Singularities and Black Holes

G W Gibbons

DAMTP, Cambridge University

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Einstein's General Relativity is incomplete because

- It predicts that gravitational collapse , both at the Big Bang and inside black holes, brings about spacetime singularities as at which the theory breaks down

- It gives no account of 'matter 'as opposed to geometry , and in particular the nature of classical 'particles '

- It is incompatible with quantum mechanics

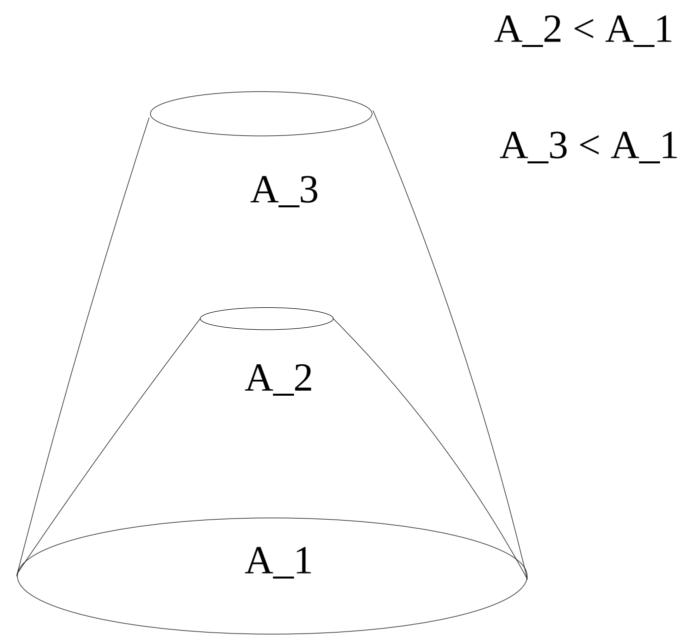
Singularities

First discovered in Friedmann-Lemaitre models, it was shown by Roger Penrose(1965) that these arise if closed trapped surfaces occur during gravitational collapse and work by Geroch , Hawking and Penrose showed that as long as matter satisfies various positive energy conditions, then spacetime singularities are inevitable in the future of certain types of Cauchy data.

Thus unlike classical Yang-Mills theory * and scalar fields theories with renormalizable potentials, Leibniz-Laplace Determinism breaks down for General Relativity. It can at best be an effective theory.

*or possibly Born-Infeld theory

The area of a closed trapped 2-surface decreases in both the inward *and* the outward directions if pushed to the future along its two lightlike normals



These theorems also apply to classical supergravity theories in all relevant dimensions since the matter fields satisfy the energy conditions. If supermatter is added, then only if potentials for scalars are positive (which cannot happen for pure supergravity) could singularities conceivably be avoided. However one may also truncate to the pure gravity sector and we are back to the same problem.

The same problem arises in String Theory in the zero slope limit. Only higher curvature terms could could conceivably evade the problem.

The Strong Energy Condition

$$\boxed{T_{\hat{0}\hat{0}} + \sum_i T_{\hat{i}\hat{i}} \geq 0} \tag{1}$$

is the most important energy condition (‘Gravity is attractive‘)

It can only fail if potentials for scalars are positive.

Unless it fails, cosmic acceleration (e.g. a positive cosmological constant, $\Lambda > 0$, is impossible)

Thus there can be no inflation in pure supergravity theories, or the zero slope limit of String theory.*

*except in models with time-dependent extra dimensions which have other problems

Einstein was also concerned about the breakdown of classical theory near point particles such as Maxwellian Linear Electrodynamics coupled to classical point particles. The self-force diverges, as does the total energy.

