Introduction to measurement-based quantum computation

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Goals of this tutorial*

- Give some details to understand basic ingredients of measurement-based quantum computation (MBQC)
- Give pointers to related development/application (fewer details)
- Will point out related talks in this conference
- Give some open problems

*This tutorial assumes little prior knowledge
Review papers


Moore’s Law:

The number of transistors on a chip doubles ~every 2 years

➔ A transistor hits the size of a few atoms in about 20 years
➔ Quantum regime is inevitable
Candidate systems* for quantum computers

*You may see many of these throughout this conference
New quantum Moore’s Law?

- Number of qubits in ion trap
  - Roughly doubles every 6 years!
    - (may depend on physical systems)
    - e.g. see Nathan Langford’s tutorial on circuit QED
ENIAC – first generation computer

Contained:
17,468 vacuum tubes,
7,200 crystal diodes,
1,500 relays,
70,000 resistors,
10,000 capacitors
5 million hand-soldered joints

Weighed 27 tons
About 8.5 by 3 by 100 feet
Took up 1800 square feet

20 ten-digit signed accumulators

When will the first-generation quantum computer appear?
Quantum computation in a nutshell

- Consider a function $f$ and a corresponding unitary $U$:

$$U_f : |k\rangle \otimes |0\rangle \rightarrow |k\rangle \otimes |f(k)\rangle$$

- Exploit quantum parallelism:

$$\left( \sum_{k=0}^{2^n-1} |k\rangle \right) \otimes |0\rangle \rightarrow \sum_{k=0}^{2^n-1} |k\rangle \otimes |f(k)\rangle$$

  - Naive measurement only gives one $f(k)$ at a time
  - Good design of measurement may reveal properties of $f$
    - e.g. Shor’s factoring algorithm

- Factoring is hard:

$$180708208868740480595165616440590556627810251676940134917012702$$
$$1450056662540244048387341127590812303371781887966563182013214880$$

$$557 = (?) \times (?)$$

$$= (39685999459597454290161126162883786067576449112810064832555157243)$$
$$\times$$
$$= (45534498646735972188403686897274408864356301263205069600999044599)$$
Quantum computation: Circuit model

- **Building blocks**
  - **Universal gates**
    - (1) One qubit gates: any rotation
    - (2) Two qubit gate: entangling
      - e.g., C-Z gate or Controlled-NOT gate

- **CNOT:**
  - 0 0 → 0 0
  - 0 1 → 0 1
  - 1 0 → 1 1
  - 1 1 → 1 0
CNOT & CZ gates

CNOT:

\[ \text{CNOT} = |0\rangle_c |0\rangle \otimes I_t + |1\rangle_c |1\rangle \otimes X_t \]

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CZ:

\[ \text{CZ} = |0\rangle_c |0\rangle \otimes I_t + |1\rangle_c |1\rangle \otimes Z_t \]

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(Models of) Quantum Computation

- Circuit:

- Adiabatic: \[ H(t) = \left(1 - \frac{t}{T}\right)H_{\text{initial}} + \frac{t}{T}H_{\text{final}} \]

- Topological: using braiding of anyons to simulate quantum gates

- Measurement-based: local measurement is the only operation needed
I. Introduction

II. One-way cluster-state (or measurement-based) quantum computers

III. Other entangled resource states: Affleck-Kennedy-Lieb-Tasaki (AKLT) family

IV. Summary
Now focus on measurement-based (or one-way) quantum computer:

which can “simulate” unitary evolution
Unitary operation by measurement?

- Intuition: entanglement as resource!

- **Controlled-Z (CZ) gate from Ising interaction**

\[
CZ_{12} = e^{-i \frac{\pi}{4} (1 - \sigma_z^{(1)}) (1 - \sigma_z^{(2)})} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

- **Entanglement is generated:**

\[
(a|0\rangle + b|1\rangle) |+\rangle \rightarrow |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle
\]
Unitary operation by measurement?

- Intuition: entanglement as resource!

\[(a|0\rangle + b|1\rangle) \langle + | \xrightarrow{\text{CZ}} |\psi\rangle = a|0\rangle \langle + | + b|1\rangle \langle - |

- Measurement on 1st qubit in basis

\[|\pm \xi\rangle = \left( e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle \right)/\sqrt{2}\]

with outcome denoted by \(\pm = (-1)^s\)

\(\Rightarrow\) Second qubit becomes

\[\langle \pm \xi |\psi\rangle_{12} \sim a e^{i\xi/2}|+\rangle_2 \pm b e^{-i\xi/2}|-\rangle_2 = H e^{i\xi Z/2} Z^s (a|0\rangle_2 + b|1\rangle_2)\]

\(\Rightarrow\) A unitary gate is induced: \(U(\xi, s) = H e^{i\xi Z/2} Z^s\)

\[Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\]
Simulating arbitrary one-qubit gates

- In terms of circuit: [Raussendorf & Wei, Ann Rev Cond-Mat ‘12]

\[
|\text{in}\rangle = (a|0\rangle + b|1\rangle)_{1} \xrightarrow[\xi, CZ]{+ \text{ or } -} U(\xi, s)|\text{in}\rangle_{2} \xrightarrow{U(\xi, s) = H e^{it\frac{Z}{2}} Z^{s}}
\]

- Can cascade this a few times:

\[
|\text{in}\rangle = (a|0\rangle + b|1\rangle)_{1} \xrightarrow[\xi_{1}, CZ]{+ \text{ or } -} U_{1}\{\{\xi\}\} = \prod_{i=4}^{1} U(\xi_{i}, s_{i})
\]

\[
|+\rangle_{2} \xrightarrow[\xi_{2}, CZ]{+ \text{ or } -} |+\rangle_{3} \xrightarrow[\xi_{3}, CZ]{+ \text{ or } -} |+\rangle_{4} \xrightarrow[\xi_{4}, CZ]{+ \text{ or } -} |+\rangle_{5}
\]
Example: arbitrary one-qubit gate

\[ |\text{in} \rangle = (a|0 \rangle + b|1 \rangle) \]

Consider: \( \xi_1 = 0 \) & construct arbitrary rotation

\[ U_1(\{\xi\}) = \prod_{i=4}^{1} U(\xi_i, s_i) \]
\[ U(\xi, s) = H e^{i\xi Z/2} Z^s \]

We realize an Euler rotation, up to byproduct Z, X operators:

\[ U_1(\{\xi\}) = (H e^{i\xi_4 Z/2} Z^s_4) (H e^{i\xi_3 Z/2} Z^s_3) (H e^{i\xi_2 Z/2} Z^s_2) (H Z^s_1) \]

