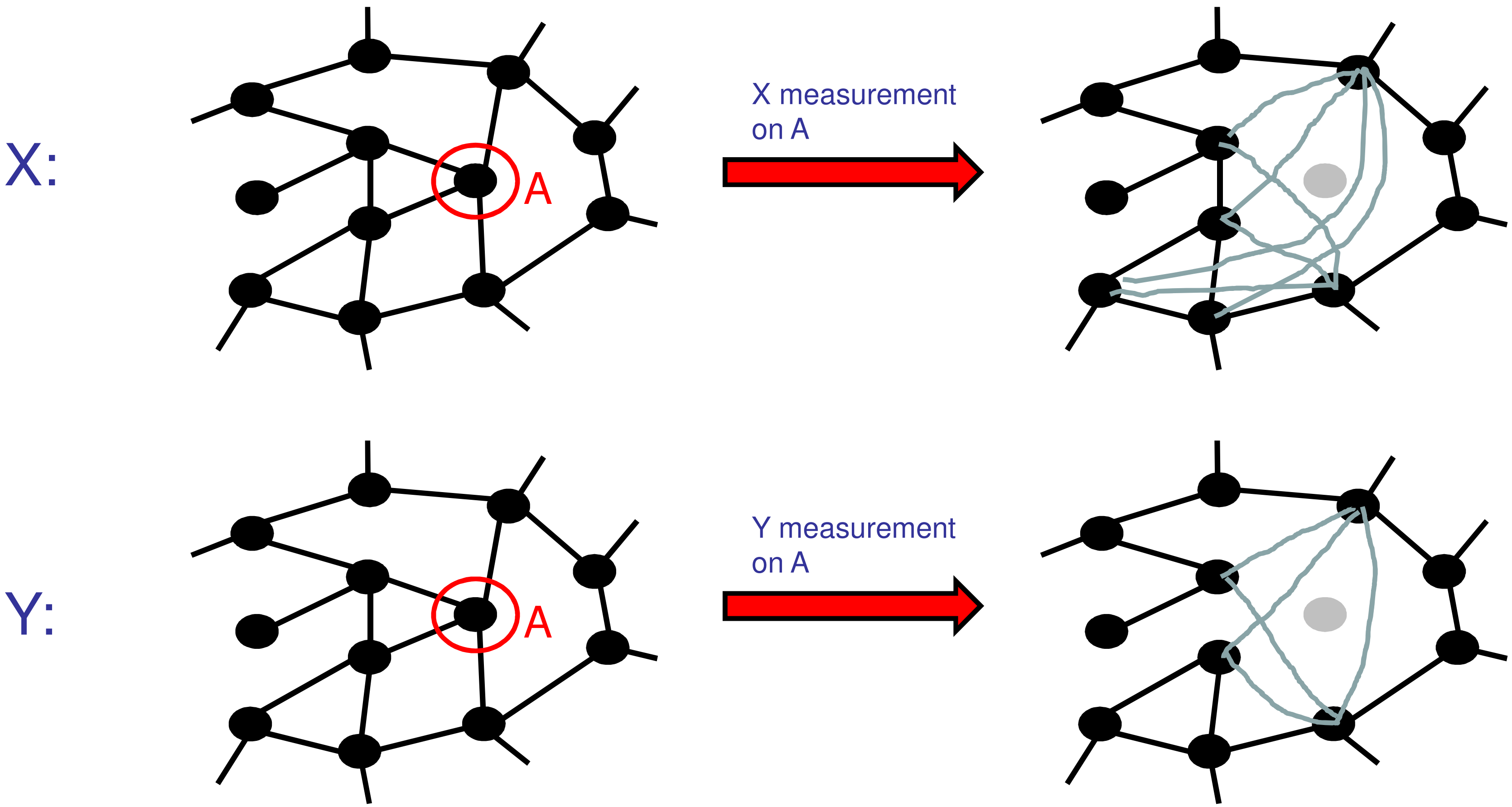
- Effect of nontrival $K_\alpha = \frac{1}{\sqrt{2}} |\phi^-_\alpha\rangle\langle\phi^-_\alpha| F_\alpha$
  ➔ non-planar graph

