

Can we extract physics from QCD with the fixed topological charge ?

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For JLQCD Collaboration

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Motivation

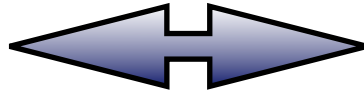
Recently JLQCD collaboration starts dynamical QCD simulations with Overlap Fermion using BG/L at KEK (57Tflops).

$$D = \frac{1}{a} [1 - \gamma_5 \text{sign}(H_W)]$$

- exact chiral symmetry
- but very CPU demanding
- Numerical cost depend on
 - ✓ Spectrum of Wilson-Dirac op. H_W
 - ✓ Approximation of $\text{sgn}(H_W)$
- **JLQCD**: take a “gauge action” which suppress near-zero modes of H_W
 - ✓ reduce numerical costs drastically for large volume
 - ✓ locality of overlap operator is guaranteed

However **topological charge never changes !**

Zero probability to have an exact zero mode



No chance to change a topological sector

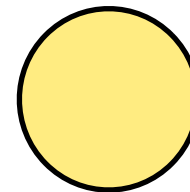
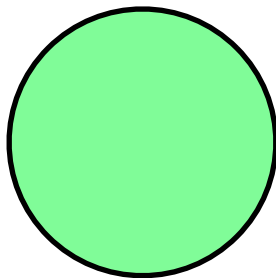
Can we trust results from such simulations ?

Optimist's opinion

It is just a finite-size effect.

Who cares the topological charge behind the moon ?

Q_{KEK}



$Q_{\text{moon}} = Q - Q_{\text{KEK}}$

This statement is inaccurate or even misleading !

Since I was very skeptical about a validity of simulations with fixed topological charges, I have worked on this problem with my collaborators.

In this talk, I discuss how we can calculate physical observables in QCD from simulations with fixed topological charges.

Brower et. al, PLB560(2003)64

- Introduction
- Derivation of general formula
- Topological susceptibility
- Preliminary numerical result from JLQCD
- Conclusion

In collaboration with

H. Fukaya(Riken), S. Hashimoto(KEK), T. Oonogi(Kyoto)

Work in progress

Derivation of general formula

Partition function

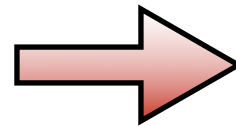
$$Z(\theta) \equiv \langle \theta | \theta \rangle = e^{-V F(\theta)}$$

$$F(\theta + 2\pi) = F(\theta) = F(-\theta)$$

Normalization $F(0) = 0$

$$F(\theta) = \frac{\chi_t}{2} \theta^2 + \dots$$

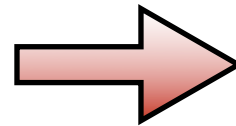
$$\chi_t = \frac{1}{V} \langle 0 | Q^2 | 0 \rangle > 0$$



local minimum

$$\theta = 0$$

$$F(\forall \theta \neq 0) > F(0) = 0$$



global minimum

$$\theta = 0$$

Vafa-Witten's theorem

$F(\theta)$ may have a singularity at π

Witten(ChPT)

$$F(\theta) = \chi_t [1 - \cos(\theta/N_f)]$$

$$-\pi < \theta < \pi$$

Partition function at fixed Q

Fourier series

$$Z_Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta Z(\theta) e^{i\theta Q} \quad \left(Z(\theta) = \sum_{Q \in \mathbb{Z}} Z_Q e^{-i\theta Q} \right)$$

$$Z(\theta) e^{i\theta Q} = e^{-VF_Q(\theta)}, \quad F_Q(\theta) = F(\theta) - i\theta \frac{Q}{V}$$