Propagating Z’s to left and use \( HZH = X \):

\[ U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{i(-1)^{s_1+s_3} \xi_4 X/2} e^{i(-1)^{s_2} \xi_3 Z/2} e^{i(-1)^{s_1} \xi_2 X/2} \]

Take \( \xi_2 = -(-1)^{s_1} \gamma \), \( \xi_3 = -(-1)^{s_2} \beta \), \( \xi_4 = -(-1)^{s_1+s_3} \alpha \)

we realize an Euler rotation, up to byproduct Z, X operators:

\[ U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{-i\alpha X/2} e^{-i\beta Z/2} e^{-i\gamma X/2} \]
Consider: $\xi_1 = 0$, & construct arbitrary rotation

Take $\xi_2 = -(-1)^{s_1}\gamma$, $\xi_3 = -(-1)^{s_2}\beta$, $\xi_4 = -(-1)^{s_1+s_3}\alpha$

we realize an Euler rotation, up to byproduct $Z$, $X$ operators:

$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{-i\alpha X/2} e^{-i\beta Z/2} e^{-i\gamma X/2}$$

Note: measurement basis can depend on prior results

Byproduct operators $Z^{s_1+s_3} X^{s_2+s_4}$ can be absorbed by modifying later measurement basis

Byproduct operators on final measurement in Z basis (readout) can be easily taken into account (only X flips 0/1)
Linear cluster state: resource for simulating arbitrary one-qubit gates

May as well take $|\text{in}\rangle = |+\rangle$ the whole state before measurement $\xi$'s is a highly entangled state $\rightarrow$ 1D cluster state
Simulating CNOT by measurement

- Consider initial state

\[ (a|0\rangle + b|1\rangle)_1 (c|0\rangle + d|1\rangle)_2 |+\rangle_3 |+\rangle_4 \]

\[ CZ_{23} CZ_{13} CZ_{34} \rightarrow |\psi\rangle_{1234} \]

\[ |\psi\rangle_{1234} = |0\rangle_3 (a|0\rangle_1 + b|1\rangle_1) (c|0\rangle_2 + d|1\rangle_2)|+\rangle_4 \\
+ |1\rangle_3 (a|0\rangle_1 - b|1\rangle_1) (c|0\rangle_2 - d|1\rangle_2)|-\rangle_4 \]

- Measurement on 2\textsuperscript{nd} and 3\textsuperscript{rd} qubits in basis \(|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}\)

If outcome=++: an effective CNOT applied:

\[ |\psi\rangle_{14} = 23\langle ++ |\psi\rangle_{1234} \sim \text{CNOT}_{14}(a|0\rangle_1 + b|1\rangle_1)(c|0\rangle_4 + d|1\rangle_4) \]

Can show: \[ |\psi_{\text{out}}\rangle \sim Z_1^{a_2} X_4^{a_3} Z_4^{a_2} \text{CNOT}_{14}|\text{in}\rangle_{14} \]

- Note the action of CZ gates can be pushed up front
(a 4-qubit “cluster” state can be used to simulating CNOT)
CNOT gate: symmetric design

Raussendorf & Briegel, PRL 01'

The following measurement pattern simulates CNOT gate (via entanglement between wires)

Q: how do I know it implements CNOT? byproduct operators=?

Ans: see Theorem I in Raussendorf, Browne & Briegel, PRA '03 (generalization to qudit: Zhou et al. PRA '03)
Theorem 1. Let $C(g) = C_f(g) \cup C_M(g) \cup C_O(g)$ with $C_f(g) \cap C_M(g) = C_f(g) \cap C_O(g) = C_M(g) \cap C_O(g) = \emptyset$ be a cluster for the simulation of a gate $g$, realizing the unitary transformation $U$, and $\ket{\phi}_{C(g)}$ the cluster state on the cluster $C(g)$.

Suppose the state $\ket{\psi}_{C(g)} = P^{(C_M(g))} (\mathcal{M}) \ket{\phi}_{C(g)}$ obeys the $2n$ eigenvalue equations

$$\sigma_x^{(C_f(g),i)}(U\sigma_x^{(i)}U^\dagger)^{(C_O(g))}\ket{\psi}_{C(g)} = (-1)^{\lambda_x,i}\ket{\psi}_{C(g)},$$  

(61)

$$\sigma_z^{(C_f(g),i)}(U\sigma_z^{(i)}U^\dagger)^{(C_O(g))}\ket{\psi}_{C(g)} = (-1)^{\lambda_z,i}\ket{\psi}_{C(g)},$$

with $\lambda_{x,i}, \lambda_{z,i} \in \{0,1\}$ and $1 \leq i \leq n$.

Then, on the cluster $C(g)$ the gate $g$ acting on an arbitrary quantum input state $\ket{\psi_{in}}$ can be realized according to Scheme 1 with the measurement directions in $C_M(g)$ described by $\mathcal{M}^{(C_M(g))}$ and the measurements of the qubits in $C_f(g)$ being $\sigma_x$ measurements. Thereby, the input and output state in the simulation of $g$ are related via

$$\ket{\psi_{out}} = U U_{\Sigma} \ket{\psi_{in}},$$  

(62)

where $U_{\Sigma}$ is a byproduct operator given by

$$U_{\Sigma} = \bigotimes_{(C_f(g) \ni i) = 1}^{n} (\sigma_z^{[i]})^{\lambda_{z,i}} (\sigma_x^{[i]})^{\lambda_{x,i}}.$$  

(63)
2D cluster state and graph states

- Can be created by applying CZ gates to each pair with edge

\[ |G\rangle = \bigotimes_{\langle i,j \rangle} CZ_{ij} \left( |+\rangle |+\rangle \cdots |+\rangle \right) \]

[Raussendorf&Briegel ’01]

- Cluster state: special case of general “graph” states

\[ K_v |G\rangle = |G\rangle, \quad \forall \text{ vertex } v \]

\[ K_v = X_v \bigotimes_{u \in \text{Nb}(v)} Z_u \]

(can show this, using above def. of G)

⇒ Uniquely define the state G, also via Hamiltonian

\[ H = - \sum_v K_v \]
Z measurement on graph state

- The effect is just to remove the measured qubit, keeping the remaining entanglement structure.

\[ |\Psi_G = |0\rangle_a |\Psi_{G\setminus a}\rangle + |1\rangle_a \left( \prod_{b \in NB(a)} Z_b \right) |\Psi_{G\setminus a}\rangle \]

Graph after Z measurement on \(a\):