$$\mathbf{E} = \frac{q}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} \to \infty\,, \tag{2}$$

$$E = \frac{1}{2}\int d^3x\,\mathbf{E}^2 \to \infty\,. \tag{3}$$

An attractive idea, pioneered by Mie and later refined by Born and Infeld was Non-Linear Electrodynamics

$$
\begin{aligned}
\mathbf{D} &= \frac{q}{4\pi}\frac{\widehat{\mathbf{r}}}{r^2} \rightarrow \infty\,, &(4)\\
\mathbf{E} &= \frac{q}{4\pi}\frac{\widehat{\mathbf{r}}}{\sqrt{r^4 + q^2 E_c^{-2}}} \rightarrow ?\,, &(5)\\
E &= \frac{1}{2}\int d^3x\,\mathbf{E}.\mathbf{D} < \infty\,. &(6)
\end{aligned}
$$

However this is *not source-free*, it has a delta-function source at the origin where the field equations are violated.

For want of a better word I call this a BIon.

More acceptable to Einstein would be solutions of the classical field equations which are of *finite total energy* and which are everywhere smooth and singularity free,*without sources*. Nowadays we would call these Soliton, or ‘classical lumps’.

The best known example is the ’t Hooft-Polyakov monopole of Yang-Mills-Higgs theory.

We now believe that quantum field theories may admit two types of particle, perturbative particles, described approximately by the Klein-Gordon, Dirac or Proca equations, and non-perturbative particles or solitons described approximately by classical field theory.

In exceptional theories, such as supersymmetric theories, there may be a symmetry or duality between these two types of particles, for example in $\mathcal{N} = 4$ SUSY Yang-Mills Theory.

Einstein was especially concerned about the gravitational field of point particles, whose exterior metric was obtained by Schwarzschild

$$\boxed{ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin\theta^2 d\phi^2)\,.} \tag{7}$$

This diverges both at $r = 0$ and $r = 2M$. The former is a space-time singularity, much much worse than a delta-function, the latter a coordinate artifact due to an event horizon.

Serini, Einstein Pauli and Lichnerowicz were able to show that there are no static or stationary soliton like solutions of the vacuum Einstein equations without horizons.

The presence of an horizon implies a singularity.

These results extend straightforwardly to include the sort of matter encountered in ungauged supergravity theories and Klauza-Klein theory* or the zero slope limits of String Theory. They follow essentially because these theories do not admit a length scale: rigid dilation

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \qquad \lambda \text{ constant}, \tag{8}$$

is a symmetry of the equations of motion.

*with the proviso that the fields in four dimensions are regular, see later

Einstein and Rosen realised that a key to understanding the Schwarzschild source was the existnce of what we now call an Einstein-Rosen bridge or wormhole connecting two asymptotically flat regions.

In isotropic coordinates

$$ds^2 = -(\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}})^2 dt^2 + (1 + \frac{M}{2\rho})^4 (d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2)) \qquad (9)$$

invariant under the involution

$$\rho \to \frac{M^2}{4\rho}\,. \qquad (10)$$

Only later, with the work of Szekeres , Kruskal, Finklestein etc did the full complexity of the maximal analytic continutation of the Schwarzschild solution become clear. This shows that the singularity, which is space-like rather than timelike as one might have suspected is still present. However it does not lie on a surface of constant t. It lives on $R^2 \times S^2$ with metric

$$ds^2 = -\frac{32M^3}{r} e^{\frac{r}{2M}} dU dV + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{11}$$

$$-UV = e^{\frac{r}{2M}} (\frac{r}{2M} - 1) \qquad -(U/V) = e^{\frac{t}{2M}} \,. \tag{12}$$

The surfaces of constant r are hyperbolae $UV = \text{constant}$.

The *two* static exterior regions have $UV < 0$

The future and past time-dependent regions inside the horizon ahve $UV > 0$

The surfaces of constant time are straight lines through the origin $U = \text{constant}\, V$.

The region $UV > 0$ contains closed trapped 2-spheres, whose null normals are $U = \text{constant}$, and $V = \text{constant}$.

The past and future singularities are inside the horizon at $UV = 1$.