Saddle point expansion

saddle point

$$\theta_Q = i \frac{Q}{\chi_t V} (1 + O(\delta^2))$$

$$V \rightarrow \infty$$

$$\delta = \frac{Q^2}{\chi_t V}$$

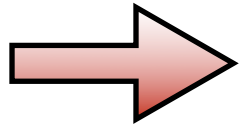
$$Z_Q = \frac{e^{-VF_Q}}{\sqrt{2\pi VF_Q^{(2)}}} \left[1 - \frac{VF_Q^{(4)}}{8(VF_Q^{(2)})^2} + O\left(\frac{1}{V^2}\right) \right]$$

$$VF_Q = \frac{Q^2}{2\chi_t V} (1 + O(\delta^2))$$

$F_Q^{(n)}$: n-th derivative at the saddle point

$$F_Q^{(2)} = \chi_t (1 + O(\delta^2))$$

$$F_Q^{(4)} = O(1) (1 + O(\delta^2))$$



$$Z_Q = \frac{1}{\sqrt{2\pi V \chi_t}} \exp \left[-\frac{Q^2}{2\chi_t V} \right]$$

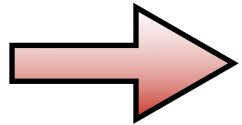
Gaussian distribution for

$$Q^2 \ll \chi_t V$$

arbitrary correlation function

$$G(\theta) = \langle \theta | O_1 O_2 \cdots O_n | \theta \rangle$$

$$G_Q = \frac{1}{2\pi Z_Q} \int d\theta Z(\theta) G(\theta) e^{i\theta Q}$$



Master formula

$$Q^2 \ll \chi_t V$$

$$G_Q = G(\theta_Q) + G^{(2)}(\theta_Q) \frac{1}{2\chi_t V} + O\left(\frac{1}{V^2}\right)$$

CP-even

$$G_Q^{\text{even}} = G(0) + G^{(2)}(0) \frac{1}{2\chi_t V} + O\left(\frac{1}{V^2}\right)$$

$$V \rightarrow \infty \quad G_Q^{\text{even}} = G(0) + O\left(\frac{1}{V}\right)$$

CP-odd

$$G_Q^{\text{odd}} = G^{(1)}(0) \frac{iQ}{\chi_t V} + O\left(\frac{1}{V^2}\right)$$

NEDM may be extracted from simulations with fixed Q .

We can take small Q at will to satisfy

$$Q^2 \ll \chi_t V$$

Topological susceptibility

$q(x)$: topological charge density i.e. $Q = \int d^4x q(x)$

$|x| \rightarrow \infty$

$$\langle q(x)q(0) \rangle_Q = \langle q(x)q(0) \rangle + \langle q(x)q(0) \rangle^{(2)} \frac{1}{2\chi_t V}$$

clustering property $\rightarrow 0$

$$\langle q(x)q(0) \rangle^{(2)} \rightarrow -2\chi_t^2$$

$$\langle q(x)q(0) \rangle_Q \rightarrow -\frac{1}{V} \chi_t + O\left(\frac{1}{V^2}\right)$$

clustering property is violated.

Weinberg's textbook

Topological charge
behind the moon matters

Fermionic correlation function

anomalous WTI

$$\langle \partial_\mu A_\mu(x) O - mP(x) O + q(x) O + \delta_x O \rangle = 0$$

$$O = mP(0) \quad \langle mP(x) mP(0) \rangle = \langle \partial_\mu A_\mu(x) mP(0) \rangle + \langle q(x) mP(0) \rangle$$

$x \neq 0$

$$O = q(x) \quad \langle mP(x) q(0) \rangle = \langle \partial_\mu A_\mu(x) q(0) \rangle + \langle q(x) q(0) \rangle$$

$$\langle mP(x) mP(0) \rangle_Q = \langle q(x) q(0) \rangle_Q + \langle \partial_\mu A_\mu(x) mP(0) \rangle_Q + \langle q(x) \partial_\mu A_\mu(0) \rangle_Q$$

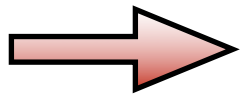
$$|x| \rightarrow \infty$$

$$\langle \partial_\mu A_\mu(x) mP(0) \rangle_Q \rightarrow \langle \partial_\mu A_\mu(x) mP(0) \rangle + \frac{1}{2\chi_t V} \langle \partial_\mu A_\mu(x) mP(0) \rangle^{(2)}$$