- If outcome =0:
  \[ |0\rangle_a |+\rangle_1 |C\rangle_{234} \quad |C\rangle_{234} : \text{linear cluster state} \]

- If outcome =1:
  \[ |0\rangle_a |-\rangle_1 Z_2 Z_3 |C\rangle_{234} \]

- For X & Y measurements, see [Hein, Eisert, Briegel '04, Hein et al. '06]
2D cluster state is a resource for quantum computation

\[ |C\rangle = \bigotimes_{\langle i,i \rangle} C^Z_{i,j} (|+\rangle|+\rangle \cdots |+\rangle) \]

- Whole entangled state is created first (by whatever means)
- Operations needed for universal QC are single-qubit measurements only

⇒ Pattern of measurement gives computation (entanglement is being consumed → one-way)

⇒ Elementary “Lego pieces” for QC:
Cluster state for universal computation

- Carve out entanglement structure by local Z measurement

(1) Each wire simulates one-qubit evolution (gates)
(2) Each bridge simulates two-qubit gate (CNOT)

2D or higher dimension is needed for universal QC & Graph connectivity is essential (percolation)
Realizations of cluster states

- Bloch’s group: controlled collision in cold atoms (Nature 2003)

- J-W Pan’s group: 4-photon 6 qubit and CNOT (PRL 2010)
Cluster state: a valence-bond picture

- Cluster state = a valence-bond state = a projected entangled pair state (PEPS) [Verstraete & Cirac '04]

  - Bond of two virtual qubits =
    \[ \text{CZ}|++\rangle = |0\rangle|+\rangle + |1\rangle|-\rangle \]

  - Projection of several virtual qubits to physical qubit =
    \[ P = |0\rangle\langle0000| + |1\rangle\langle1111| \]

- Quantum computation via teleportation [see also Gottesman & Chuang '99]

  - 1-qubit gate:
    \[ \langle \alpha | \langle \psi | \sigma_\alpha U | \psi \rangle \]

  - 2-qubit gate:
QC in correlation space

- Previous picture of valence bond was generalized by Gross and Eisert using matrix product states (MPS) and PEPS

- Illustrate with 1D cluster state:

\[
|\Psi\rangle = \sum_{\{s\}^s} \bar{L} \cdot A_{s_1} \cdots A_{s_{i+1}} \cdot A_{s_i} \cdots A_{s_1} \cdot \bar{R} |s_1, \ldots, s_i, \ldots s_1\rangle
\]

- Measurement outcome \(\varphi_i\) at site \(i\):

\[
A(\varphi_i) \equiv \sum_{s_i} \langle \varphi_i | s_i \rangle A_{s_i}
\]

\[
\langle \varphi_n, \ldots \varphi_i, \ldots, \varphi_1 | \Psi \rangle = \bar{L} \cdot A(\varphi_n) \cdots A(\varphi_i) \cdots A(\varphi_1) \cdot \bar{R}
\]
Cluster state QC: in correlation space

- Measurement outcome $\varphi_i$ at site $i$: $A(\phi_i) \equiv \sum_{s_i} \langle \phi_i | s_i \rangle A_{s_i}$

$$\langle \phi_n, \ldots \phi_i, \ldots, \phi_1 | \Psi \rangle = \vec{L} \cdot A(\phi_n) \cdots A(\phi_i) \cdots A(\phi_1) \cdot \vec{R}$$

- As spins are measured, the boundary vector $R$ is operated by gates
  $$|R\rangle \rightarrow A_1(\phi_1)|R\rangle \rightarrow A_1(\phi_2)A_1(\phi_1)|R\rangle \rightarrow \cdots$$

- For 1D cluster state: $A(0) = |+\rangle\langle 0|$, $A(1) = |-\rangle\langle 1|$
  - measure in basis $|\pm \xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle) / \sqrt{2}$
  - obtain same 1-qubit gate as before:
    $$A(\xi, s) = e^{i\xi/2} |+\rangle\langle 0| + (-1)^s e^{-i\xi/2} |-\rangle\langle 1| = He^{i\xi Z/2}Z^s$$

- 2-qubit gates use 2D PEPS → see Gross & Eisert ‘07
Comment: deriving MPS for cluster state

\[ |0+\rangle + |1-\rangle = \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \]

\[ \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} = \begin{pmatrix} |+0\rangle & |+1\rangle \\ |0-\rangle & |-1\rangle \end{pmatrix} \]

\[ P_v = |0\rangle \langle 00| + |1\rangle \langle 11| \]

- MPS form:

\[ P_v \left( \begin{pmatrix} |+0\rangle & |+1\rangle \\ |0-\rangle & |-1\rangle \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} |0\rangle & |1\rangle \\ |0\rangle & |-1\rangle \end{pmatrix} \right) = |0\rangle \langle +| \langle 0| + |1\rangle \langle -| \langle 1| \]

\[ A(0) = |+\rangle \langle 0|, \quad A(1) = |-\rangle \langle 1| \]
Related talk:

Cluster states: not unique ground state of 2-body Hamiltonians

- First proved by Nielsen

- Van den Nest et al. proved for general (connected) graph states \( G \):
  - For approximation: ground-state of 2-body Hamiltonian can be \( \epsilon \)-close to \( G \), but the gap is proportional to \( \epsilon \)

- Bartlett & Rudolph constructed a two-body Hamiltonian such that the ground state is approximately an encoded cluster state

- Darmawan & Bartlett constructed encoded cluster state by deforming the AKLT Hamiltonian
Linear optic QC & cluster state

- Linear optic universal QC possible with single photon source, linear optic elements (beam splitters, mirrors, etc) & photon counting
  - High overhead in entangling gates [Knill, Laflamme & Milburn '01]

- Cluster state helps reduce this overhead [Yoran & Reznik '03; Nielsen '04; Browen & Ruloph '05; Kieling, Rudolph & Eisert '07]
  - Grow cluster states efficiently

- Experiments: see e.g. [O'Brien Science '07]
Create continuous-variable cluster states

- Use frequency comb and parametric amplifier in cavity

  - Theory: [Menicucci et al ‘06, ’08]
  - Experiment I: [60 modes in Pfister group ’11]
  - Experiment II: [> 10,000 modes in Furusawa group ‘12]
Related talks:

Thursday Session A:

6. [3:40-4:00] **Alessandro Ferraro, Oussama Houhou, Darren Moore, Mauro Paternostro and Tommaso Tufarelli. Measurement-based quantum computation with mechanical oscillators**
Fault tolerant cluster-state QC

- Uses a 3d cluster state and implements surface codes in each 2d layer

  ![3D Cluster State Diagram]

  [Raussendorf, Harrington & Goyal '07]