□ :logical X measurement

⬡ :logical Y measurement

Non-planarity from X/Y measurement

[See e.g. *Hein et '06*]

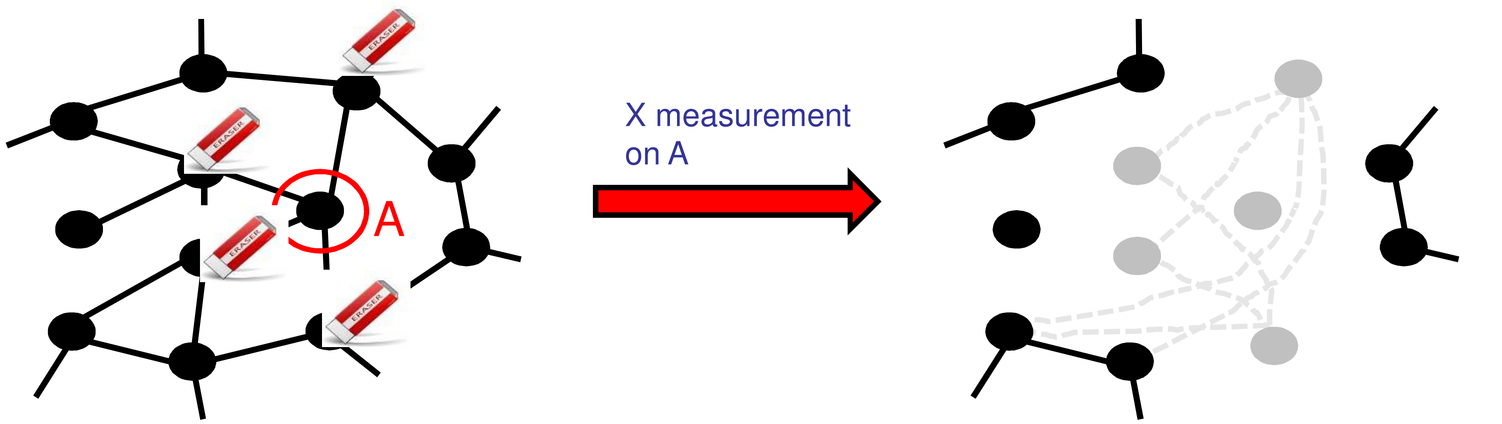


➔ Effect of X measurement is more complicated than Y measurement

Restore planarity: further measurement

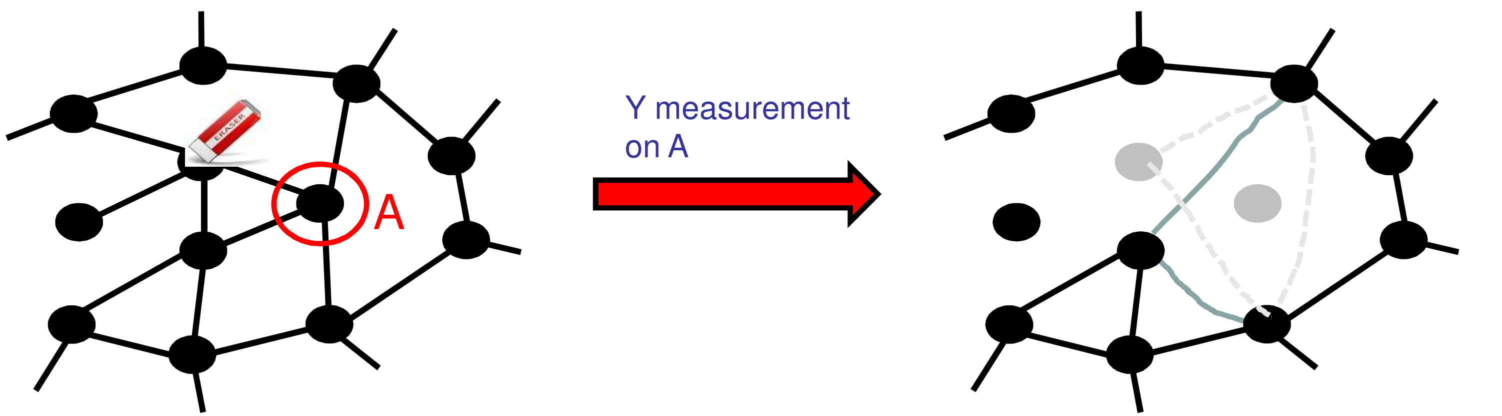
- Deal with non-planarity due to Pauli X measurement:
  remove all vertices surrounding that of X measurement (via Z measurement)