The no-go-theorems of Serini, Einstein, Pauli and Lichnerowicz were extended to uniqueness or no hair theorems by Israel, Robinson , Carter and Hawking etc to show that the final state of gravitational collapse is given by by a member of the Kerr-Newman family of metrics which are complete specified once one has given the total mass M, angular momentum J and electric and magnetic charges Q and P subject to the constraint

$$\boxed{M \geq \frac{1}{2}\sqrt{Q^2 + P^2 + \sqrt{4J^2 + Q^2 + P^2}}\,.} \tag{13}$$

This inequality ensures that the *singularities and sources of the stationary solutions are hidden inside an event horizon*. The conjecture that they are always so hidden, even in dynamic situations, is called Cosmic Censorship.

The current status, and indeed precise formulation, of the Cosmic Censorship Conjecture in classical General Relativity remains unclear. It does seem however that it can fail in certain, rather special circumstances(massless scalar field). .Thus at the very least, the word generic must be included. It certainly seems to be true for an open set in the space of all Cauchy Data.

The same uncertainty still clouds the nature of the spacetime singularities. However recent work is compatible with the idea that the chaotic, oscillatory, Mixmaster type singularities of the type suggested by Belinsky Lifshitz and Khalitnikov arise from an open set of Cauchy data.

A striking feature of the classical mechanics of black holes are analogies to the laws of thermodynamics. One may associate with every stationary solution a surface area A, surface gravity κ, angular velocity Ω and electric and magnetic potentials ψ and χ. One has

$$\boxed{\kappa\,,\Omega\,,\psi\,,\chi\,, \qquad \text{are constant on the horizon}} \tag{14}$$

$$\boxed{dM = \frac{1}{4\pi}\kappa dA + \Omega dJ + \psi dQ + \chi dP \qquad \text{in a quasi stationary process}} \tag{15}$$

$$\boxed{dA \geq 0} \tag{16}$$

$$\boxed{\kappa \qquad \text{cannot be reduced to zero by a finite process}} \tag{17}$$

The surface gravity κ is defined by

$$l^\beta l^\alpha_{;\beta} = \kappa l^\alpha \,, \tag{18}$$

and measures the extent to which the null generators l^α, $l^\alpha l_\alpha = 0$ of the horizon fail to be affinely parameterised

$$\boxed{U = \exp \kappa u} \tag{19}$$

U is Kruskal time, valid near the horizon. u is Eddington- Finkelstein time (same as Killing time) , valid near infinity.

$$U = T - R^\star \qquad u = t - r^*, \tag{20}$$

$$r^\star = r + 2M \ln(\frac{r}{2M} - 1)\,. \tag{21}$$

The formal similarity between temperature and surface gravity and entropy and area suggested to Bekenstein among others an extension of thermodynamics to black holes

$$S \propto A\,. \tag{22}$$

The identification was clinched and the factor of proportionality fixed by Hawking's work on particle creation in the neighbourhood of horizons.

$$\boxed{S = \frac{1}{4}A\,, \qquad T = \frac{\kappa}{2\pi}\,.} \tag{23}$$

Hawking's calculation used Free Quantum Field Theory in Fixed Curved Spacetime background (FQFTICS).

It rests on an assumption about positive frequency during gravitational collapse. Hawking argued that gravitational collapse brings about a quantum state with no 'particles'defined using the Kruskal coordinate U. It follows that there is a thermal distribution of particles defined using the Eddington-Finkelstein coordinate u.

To see this, Fourier transform

$$e^{-i\omega U} = \int_{-\infty}^{+\infty} d\omega' e^{-i\omega' u} f(\omega, \omega') \tag{24}$$

$$N(\omega') \propto \int_{-\infty}^{0} d\omega |f(\omega, \omega')|^2 \propto \frac{1}{e^{\frac{2\pi\omega'}{\kappa}} \pm 1} . \tag{25}$$

Hartle and Hawking extended this argument to establish the Einstein A and B coefficient relations for the spontaneous and induced emission rates, making use of the periodicity of the formula relating u and U in imaginary times. Gibbons and Perry pointed out that one this periodicity implied that the Green's functions for a black hole in equilibrium with its products must be Thermal Green's functions with periodicity in imaginary time given by

$$\boxed{\beta = T^{-1} = \frac{2\pi}{\kappa}.} \tag{26}$$

This shows that Hawking's result remains true for Interacting Quantum Field theory in Curved Spacetime (ICFTICS).