$\rightarrow 0$

$$\langle \partial_\mu A_\mu(x) mP(0) \rangle^{(2)} \rightarrow -2 \langle \partial_\mu A_\mu(x) Q \rangle \langle mP(0) Q \rangle$$

$$\langle \partial_\mu A_\mu(x) Q \rangle = \frac{1}{V} \int d^4x \langle \partial_\mu A_\mu(x) Q \rangle = 0$$



$$\langle \partial_\mu A_\mu(x) mP(0) \rangle_Q \rightarrow 0$$

Similarly

$$\langle \partial_\mu A_\mu(x) q(0) \rangle_Q \rightarrow 0$$

Therefore

$$\langle mP(x) mP(0) \rangle_Q \rightarrow \langle q(x) q(0) \rangle_Q \rightarrow -\frac{1}{V} \chi_t$$

$$\langle m^a P(x) mP^a(0) \rangle_Q \rightarrow 0$$

From non-singlet WTI

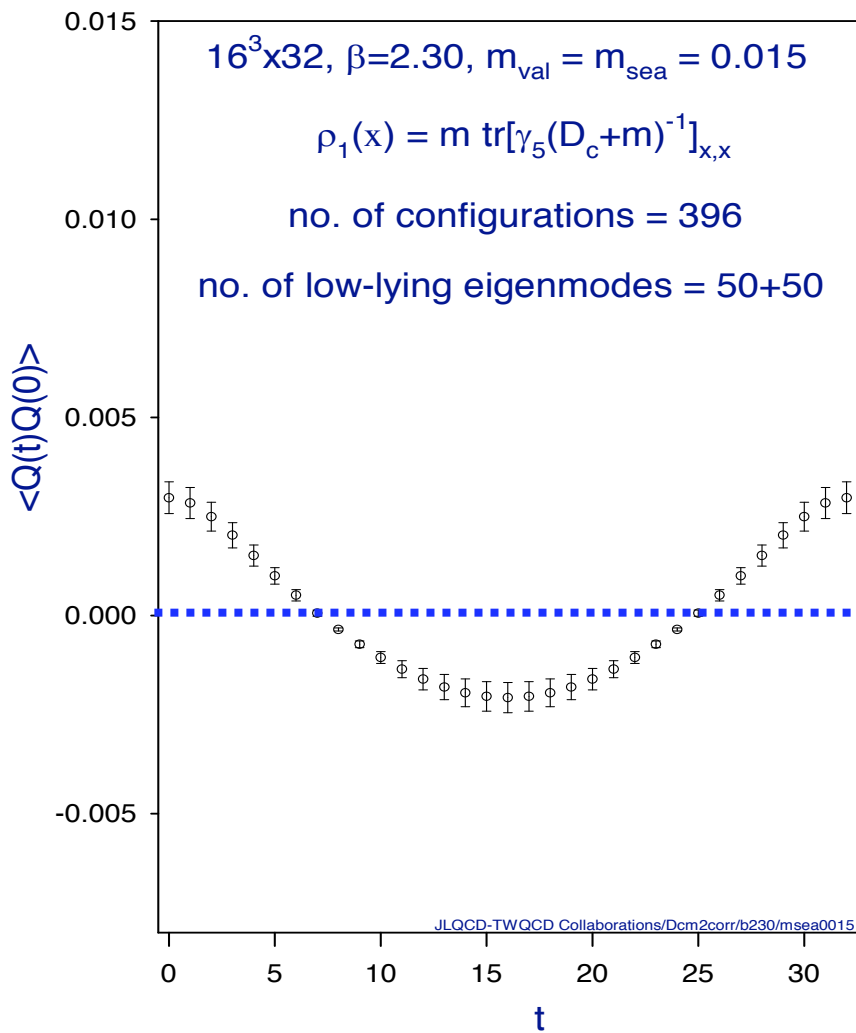
Final formula

$$|x| \rightarrow \infty$$

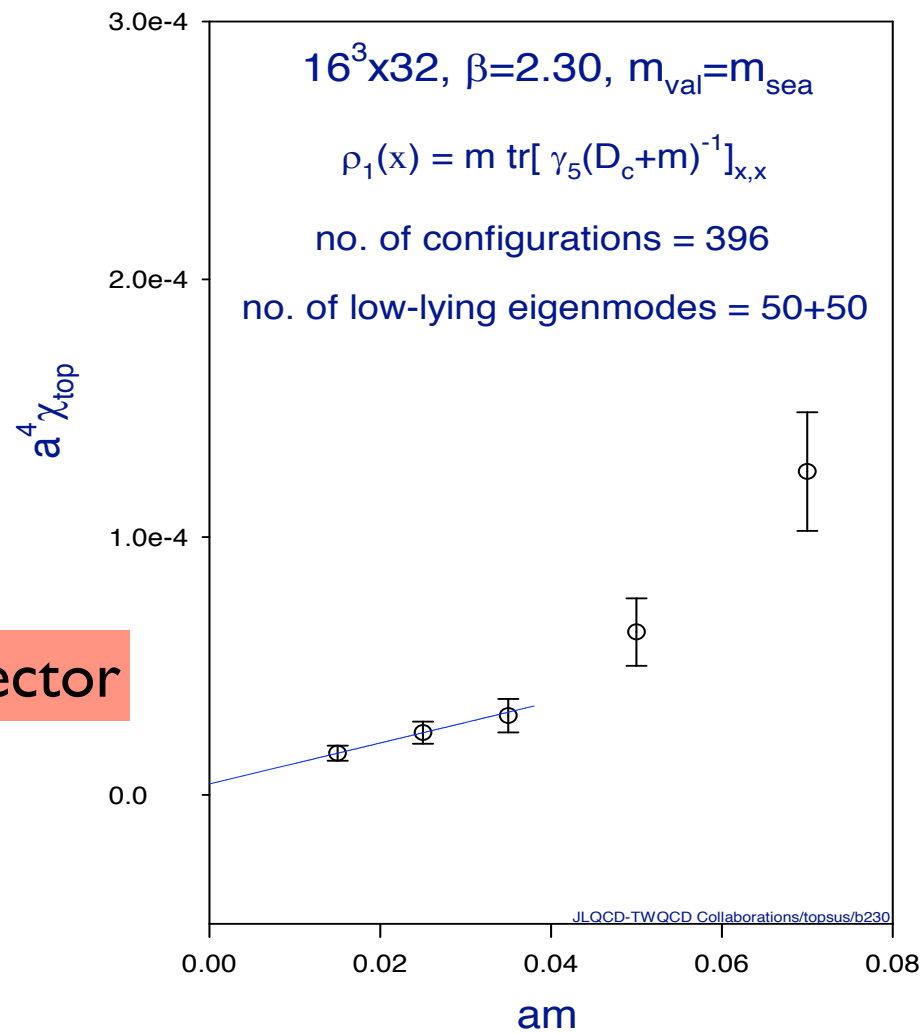
$$\langle mP(x) mP(0) \rangle_Q^{\text{disconnected}} = -\frac{1}{V} \chi_t + O\left(\frac{1}{V^2}\right)$$