X:



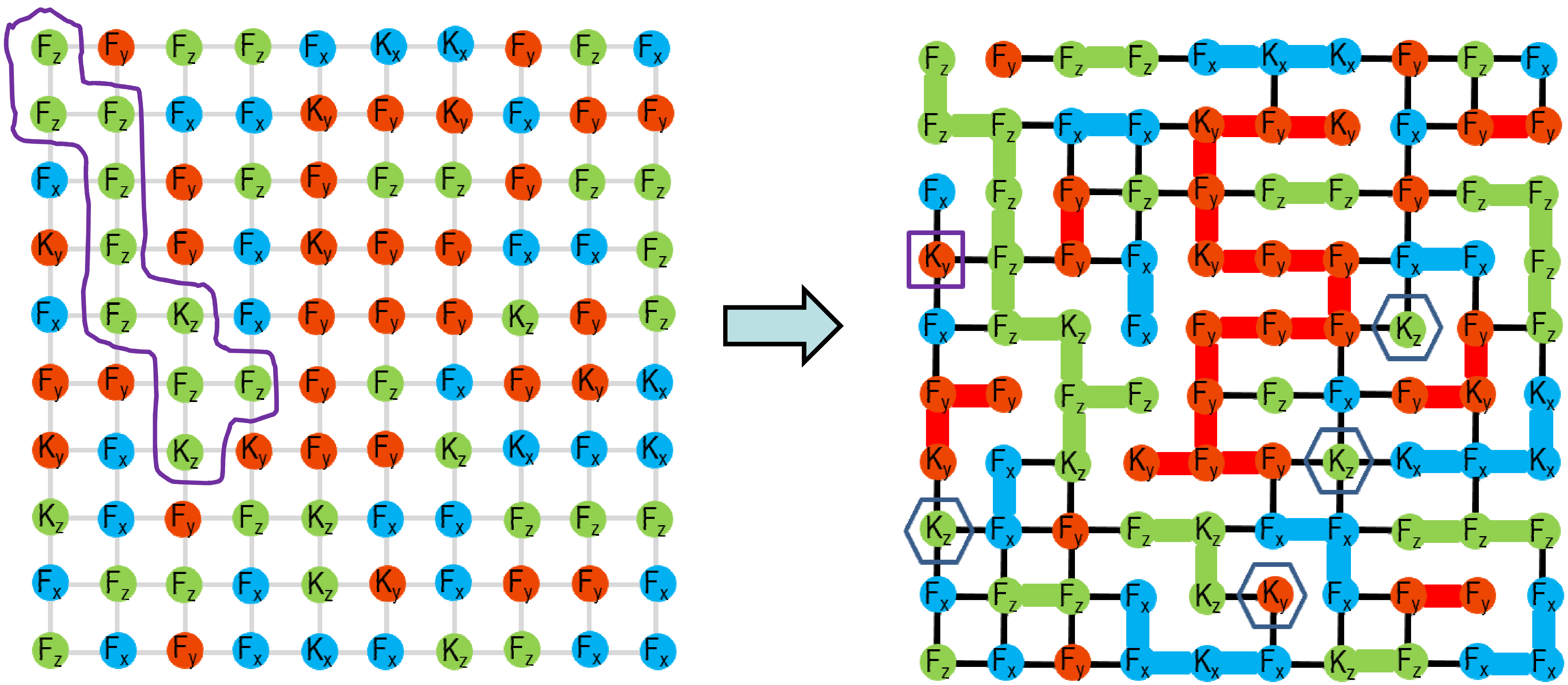
- Deal with non-planarity due to Pauli Y measurement:
  remove only subset of vertices surrounding that of Y measurement