Gibbons and Hawking extended this idea to Semi-Classical Quantum Gravity (SCQG) by considering a path integral over Riemannian metrics which are periodic in imaginary time

$$Z = e^{-\beta\Phi} = \int d[g] e^{-I[g]}, \tag{27}$$

where

$$I = -\frac{1}{16\pi}\int \sqrt{g} R d^4 x + \ldots \tag{28}$$

is the classical Euclidean action and Φ is the thermodynamic potential

$$\Phi = M - TS - \Omega J - \psi Q \tag{29}$$

At the lowest semi-classical level, the Riemannian Black Hole solution gives the saddle point and one obtains what may be called the Quantum Statistical Relation

$$\boxed{\beta\Phi = I^{\text{Euclidean}}} \tag{30}$$

For example

$$ds^2 = (1 - \frac{2M}{r})d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin\theta^2 d\phi^2)\,. \tag{31}$$

$$\beta = 8\pi M\,, \qquad I^{\text{Euclidean}} = \frac{M}{2T} \tag{32}$$

These calculations may be extended (De Wit, Dabholkar,Sen,...) to include higher curvature terms in the action, such as arise at higher order in α' in String theory, provided that one uses an expression for the black hole entropy in due to Wald. In general however, to control quantum fluctuations, one should presumably go to higher order in string loops, i.e. in powers of $g_s = e^{\phi}$.

One may avoid doing this for special supersymmetric, zero temperature, extreme or BPS black holes.

Another problem is the nature of the perfectly diathermic box box in which the black hole must be contained. One possibility is to use special the properties of Anti-De-Sitter spacetime.

Both ideas are essential components of our current theoretical understanding of black holes in M/String theory.

Although they may be stable classically, Hawking's results imply that black holes are unstable quantum-mechanically because they may emit gravitons, photons etc which carry of energy and angular momentum, leading to well known uncertainties about the nature of the final state. *

In the charged case the mechanism is the Schwinger process. This is energetically favourable if

$$\frac{eQ}{r_+} > m \qquad \text{super} - \text{radiant condition} \tag{33}$$

And rapid if

$$\frac{Q}{r_+^2} > \frac{m^2}{e} \qquad \text{tunneling condition} \tag{34}$$

*Wovon man nicht sprechen kann, darüber muß man schweigen

$$2E = r_+ + \frac{Q^2}{r_+} \tag{35}$$

In the charged case however it may be that no physical field carries the relevant charge * in which case the evolution is towards the black hole with least mass for fixed charge. This has $M = |Q| = |Z|$, zero temperature and will non-longer radiate.

$$T = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} \tag{36}$$

*or is to massive

Gibbons and Hajicek realised that *Extreme-Reissner Nordström Black holes are the solitons of Einstein's theory*.

$$ds^2 = (1 - \frac{|Z|}{r})^2 dt^2 + \frac{dr^2}{(1 - \frac{|Z|}{r})^2} + r^2(d\theta^2 + \sin^2\theta^2 d\phi^2) \qquad (37)$$

Earlier Hartle and Hawking pointed out that Majumdar and Papapetrou had discovered that Reissner-Norstrøm black holes satisfy a No Force Condition

$$\begin{aligned} ds^2 &= -H^{-2}dt^2 + H^2 d\mathbf{x}^2\,, && (38)\\ H &= 1 + \sum_a \frac{M_a}{|\mathbf{x} - \mathbf{x}_a|}\,, && (39)\\ F &= \cos\alpha\, d(\frac{dt}{H}) + \sin\alpha \star (\frac{dt}{H}) && (40)\\ M_a &= \sqrt{Q_a^2 + P_a^2} = |Z_a|\,, \qquad Z_a = \cos\alpha\, Q_a + i \sin\alpha\, P_a && (41) \end{aligned}$$