- CNOT is achievable

  ![CNOT Diagram]

- Uses magic-state distillation to achieve non-Clifford gate

  ![Magic-State Distillation Diagram]

  [Raussendorf, Harrington & Goyal '07]

  [Barrett & Stace '10]

- Error threshold 0.75%, qubit loss threshold 24.9%
Related talk:

Friday 10:30-11:00 [Long] Guillaume Dauphinais and David Poulin. *Fault Tolerant Quantum Memory for non-Abelian Anyons*
Universal blind quantum computation

Using the following cluster state (called brickwork state)

- Alice prepares
  \[ |\Psi\rangle = \otimes_{x,y} (|0\rangle_{x,y} + e^{i\theta_{x,y}} |1\rangle_{x,y}) \]
  with random \( \theta_{x,y} = 0, \pi/4, \ldots, 7\pi/4 \)
- Bob entangles all qubits according to the brickwork graph via CZ gates
- Alice tells Bob what measurement basis for Bob to perform and he returns the outcome (compute like one-way computer)
  ➔ Alice can achieve her quantum computation without Bob knowing what she computed!!

1. Alice computes \( \phi'_{x,y} \) where \( s^X_{0,y} = s^Z_{0,y} = 0 \).
   \[ \phi'_{x,y} = (-1)^{s^X_{x,y} \phi_{x,y} + s^Z_{x,y}} \]
2. Alice chooses \( r_{x,y} \in_R \{0, 1\} \) and computes \( \delta_{x,y} = \phi'_{x,y} + \theta_{x,y} + \pi r_{x,y} \).
3. Alice transmits \( \delta_{x,y} \) to Bob. Bob measures in the basis \( \{|+\delta_{x,y}\rangle, \{-\delta_{x,y}\rangle\} \).
4. Bob transmits the result \( s_{x,y} \in \{0, 1\} \) to Alice.
5. If \( r_{x,y} = 1 \) above, Alice flips \( s_{x,y} \); otherwise she does nothing.
We have seen the cluster states on the square lattice and the brickwork lattice for universal for quantum computation

Q: How much do we know about the general cluster/graph states?
Universality in graph/cluster states

- Beyond square & brickwork: other 2D graph/cluster states on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal
  
  ➔ Can use local measurement to convert one to the other (with fewer qubits, but still macroscopic)  

[Van den Nest et al. ‘06]
Graph states on regular lattices

- Beyond square & brickwork: other 2D graph/cluster states on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal

[Van den Nest et al. '06]

Local measurement converts one to another
Universality in graph/cluster states

- Beyond square & brickwork: other 2D graph/cluster states on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal
  - Can use local measurement to convert one to the other (with fewer qubits, but still macroscopic) [Van den Nest et al. ‘06]

- Faulty square lattice (degree ≤ 4) [Browne et al. ‘08]
  - As long as it is sufficiently connected (a la percolation), can find sub-graph ~ honeycomb
Cluster state on faulty lattice

- No qubits on empty sites (degree $\leq 4$) $\iff$ site percolation

- But assume perfect CZ gates $|G\rangle = \bigotimes_{(i,i)} CZ_{ij} (|+\rangle|+\rangle \cdots |+\rangle)$

- As long as probability of occupied sites $> \text{site percolation threshold}$ $\implies$ still universal for MBQC
Universality in graph/cluster states

- Beyond square & brickwork: other 2D graph/cluster states on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal
  - Can use local measurement to convert one to the other (with fewer qubits, but still macroscopic)  
  
- Faulty square lattice (degree ≤ 4)
  - As long as it is sufficiently connected (a la percolation), can find sub-graph ~ honeycomb  

- Any 2D planar random graphs in supercritical phase of percolation are universal
Other universal states

- So far no complete characterization for resource states
- Can they be unique ground state with 2-body Hamiltonians with a finite gap?
  - If so, create resources by cooling!

- TriCluster state [Chen et al. ’09]
- Affleck-Kennedy-Lieb-Tasaki (AKLT) family of states [AKLT ’87, ’88]
  - 1D (not universal): [Gross & Eisert ’07, ’10] [Brennen & Miyake ’08?]
  - 2D (universal): [Wei, Affleck & Raussendorf ’11] [Miyake ’11] [Wei et al. ‘13-’15]
- Symmetry-protected topological states
  - 1D (not universal): [Else, Doherty & Bartlett ’12] [Miller & Miyake ’15] [Prakash & Wei ’15]
  - 2D (universal, but not much explored): [Poulsen Nautrup & Wei ’15] [Miller & Miyake ’15]
Example ground state of two-body Hamiltonian as computational resource

- TriCluster state (6-level) [Chen, Zeng, Gu, Yoshida & Chuang, PRL’09]

\[
\begin{align*}
    |00\rangle &+ |01\rangle + |10\rangle - |11\rangle \\
    P_{\text{Tic}} &= |\bar{0}\rangle \langle 000| + |\bar{1}\rangle \langle 111| + |\bar{2}\rangle \langle 100| \\
                     &+ |\bar{3}\rangle \langle 011| + |\bar{4}\rangle \langle 010| + |\bar{5}\rangle \langle 101| \\
    H_{\text{triC}} &= \sum_a \left( h_{ab} + h_{ba} + h_{a} \right) \\
    h_{ab} &= \frac{2(2S_{a_1} - 5)(2S_{a_2} - 3)(2S_{a_3} - 1)(2S_{a_4} + 1)(4S_{a_5} + 11)}{2S_{b_1} + 5}(2S_{b_2} + 3)(2S_{b_3} - 1)(2S_{b_4} + 1)(4S_{b_5} - 11) \\
    &= 75\sqrt{2}S_{a_1}S_{a_2}S_{a_3}S_{a_4}S_{a_5} \left( \frac{48S_{b_1}^4 - 64S_{b_2}^3 - 280S_{b_3}^2 - 272S_{b_4} - 67}{(2S_{b_2} - 5)(2S_{b_3} - 3)(2S_{b_4} - 1)(2S_{b_3} + 3)} \right) \\
    &+ \frac{4\sqrt{10}S_{a_1}^3S_{a_2}S_{a_3}S_{a_4}S_{a_5}}{(128S_{b_2}^5 + 560S_{b_2}^4 - 2840S_{b_2}^3 - 3848S_{b_2}^2 + 675)} \\
    &+ \frac{4\sqrt{10}(128S_{a_2}^5 - 560S_{a_2}^4 + 2840S_{a_2}^3 - 3848S_{a_2}^2 - 675)}{S_{b_1}(2S_{b_2} - 5)(2S_{b_3} - 3) + \text{h.c.}} \\
    h_{b_a} &= \frac{-25S_{a_1}S_{a_2}S_{a_3}S_{a_4}S_{a_5}}{(2S_{b_2} - 5)(2S_{b_3} - 3)(2S_{b_4} + 3)(2S_{b_3} + 5)} \\
    &+ \frac{25S_{b_1}^4S_{b_2}S_{b_3}S_{b_4}S_{b_5}}{(2S_{b_2} - 5)(2S_{b_3} - 3)} \left( 224S_{b_2}^5 - 16S_{b_2}^4 - 1968S_{b_2}^3 + 40S_{b_2}^2 + 3550S_{b_2} - 9 \right) \\
    &+ \frac{12S_{b_1}^5}{(2S_{b_2} - 5)(2S_{b_3} - 3)} \left( 416S_{b_2}^5 - 80S_{b_2}^4 - 3600S_{b_2}^3 + 520S_{b_2}^2 + 5994S_{b_2} - 125 \right) + \text{h.c.} (a \Leftrightarrow b),
\end{align*}
\]
Too much entanglement is useless