Y:



POVM ➔ Graph of the graph state

Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)



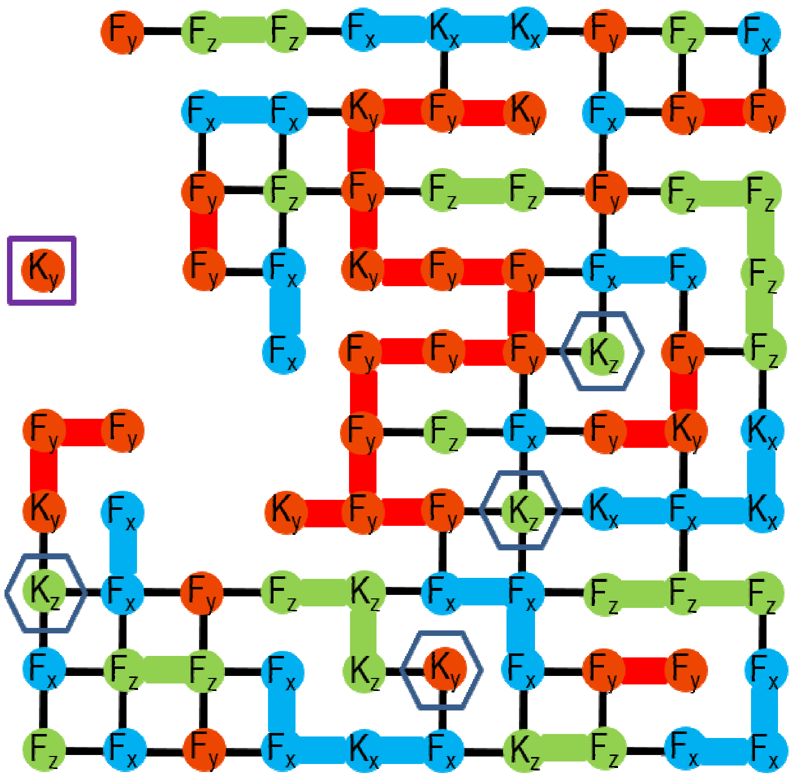
- Pauli X or Y measurement on planar graph state ➔ non-planar graph