Model independent / No short distance singularity

Preliminary numerical results from JLQCD

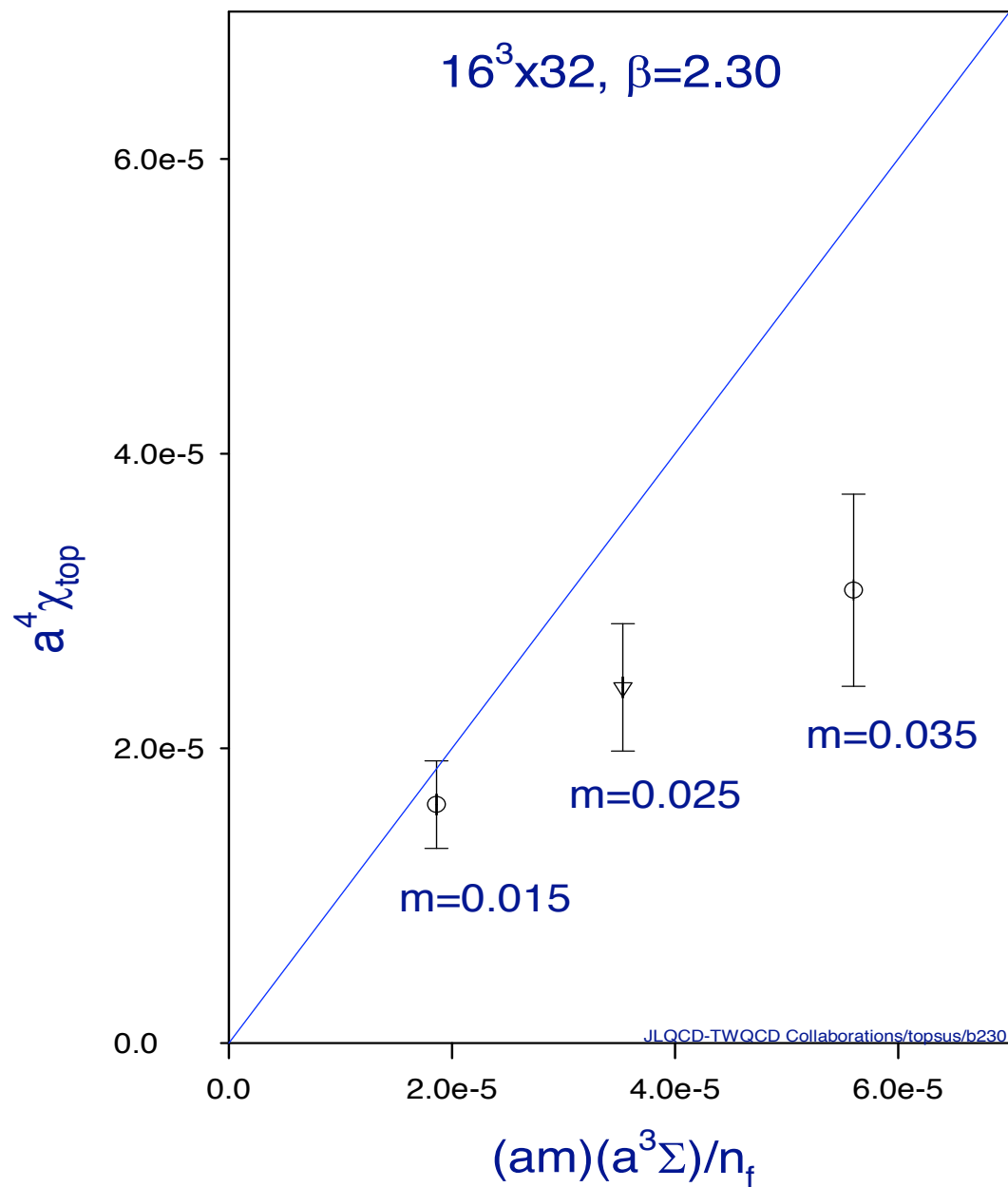


Q=0 sector



$$\langle mP(x)mP(0) \rangle_Q^{\text{disconnected}} = -\frac{1}{V} \chi_t + O\left(\frac{1}{V^2}\right)$$