- States (n-qubit) possessing too much geometric entanglement $E_g$ are not universal for QC (i.e. if $E_g > n - \delta$).

$$E_g(\vert \Psi \rangle) = -\log_2 \max_{\phi \in P} |\langle \phi | \Psi \rangle|^2$$

$P = \text{set of product states}$

- Intuition: if state is very high in geometric entanglement, every local measurement outcome has low probability.
  - Whatever local measurement strategy, the distribution of outcomes is so random that one can simulate it with a random coin (thus not more powerful than classical random string).

- Moreover, states with high entanglement are typical:
  - Those with $E_g < n - 2 \log_2(n) - 3$ is rare, i.e. with fraction $< e^{-n^2}$
  - Universal resource states are rare!!
I. Introduction

II. One-way (measurement-based) quantum computers

III. Other entangled resource states: AKLT family

IV. Summary
A new direction: valence-bond ground states of isotropic antiferromagnet

- **AKLT (Affleck-Kennedy-Lieb-Tasaki) states/models**
  - Importance: provide strong support for Haldane’s conjecture on spectral properties of spin chains [AKLT ’87, 88]
  - Provide concrete example for symmetry-protected topological order [Gu & Wen ’09, ’11]

- **States of spin S=1,3/2, 2,.. (defined on any lattice/graph)**
  - Unique* ground states of gapped# two-body isotropic Hamiltonians
  
  \[ H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j) \] 
  \[ f(x) \text{ is a polynomial} \]

* with appropriate boundary conditions; # gap proved in 1D; evidence in 2D: Garcia-Saez, Murg, Wei ’12
(hybrid) AKLT state defined on any graph

- S = # neighbors / 2
- # virtual qubits = # neighbors
- S = # neighbors / 2
- Physical spin Hilbert space = symmetric subspace of qubits

\[ P_v = \text{projection to symmetric subspace of } n \text{ qubit} \equiv \text{spin } n/2 \]
1D AKLT state for simulating 1-qubit gates

- Easy to see from its matrix product state (MPS)

\[ |01\rangle - |10\rangle = \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} |1\rangle \\ -|0\rangle \end{pmatrix} \]

\[ P_v = |+1\rangle \langle 00| + |0\rangle (\langle 01| + \langle 10|)/\sqrt{2} + |-1\rangle \langle 11| \]

- MPS form:

\[ |0\rangle \equiv |z\rangle, |+1\rangle \equiv -(|x\rangle + i|y\rangle)/\sqrt{2}, |-1\rangle \equiv (|x\rangle - i|y\rangle)/\sqrt{2} \]

\[ P_v \begin{pmatrix} |10\rangle \\ -|00\rangle \\ |11\rangle \\ -|01\rangle \end{pmatrix} = \begin{pmatrix} |0\rangle/\sqrt{2} \\ -|+1\rangle \\ |1\rangle \\ -|0\rangle\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(|x\rangle X + |y\rangle Y + |z\rangle Z) \]

- Gates with superposition of X, Y, Z are achievable

- Arbitrary 1-qubit gates possible (but universal QC requires 2-qubit gates) ➔ any 2D AKLT states universal?
Hamiltonian & SPT order

- 1D spin-1 AKLT state \( |x\rangle X + |y\rangle Y + |z\rangle Z \)
  is ground state of the gapped 2-body Hamiltonian

\[
H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2
\]

- AKLT is a symmetry-protected topological (SPT) state, e.g. by \( Z_2 \times Z_2 \) symmetry (rotation around x or z by 180°)

- Under transformation on physical spins:

  \[
  |0\rangle \equiv |z\rangle, \ |+1\rangle \equiv -(|x\rangle + i|y\rangle)/\sqrt{2}, \ |-1\rangle \equiv (|x\rangle - i|y\rangle)/\sqrt{2}
  \]

  \[
  U_z(\pi) = \begin{pmatrix}
  -1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & -1
\end{pmatrix}
  \begin{align*}
  |z\rangle & \rightarrow |z\rangle, \ |x\rangle \rightarrow -|x\rangle, \ |y\rangle \rightarrow -|y\rangle \\
  A & \rightarrow Z \cdot A \cdot Z
\end{align*}
\]

  \[
  U_x(\pi) = \begin{pmatrix}
  0 & 0 & 1 \\
  0 & -1 & 0 \\
  -1 & 0 & 0
\end{pmatrix}
  \begin{align*}
  |z\rangle & \rightarrow -|z\rangle, \ |x\rangle \rightarrow |x\rangle, \ |y\rangle \rightarrow -|y\rangle \\
  A & \rightarrow X \cdot A \cdot X
\end{align*}
\]

⇒ Projective representation (e.g. \( Z & X \)) of symmetry implies SPT order
SPT order of cluster state

- MPS for cluster state (single site):
  \[ A(0) = |+\rangle\langle 0|, \quad A(1) = |--\rangle\langle 1| \]
  \[ \Rightarrow \text{ +/- basis: } A(+) \sim A(0) + A(1) = H, \quad A(--) = HZ \]

- Two sites:
  \[ A(++) = H^2 = 1, \quad A(+-) = H(HZ) = Z \]
  \[ A(--)= (HZ)H = X, \quad A(-+) = (HZ)^2 = XZ \]

- Under XIXI… on physical spins:
  \[ A(++) \rightarrow A(++) , \quad A(+-) \rightarrow A(+-) \]
  \[ A(--) \rightarrow -A(+-), \quad A(-+) \rightarrow -A(--) \]
  \[ A(\alpha, \beta) \rightarrow Z \cdot A(\alpha, \beta) \cdot Z \]
  \[ A(\alpha, \beta) \rightarrow X \cdot A(\alpha, \beta) \cdot X \]