In isotropic coordinates, in the sub-extreme case, the metric is

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Z^2}{r^2})dt^2 + (1 + \frac{M}{\rho} + \frac{M^2 - Z^2}{4\rho^2})^2 (d\rho^2 + \rho^2 (d\theta^2 + \sin^2 d\phi^2)) \tag{42}$$

$$r = \rho + M + \frac{M^2 - Z^2}{4\rho}\,, \tag{43}$$

and is invariant under the isometric involution

$$\rho \to \frac{M^2 - Z^2}{4\rho}\,. \tag{44}$$

We get two regions joined by an Einstein-Rosen bridge.

In the extreme case, $M = |Z|$, things are different.

There is just one sheet, with an infinitely long throat which asymptotes the homogenous Bertotti-Robinson product metric on $S^2 \times AdS_2$.

In fact Couch and Torrence found an involution

$$r \rightarrow \frac{M^2}{r} \tag{45}$$

which interchanges the horizon and infinity *conformally* *

$$ds^2 \rightarrow \frac{M^4}{r^4} ds^2 \,. \tag{46}$$

*Exactly the same structure arises in the case of the D3-brane

Einstein-Maxwell theory has a supersymmetric extension, $\mathcal{N} = 2$ Supergravity theory. Gibbons realised, that while finite temperature black holes are not supersymmetric, the Majumdar-Papapetrou solutions admit a Killing spinor

$$\boxed{\nabla_\alpha \epsilon^i + \frac{1}{4}\epsilon^{ij} F_{\mu\nu}\gamma^\mu\gamma^\nu\gamma_\alpha\epsilon^j = 0\,.} \tag{47}$$

Thus the charge Z was a central charge and that the Cosmic Censorship inequality is a Bogomlonyi Bound

$$M \geq |Z|, \tag{48}$$

with equality if and only if the solution admits a Killing spinor Gibbons and Hull established this in detail using a Witten-Nester argument. and Tod completed the task of finding all supersymmetric solutions of this theory.

The basic idea, which generalises some in sights of Olive and Witten about solitons in flat space time

$$\{Q^i, \bar{Q}^j\} = \gamma\mu P^\mu \delta^{ij} + Z^{ij} C + \dots \tag{49}$$

extends to all supergravity theories in all dimensions and to p-branes, extended p-dimensional objects.

Rather generally, it is believed that the associated BPS states suffer no quantum corrections and hence semi-classical calculations of such things as entropy should be reliable.

An important fact is that Newton's constant G does not enter in the relation between entropy S and charge Q.

$$S = \pi Q^2\,. \tag{50}$$

By suitably changing the boundary conditions used in open string theory, Polchinksi was able to make contact with 2-dimensional conformal field theory and string theory. The equivalent of the soliton states are Dirichlet branes. If gravity is negligible, they have a low energy description using a Dirac-Born-Infeld Lagrangian.

The non-perturbative Ramond-Ramond charges carried by the D-branes are not carried by elementary strings states. They resemble Wheeler's Charge without Charge and are absolutely conserved and central.

Gibbons and Callan and Maldacena showed that if a string ends on a 3-brane it looks, from the standpoint of those living on the brane, like a BIon, an idea which had been partially anticipated in the nineteenth century but which may not have appealed to Einstein.

The development of D-brane theory allowed Strominger and Vafa to give a microscopic description of certain BPS black holes in terms of the intersection of various D-branes. Microstates could be counted and Boltzmann's formula verified.

$$S = \ln N \,. \tag{51}$$

This is meaningful because

- Quantum corrections are under control because we are dealing with BPS states and so we may extrapolate from microstates to macroscopic black branes described by classical supergravity theory.