Leutwyler-Smilga relation

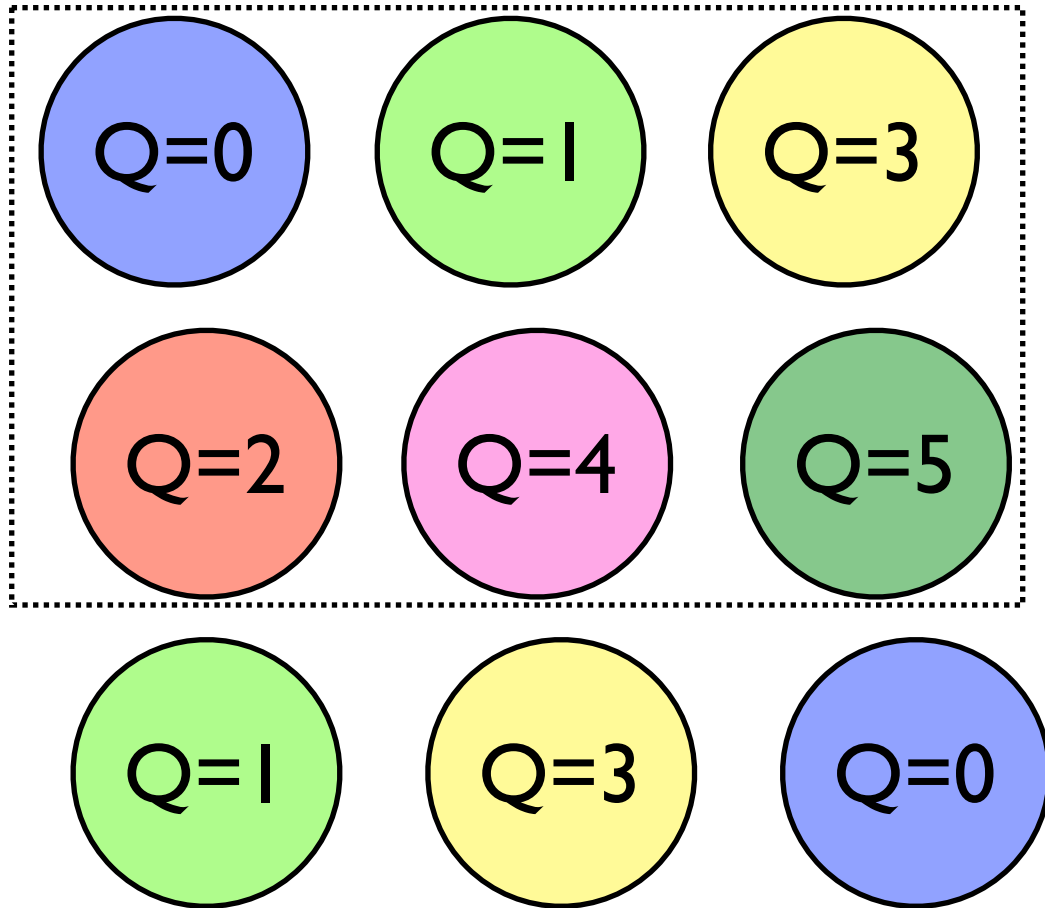


Conclusions

- We systematically derive general relations between normal QCD and QCD with fixed topology in the model-independent way.
- QCD with fixed topology is unphysical. However, if the volume is large enough, we can extract useful informations.
- We may extract the susceptibility from QCD with fixed topology. No short-distance divergence for the susceptibility.
- Preliminary numerical results of susceptibility at $Q=0$.
- Outlook
 - ✓ susceptibility from $Q=2$ and $Q=4$ sectors to check consistency
 - ✓ check of higher order corrections
 - ✓ other quantities, informations from ChPT ?

Possible problems raised by T. Izubuchi

Is a topological sector simply connected ?