- Similarly for IXIX…:
  \[ A(\alpha, \beta) \rightarrow X \cdot A(\alpha, \beta) \cdot X \]

\[ \Rightarrow \text{ projective representation } \Rightarrow \text{ SPT order} \]
AKLT is a symmetry-protected topological (SPT) state, e.g. by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (rotation around x or z by 180°) with Hamiltonian

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2$$

1D cluster state is also a SPT state, e.g. by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry ($XIXI\ldots$ or $IXIX\ldots$) with Hamiltonian

$$H = -\sum_i Z_{i-1}X_iZ_{i+1}$$

Generic states in such 1D SPT phase

$$A_\alpha = \sigma_\alpha \otimes B_\alpha$$

[Else et al. ‘12]

[Prakash & Wei ‘15]

⇒ Only identity gate (up to Pauli) is protected

⇒ But arbitrary 1-qubit gate is possible, e.g. with $S_4$ symmetry

[Miller & Miyake ‘15]
2D SPT states for universal QC

- A “Control-control-Z state”: [Miller & Miyake ’15]
  \( \psi = \text{CCZ (Control-Control-Z) gates applied to all triangles with } |\text{+++ \ldots++} \rangle \)

- Fixed-point wavefunctions of generic SPT states (with any nontrivial SPT order) are universal resource; see

In the remaining, we will focus on AKLT family of states for universal quantum computation.
Converting 1D AKLT state to cluster state

\[ P_v = |+1\rangle\langle 00| + |0\rangle(\langle 01| + \langle 10|)/\sqrt{2} + |-1\rangle\langle 11| \]

- Via adaptive local measurement (i.e. state reduction)  
  \[ \text{[Chen, Duan, Ji & Zeng '10]} \]

- Via fixed POVM  
  \[ \text{[Wei, Affleck & Raussendorf '11]} \]

  \[ \text{\textit{generalizable to 2D AKLT:}} \quad F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I \]

  \[ F_x \sim |S_x = 1\rangle\langle S_x = 1| + |S_x = -1\rangle\langle S_x = -1| \sim |++\rangle\langle ++| + |--\rangle\langle --| \]

  \[ F_y \sim |S_y = 1\rangle\langle S_y = 1| + |S_y = -1\rangle\langle S_y = -1| \sim |i,i\rangle\langle i,i| + |-i,-i\rangle\langle -i,-i| \]

  \[ F_z \sim |S_z = 1\rangle\langle S_z = 1| + |S_z = -1\rangle\langle S_z = -1| \sim |00\rangle\langle 00| + |11\rangle\langle 11| \]

  \[ \text{\textit{Outcome labeled by } x,y, z: } \quad |\psi\rangle \rightarrow F_\alpha |\psi\rangle \]
POVM: 1D AKLT state $\rightarrow$ cluster state

$|01\rangle - |10\rangle$

POVM: $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$

e.g. for the outcome (labeled x, y, z)

→ the post-measurement state is an encoded 1D cluster state with graph:

1 logical qubit = 1 domain = consecutive sites with same outcome

→ This generalizes to some 2D AKLT states (with $S \leq 2$)
Realizations of 1D AKLT state

- Resch’s group: photonic implementation (Nature Phys 2011)
2D AKLT states for quantum computation?

- On various lattices

- Honeycomb
- Square-octagon
- 'Cross'
- Star

- Square-hexagon (spin-2 spin-3/2 mixture)
- Decorated-square (spin-2 spin-1 mixture)
- Square (spin-2)
- Kagome (spin-2)

Wei, Affleck & Raussendorf, PRL ’11; Miyake ‘11;
Wei, PRA ’13, Wei, Haghnegahdar & Raussendorf, PRA ’14
Wei & Raussendorf ‘15
Proposal for 2D AKLT states

- Liu, Li and Gu [JOSA B 31, 2689 (2014)]

- Koch-Janusz, Khomskii & Sela [PRL 114, 247204 (2015)]

$t_{2g}$ electrons in Mott insulator
AKLT states on trivalent lattices

- Each site: three virtual qubits $\bigcirc \equiv$ spin $3/2$ (in general: $S = \#\text{nbr} / 2$)
  - physical spin = symmetric subspace of qubits

- Two virtual qubits on an edge form a singlet

$$P = |3/2\rangle\langle 000| + | -3/2\rangle\langle 111| + |1/2\rangle\langle W| + | -1/2\rangle\langle \overline{W}|$$

\[
|000\rangle \leftrightarrow |S = \frac{3}{2}, S_z = \frac{3}{2}\rangle \\
|111\rangle \leftrightarrow |\frac{3}{2}, -\frac{3}{2}\rangle \\
|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow |\frac{3}{2}, \frac{1}{2}\rangle \\
|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow |\frac{3}{2}, -\frac{1}{2}\rangle
\]
Use generalized measurement (POVM)

\[
F_z = \sqrt{\frac{2}{3}} \left( \left| \frac{3}{2} \right> \left< \frac{3}{2} \right|_z + \left| -\frac{3}{2} \right> \left< -\frac{3}{2} \right|_z \right) \\
F_x = \sqrt{\frac{2}{3}} \left( \left| \frac{3}{2} \right> \left< \frac{3}{2} \right|_x + \left| -\frac{3}{2} \right> \left< -\frac{3}{2} \right|_x \right) \\
F_y = \sqrt{\frac{2}{3}} \left( \left| \frac{3}{2} \right> \left< \frac{3}{2} \right|_y + \left| -\frac{3}{2} \right> \left< -\frac{3}{2} \right|_y \right)
\]

[Wei, Affleck & Raussendorf ’11, Miyake ’11]

Completeness:

\[F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I\]

- POVM gives random outcome \(x\), \(y\) and \(z\) at each site

\[|\Phi\rangle \rightarrow F_{\alpha=x,y,or\ z} |\Phi\rangle\]

- Can show POVM on all sites converts AKLT to a graph state
  (graph depends on random \(x\), \(y\) and \(z\) outcomes)
Proving graph state