□ :logical X measurement

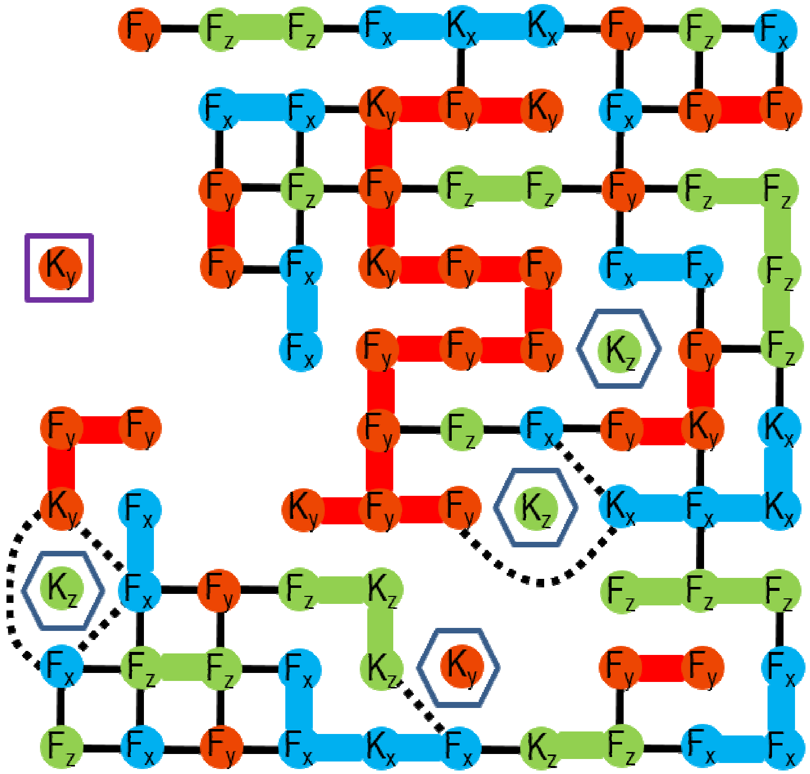
⬡ :logical Y measurement

Restore Planarity by
Another round of measurement

Deal with X measurement

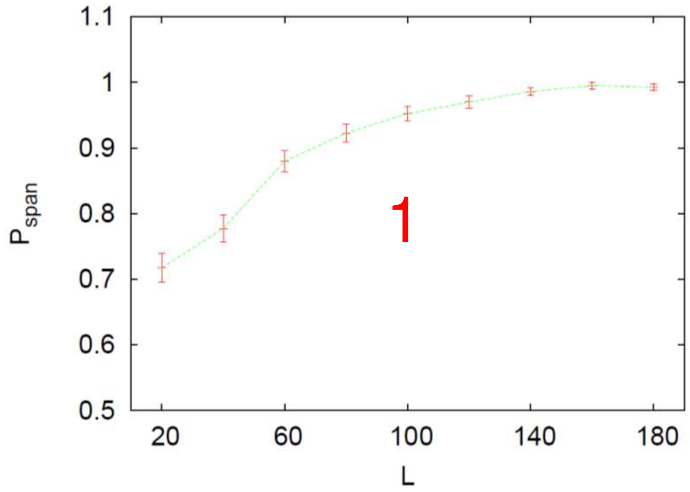


Deal with Y measurement



Examining percolation of typical graphs

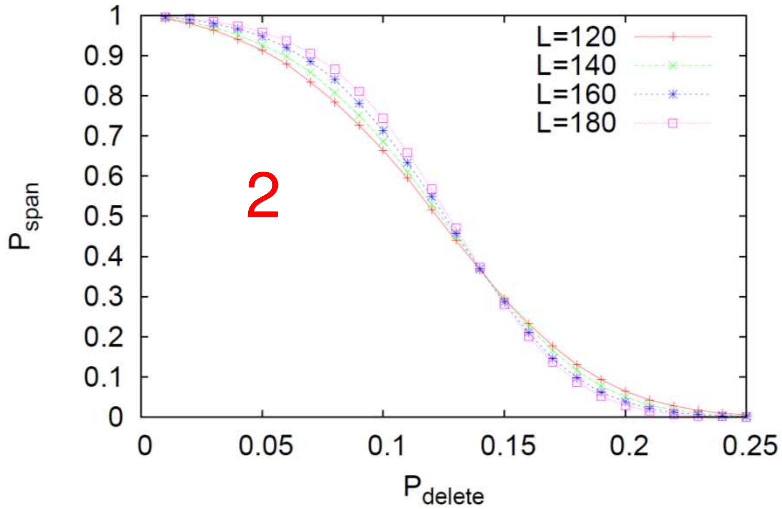
(resulting from POVM and active logical Z measurement)