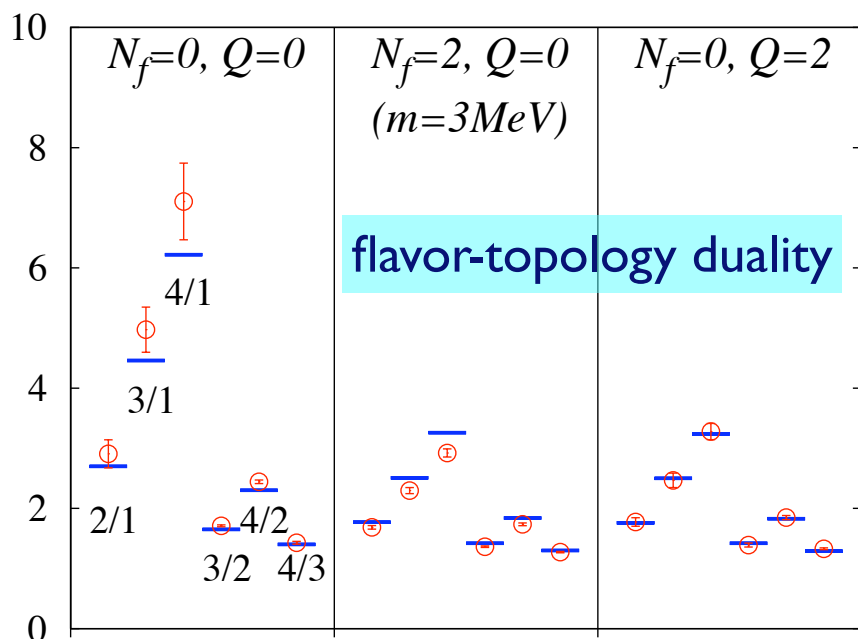


Valid example

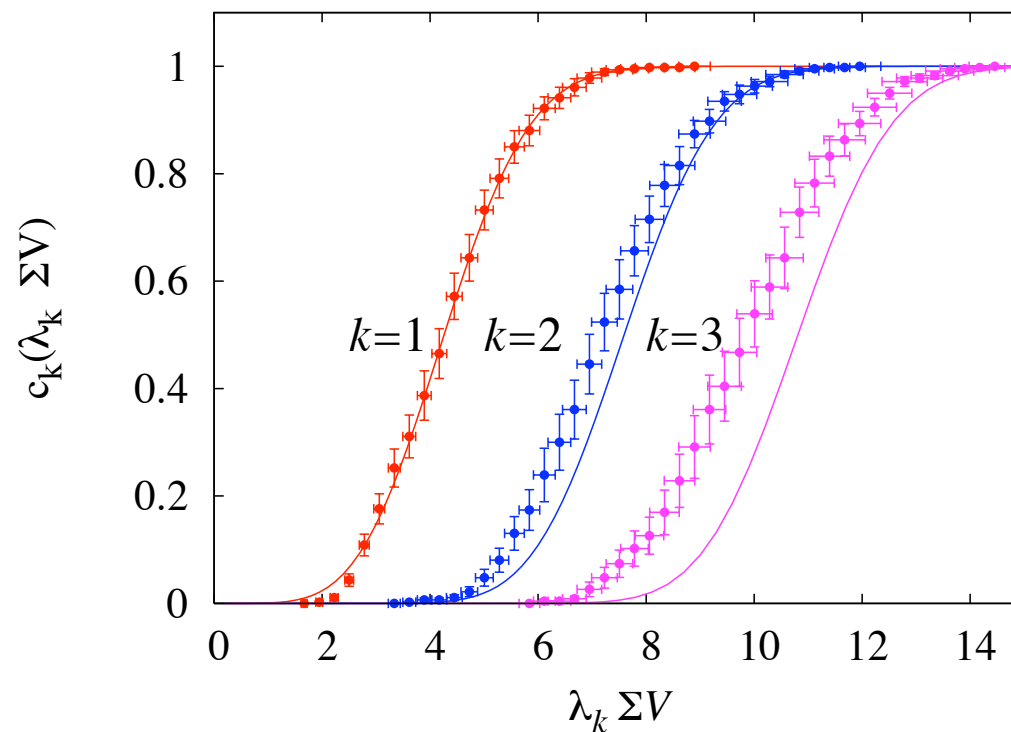
$$\varepsilon\text{-regime: } m_\pi L \leq 1 \quad (\Lambda_{\text{QCD}} L \gg 1)$$

Comparison with Random Matrix Theory with fixed topological charge

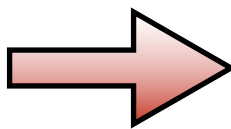
eigenvalue ratios



cumulative density



$$\Sigma = \frac{4.30}{\langle \lambda_1 \rangle V}$$



$$\Sigma^{\overline{MS}}(2\text{ GeV}) = [251(7)(11)\text{ MeV}]^3$$