Let us first explain the notation. Consider a central vertex \( C \in V(G_0(\{F\})) \) and all its neighboring vertices \( C_\mu \in V(G_0) \). Denote the POVM outcome for all \( L \) sites \( v \in C, C_\mu \) by \( a_c \) and \( a_\mu \), respectively. Denote by \( E_\mu \) the set of \( L \) edges that run between \( C \) and \( C_\mu \). Denote by \( E_c \) the set of \( L \) edges internal to \( C \). Denote by \( V_c \) the set of all qubits in \( C \), and by \( V_\mu \) the set of all qubits in \( C_\mu \). (Recall that there are four qubit locations per \( L \) vertex \( v \in C, C_\mu \).) Extending Eq. (33) of Ref. [17] to the spin-2 case, we have

\[
K_C = (\prod_{\mu} \prod_{e \in E_\mu} \sigma_{a_\mu}^{(v(e))} \sigma_{a_\mu}^{(v(e))} ) \prod_{e \in E_c} \sigma_{a_\mu}^{(v(e))} \sigma_{a_\mu}^{(v(e))} )
\]

We take the following convention for \( b \) as reported in Table II. For POVM outcome \( a_c = z \), we take \( b = x \); for \( a_c = x \), we take \( b = z \); for \( a_c = y \), we take \( b = z \). With this choice we have

\[
K_C = (-1)^{|E_c|+\sum_\mu |E_\mu|} \prod_{e \in E_\mu} \sigma_{a_\mu}^{(v(e))} \sigma_{a_\mu}^{(v(e))} ) \prod_{e \in E_c} \sigma_{a_\mu}^{(v(e))} \sigma_{a_\mu}^{(v(e))} )
\]

### Table II. The choice of \( b \) and \( a_{\mu \neq b} \).

| \( a_c \) | \( z \) | \( x \) | \( y \) |
| \( b \) | \( x \) | \( z \) | \( z \) |
| \( a_{\mu \neq b} \) | \( y \) | \( y \) | \( x \) |

| POVM outcome | \( \lambda_1 \lambda_2 \sigma_z^{[1]} \sigma_z^{[1]} \) | \( \lambda_1 \lambda_2 \sigma_x^{[1]} \sigma_x^{[1]} \) | \( \lambda_1 \lambda_2 \sigma_y^{[1]} \sigma_y^{[1]} \) |
| Logical \( \overline{X} \) operator | \( \otimes_{j=1}^{4C} \sigma_x^{[j]} \) | \( \otimes_{j=1}^{4C} \sigma_x^{[j]} \) | \( \otimes_{j=1}^{4C} \sigma_x^{[j]} \) |
| Logical \( \overline{Z} \) operator | \( \lambda_1 \sigma_x^{[1]} \) | \( \lambda_1 \sigma_x^{[1]} \) | \( \lambda_1 \sigma_y^{[1]} \) |
Probability of POVM outcomes

- Measurement gives random outcomes, but what is the probability of a given set of outcomes?

\[ P(\{\alpha(v)\}) \sim \langle \psi_{\text{AKLT}} | \bigotimes_v F_{\alpha(v)}^\dagger F_{\alpha(v)} | \psi_{\text{AKLT}} \rangle \]

- Can evaluate this using coherent states; alternatively use tensor product states

- Turns out to be a geometric object

\[ P(\{\alpha(v)\}) \sim 2^{|V| - |\epsilon|} \]

[Wei, Affleck & Raussendorf, PRL ’11 & PRA ’12]
Difference from 1D case: graph & percolation

1. What is the graph? which determines the graph state
   ➔ How to identify the graphs?

2. Are they percolated? (if so, universal resource)

[Wei, Affleck & Raussendorf PRL’11]
Recipe: construct graph for ‘the graph state’

- Examples: random POVM outcomes $x, y, z$

\[ P(\{\alpha(v)\}) \sim 2^{|V| - |E|} \]
Step 1: Merge sites to “domains” → vertices

- 1 domain = 1 logical qubit

honeycomb square octagon

: encoding of a logical qubit
Step 2: edge correction between domains

- Even # edges = 0 edge, Odd # edges = 1 edge
  (due to $\sigma_z^2 = I$ in the C-Z gate)

honeycomb

square octagon
Step 3: Check connections (percolation)

- Sufficient number of wires if graph is in supercritical phase (percolation)

- Verified this for honeycomb, square octagon and cross lattices
  - AKLT states on these are universal resources
How robust is connectivity?

- Characterized by artificially removing domains to see when connectivity collapses (phase transition)

![Graphs showing supercritical and subcritical transitions](image)

[Wei’13]
Frustration on star lattice

- Cannot have POVM outcome xxx, yyy or zzz on a triangle

- Consequences:

  1. Only 50% edges on triangles occupied < $p_{th} \approx 0.5244$ of Kagome
     → disconnected graph

  2. Simulations confirmed: graphs not percolated
     → AKLT on star likely NOT universal
Difficulty for spin-2

- Technical problem: trivial extension of POVM does NOT work!

\[
F_z = \begin{pmatrix} 2 \langle 2 |_z \rangle \langle 2 \rangle_z + \langle -2 \rangle_z \langle -2 |_z \rangle 
\end{pmatrix} 
\]

\[
F_x = \begin{pmatrix} 2 \langle 2 \rangle_x \langle 2 |_x \rangle + \langle -2 \rangle_x \langle -2 |_x \rangle 
\end{pmatrix} 
\]

\[
F_y = \begin{pmatrix} 2 \langle 2 |_y \rangle \langle 2 \rangle_y + \langle -2 \rangle_y \langle -2 |_y \rangle 
\end{pmatrix} 
\]

\[
F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z \neq c \cdot I
\]

\[\rightarrow \text{ Leakage out of logical subspace (error)}\]

- Fortunately, can add elements K’s to complete the identity

\[
F_\alpha = \sqrt{\frac{2}{3}} (|S_\alpha = +2 \rangle \langle S_\alpha = +2 | + |S_\alpha = -2 \rangle \langle S_\alpha = -2 |)
\]

\[
K_\alpha = \sqrt{\frac{1}{3}} (\langle \phi^-_\alpha \rangle \langle \phi^-_\alpha |)
\]

\[
\alpha = x, y, z
\]

\[
\sum_{\alpha = x, y, z} F_\alpha^\dagger F_\alpha + \sum_{\alpha = x, y, z} K_\alpha^\dagger K_\alpha = I
\]

[Wei, Haghneghahdar, Raussendorf’14]
Another difficulty: sample POVM outcomes

\[ p(\{F, K\}) = \langle \text{AKLT} | \bigotimes_u F_{\alpha(u)}^{\dagger} F_{\alpha(u)} \bigotimes_v K_{\beta(v)}^{\dagger} K_{\beta(v)} | \text{AKLT} \rangle = ? \]  

[Wei, Raussendorf´15]

- How to calculate such an \( N \)-body correlation function?