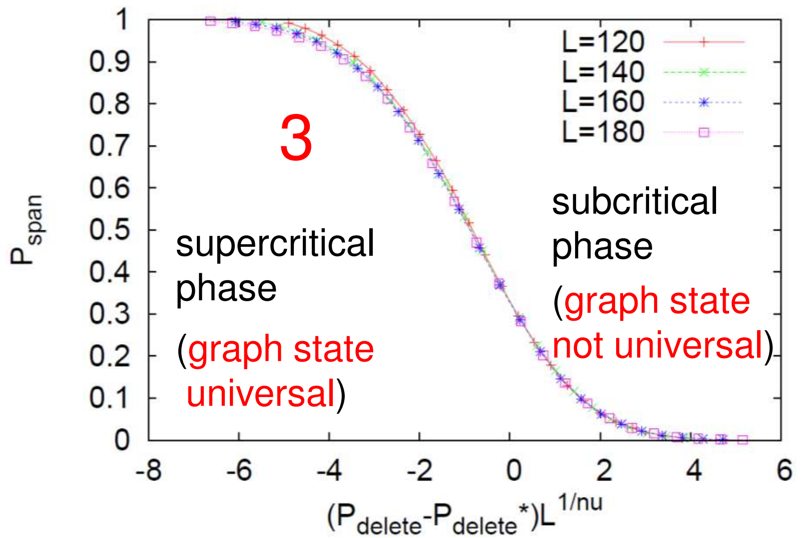


✓ 1. As system size N=L x L increases, exists a spanning cluster with high probability

✓ 2. Robustness of connectivity: finite percolation threshold (deleting each vertex with increasing probability)

✓ 3. Data collapse: verify that transition is continuous (critical exponent $\nu = 4/3$)





Spin-2 AKLT on square is universal for quantum computation

- Because the typical graph states (obtained from local measurement on AKLT) are universal ➔ hence AKLT itself is universal

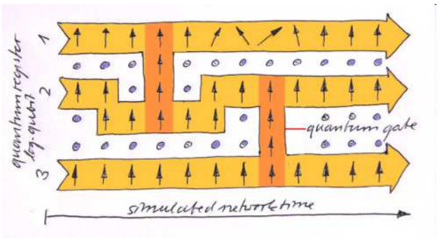
- Difference from spin-3/2 on honeycomb: ***not all*** randomly assigned POVM outcomes are allowed
  ➔ weight formula is crucial

- Emerging (partial) picture for AKLT family:

  AKLT states involving spin-2 and other lower spin entities are universal if they reside on a 2D frustration-free regular lattice with any combination of spin-2, spin-3/2, spin-1 and spin-1/2

Summary

- Introduced one-way (cluster-state) quantum computation



➔ Measurement-based QC uses entanglement

➔ Teleportation viewpoint and tensor-network approach (correlation space QC)

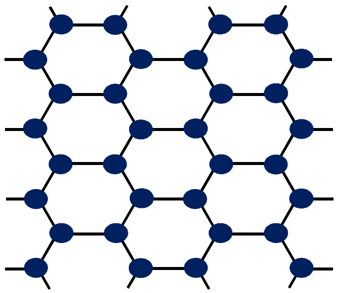
➔ Universality in graph states

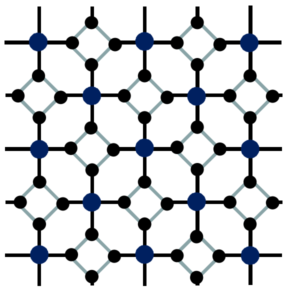
➔ Fault tolerance & surface code

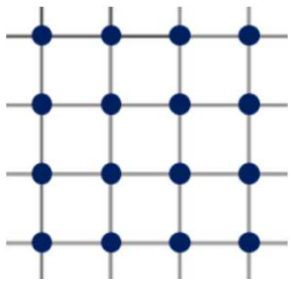
➔ Blind quantum computation

➔ Possible connection to SPT order

- Showed various AKLT states (on different 2D lattices) provide universal resource for quantum computation







Not covered

- MBQC, classical spin models & complexity

  [*Van den Nest, Dur & Briegel* '07, '08]

- Thermal phase diagram of MBQC

  [*Fujii, Nakata, Ohzeki & Murao*' '13]

  [*Li et al.* '11, *Wei, Li & Kwek* '14]

- Deformed AKLT models & transition in QC power

  [*Darmawan, Brennen & Bartlett '12,*
  Huang & Wei '16]

- Verifiable blind QC [*Hayashi & Morimae* '15]

Open problems

- ❑ Complete characterization of all universal resource states?
  - ➢ Even for AKLT family?
- ❑ Universal resource in an entire SPT phase?
  - ➢ Even for just 1D SPT phase and arbitrary 1-qubit gate?
- ❑ Deeper connection of topological QC to MBQC?