- The (Ramond-Ramond) charges Q, P are quantised because we demand consistent string propagation in these backgrounds.The charges of solitons in ungauged supergravity theories need not carry quantised charges.

Einstein would probably not have approved of this.

The situation improves if one considers the Kaluza-Klein monopoles of Gross Perry and Sorkin.

$$ds^2 = dt^2 + V^{-1}(dx^5 + \omega_i dx^i)^2 + V d\mathbf{x}^2 \,, \tag{52}$$

$$\text{grad}\, V = \text{curl}\, \omega \tag{53}$$

$$V = 1 + \sum_a \frac{M_a}{|\mathbf{x} - \mathbf{x}_a|} \tag{54}$$

Periodicity of x^5 imposes a quantisation condition on the Klauza-Klein charges and on the magnetic monopole moment.

In addition, the singularies of the four-dimensional metric receive a Higher Dimensional Resolution: they are mere coordinate artefacts in five dimensions. In this way, they evade the Pauli-Einstein theorem.

Gibbons, Horowitz and Townsend have shown that higher dimensional resolutions are quite common. However the problem of singularities and the ultimate outcome of gravitational collapse and Hawking evaporation cannot be solved in this way.

- One should consider the interior as well as the exterior of eternal BPS black holes. Israel's ideas of Mass Inflation may be relevant here.

- One should also consider neutral black holes, made out of just gravitons say. These cannot be BPS. Our best current hope of understanding them at present seems to be via Maldacena's AdS/CFT correspondence.

As pointed out by Gibbons and Townsend, Anti-Sitter spacetime frequently arises as the near horizon geometry of classical p-brane metrics. According to D-brane theory N coincident D-branes have a non-abelian version of the $SU(N)$ Born-Infeld action on their world volume. Maldacena argued in the $D3$-brane case that

$$\boxed{\text{String theory in the bulk of } AdS_5 \equiv \mathcal{N} = 4\, SU(N) \text{ Yang} - \text{Mills on the confor}} \tag{55}$$

In the $N \to \infty$ limit we relate back holes in supergravity to the large coupling limit of quantum mechanical Yang-Mills gauge theory. Since this is believed to be well defined, one suspects that there should be no singularities in the bulk, at least at the quantum level.

Much remains to be done to check this idea.

A remarkable application is due to Witten in which he relates the Hawking Page Black Hole phase Transition in the bulk to quark confinement on the boundary.

Thus vindicating the use of Euclidean techniques and the sudy of Higher Dimensional in Quantum Gravity.

$$ds^2 = (1 - \frac{2M}{r^2} + \frac{r^2}{l^2})d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r^2} + \frac{r^2}{l^2}} + r^2 d\Omega_3^2 \,. \tag{56}$$

The temperature

$$2\pi T = 2(\frac{1}{r_+} + \frac{2r_+}{l^2}) \,. \tag{57}$$

has a minimum at $r_+ = \sqrt{2}l$

For low T there are now solutions.

For high T there are two solutions.

The solution with bigger radius and smaller temperature has lower Euclidean action $I^{\text{Euclidean}}$ than AdS_5 provided

$$r_+ > l\,. \tag{58}$$

Since

$$\frac{lM}{r_+^3} = \frac{1}{2}(\frac{l}{r_+} + \frac{r_+}{l})\,, \tag{59}$$

the AdS_n Bekenstein Bound

$$2\pi l E \geq (n-2)S \tag{60}$$

is saturated at the Hawking-Page transition.

Conclusion

- The problem of the final state gravitational collapse is still far from being fully understood.

- Recent mathematical advances have justified much of what was merely conjecture on the classical side of things. We can expect more progess on this front in the near future.

- On quantum side, we now understand how certain supersymmetric black holes may be understood as non-perturbative solitons in String/M-theory and we can count micro-states.

- The AdS/CFT corresponence encourages the view that neutral, non-supersymmetric black holes evolve in a non-singular unitary fashion at the quantum level but much needs to be done to flesh out this idea, and indeed even to construct a consistent quantum theory of gravity.