**Lemma.** If there exists a set \( Q \) (subset of \( D_K \)) such that \( - \bigotimes_{\mu \in Q} (-1)^{|V_\mu|} X_\mu \) is in the stablizer group \( S(|G_0\rangle) \) of the state \( |G_0\rangle \), then \( p(\{F, K\}) = 0 \). Otherwise,

\[ p(\{F, K\}) = c \left( \frac{1}{2} \right)^{|E| - |V| + 2|J_K| - \text{dim}(\text{ker}(H))} \]

where \( c \) is a constant.

\[
\begin{align*}
|G_0\rangle &\sim \bigotimes_v F_{\alpha(v)} | \text{AKLT} \rangle \\
D_K &\text{: set of domains having all sites POVM } K \\
(H)_{\mu\nu} &\text{ = 1 if } \{K_\mu, X_\nu\} = 0, \text{ and } (H)_{\mu\nu} = 0 \text{ otherwise}
\end{align*}
\]

\( \Rightarrow \) Bottom line: can use Monte Carlo sampling
Local POVM: 5-level to (2 or 1)-level

\[
F_\alpha = \sqrt{\frac{2}{3}}(|S_\alpha = +2\rangle\langle S_\alpha = +2| + |S_\alpha = -2\rangle\langle S_\alpha = -2|)
\]

\[
K_\alpha = \sqrt{\frac{1}{3}}(|\phi_\alpha^+\rangle\langle \phi_\alpha^+|) = \frac{1}{\sqrt{2}}|\phi_\alpha^+\rangle\langle \phi_\alpha^+| F_\alpha \quad |\phi_\alpha^\pm\rangle = \sqrt{\frac{1}{2}}(|S_\alpha = 2\rangle \pm |S_\alpha = -2\rangle)
\]

\[\alpha = x, y, z\]

Completeness: \[\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I\]

- POVM gives random outcome \(F_x, F_y, F_z, K_x, K_y, K_z\) at each site

\[\Rightarrow\] Local action (depends on outcome):

\[|\Phi\rangle \rightarrow F_{\alpha=x,y, or z}|\Phi\rangle\]

or

\[|\Phi\rangle \rightarrow K_{\alpha=x,y, or z}|\Phi\rangle\]
Post-POVM state: graph state

\[ F_\alpha = \sqrt{\frac{2}{3}} (|S_\alpha = +2\rangle \langle S_\alpha = +2| + |S_\alpha = -2\rangle \langle S_\alpha = -2|) \]

\[ K_\alpha = \sqrt{\frac{1}{3}} (|\phi_\alpha^-\rangle \langle \phi_\alpha^-|) = \frac{1}{\sqrt{2}} |\phi_\alpha^-\rangle \langle \phi_\alpha^-| F_\alpha \quad |\phi_\alpha^\pm\rangle = \sqrt{\frac{1}{2}} (|S_\alpha = 2\rangle \pm |S_\alpha = -2\rangle) \]

\( \alpha = x, y, z \)

- If \( F \) outcome on all sites
  \( \Rightarrow \) a \textit{planar} graph state

\[ |G_0\rangle = \bigotimes_v F_{\alpha_v}^{(v)} |\text{AKLT}\rangle \]

- Vertex = a domain of sites with same color (\( x, y \) or \( z \))

- \( K \) outcome = \( F \) followed by measurement (then \textit{post-selecting} ‘-’ result)

  \( \Rightarrow \) Either
  (1) shrinks domain size [trivial] or
  (2) logical X or Y measurement [\textit{nontrivial}]

[Wei, Haghnegahdar, Raussendorf’14]
POVM ➔ Graph of the graph state

Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)

$|G_0\rangle = \bigotimes_v F^{(v)}_{\alpha_v} |\text{AKLT}\rangle$

- Effect of nontrivial non-planar graph

$K_{\alpha} = \frac{1}{\sqrt{2}} |\phi_{\alpha}^\rangle \langle \phi_{\alpha}^| \cdot F_{\alpha}$
Non-planarity from X/Y measurement

Effect of X measurement is more complicated than Y measurement

[See e.g. Hein et ‘06]
Restore planarity: further measurement

- Deal with non-planarity due to Pauli $X$ measurement:
  remove all vertices surrounding that of $X$ measurement (via $Z$ measurement)

- Deal with non-planarity due to Pauli $Y$ measurement:
  remove only subset of vertices surrounding that of $Y$ measurement
POVM ➔ Graph of the graph state

Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)

- Pauli X or Y measurement on planar graph state ➔ non-planar graph
Restore Planarity by Another round of measurement

Deal with X measurement

Deal with Y measurement
Examining percolation of typical graphs
(resulting from POVM and active logical Z measurement)

1. As system size $N=L \times L$ increases, exists a spanning cluster with high probability

2. Robustness of connectivity: finite percolation threshold (deleting each vertex with increasing probability)

3. Data collapse: verify that transition is continuous (critical exponent $\nu = 4/3$)
Spin-2 AKLT on square is universal for quantum computation

- Because the typical graph states (obtained from local measurement on AKLT) are universal \(\Rightarrow\) hence AKLT itself is universal

- Difference from spin-3/2 on honeycomb: *not all* randomly assigned POVM outcomes are allowed
  \(\Rightarrow\) weight formula is crucial

- Emerging (partial) picture for AKLT family:

  AKLT states involving spin-2 and other lower spin entities are universal if they reside on a 2D frustration-free regular lattice with any combination of spin-2, spin-3/2, spin-1 and spin-1/2
Summary

- Introduced one-way (cluster-state) quantum computation
  - Measurement-based QC uses entanglement
  - Teleportation viewpoint and tensor-network approach (correlation space QC)
  - Universality in graph states
  - Fault tolerance & surface code
  - Blind quantum computation
  - Possible connection to SPT order

- Showed various AKLT states (on different 2D lattices) provide universal resource for quantum computation
Not covered

- MBQC, classical spin models & complexity  
  [Van den Nest, Dur & Briegel '07, '08]

- Thermal phase diagram of MBQC  
  [Fujii, Nakata, Ohzeki & Murao '13]
  [Li et al. '11, Wei, Li & Kwek '14]

- Deformed AKLT models & transition in QC power  
  [Darmawan, Brennen & Bartlett '12, Huang & Wei '16]

- Verifiable blind QC  
  [Hayashi & Morimae '15]
Open problems

- Complete characterization of all universal resource states?
  - Even for AKLT family?

- Universal resource in an entire SPT phase?
  - Even for just 1D SPT phase and arbitrary 1-qubit gate?

- Deeper connection of topological QC to MBQC?