Chiral Potts Model

Outline:

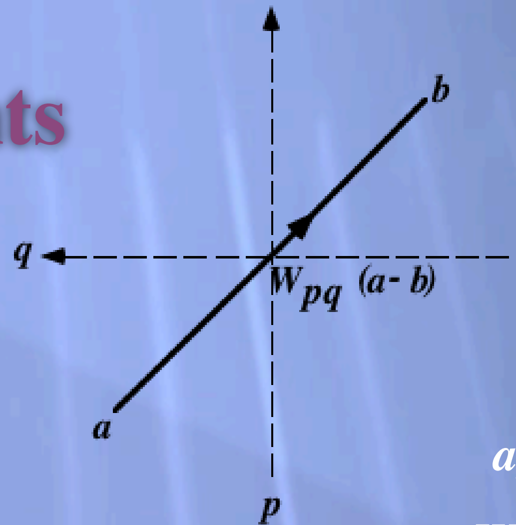
Introduction:

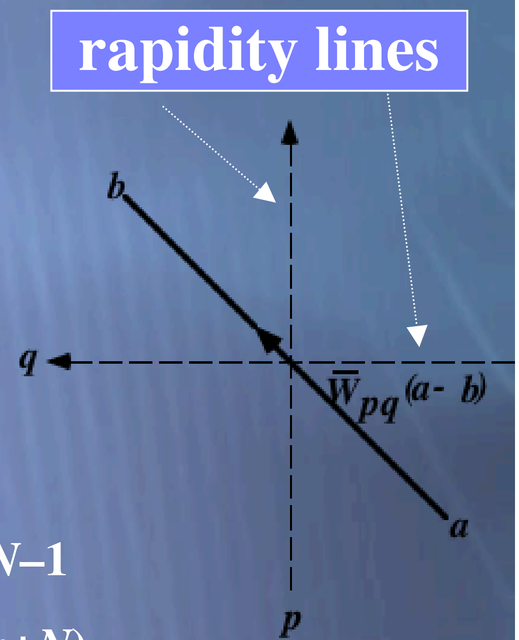
* The model
* Special properties

Recent progress:

* Correlation function
* degenerate eigenspace

Introduction





✦ Boltzmann weights

$W(a\text{-}b) \neq W(b\text{-}a)$

$\omega^N = 1$

$a,b = 0,\ldots N-1$

$W(n) = W(n+N)$

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - x_p\omega^j}{y_p - x_q\omega^j}, \qquad \overline{W}_{pq}(n) = (\mu_p\mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p - x_q\omega^j}{y_q - y_p\omega^j}$$

$$\mu_p^N = k'/(1-kx_p^N) = (1-ky_p^N)/k'$$

To each rapidity line p, one assigns two variables x_p and y_p

High genus curve

$$x_p^N + y_p^N = k(1 + x_p^N y_p^N), \; k'^2 = 1 - k^2$$

$x_p \to \omega^j y_p$

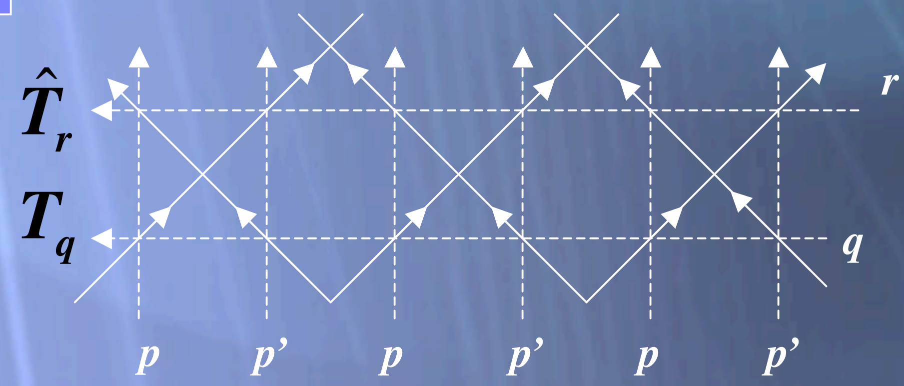
Functional Relations

✦ Commuting Transfer Matrices

***L* is the number of sites**

$$T(x_q,y_q)_{\sigma}^{\sigma'} = \prod_{J=1}^{L} W_{pq}(\sigma_J-\sigma'_J)\overline{W}_{p'q}(\sigma_{J+1}-\sigma'_J),$$

$$\hat{T}(x_r,y_r)_{\sigma}^{\sigma'} = \prod_{J=1}^{L} \overline{W}_{pr}(\sigma_J-\sigma'_J)W_{p'r}(\sigma_J-\sigma'_{J+1})$$



When the two rapidities are related by the relation $x_r=y_q$, $y_r=\omega^j x_q$, the product of two transfer matrices, becomes block triangular.

$$\begin{bmatrix} \tau_j(t_q) & * \\ 0 & \tau_{N-j}(\omega^j t_q) \end{bmatrix}$$

where $t_q=x_q y_q$

Shift operator

$$\Gamma_q T(x_q,y_q)\hat{T}(y_q,\omega^j x_q) = \alpha_q \tau_j(t_q) + \beta_q X^j \tau_{N-j}(\omega^j t_q)$$

Constants

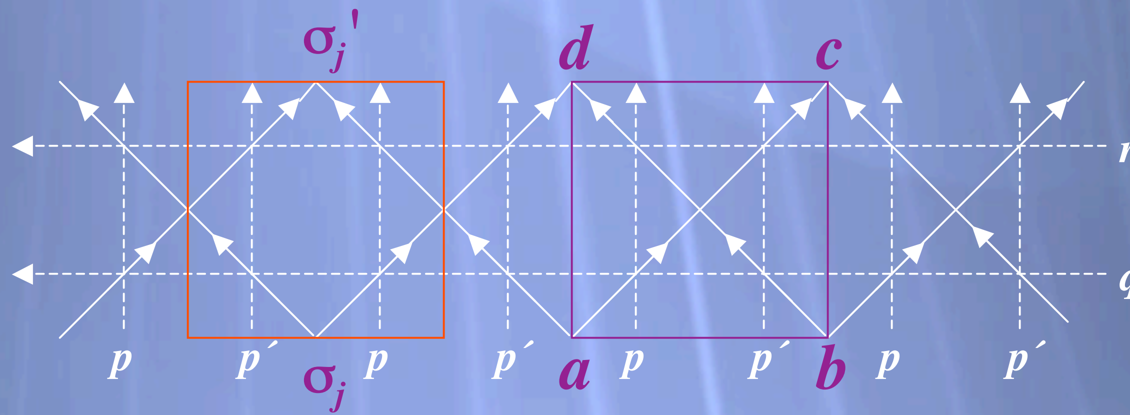
continue

$$\tau_j(t_q)\tau_2(\omega^j t_q)X^j = \boxed{\eta_q}\,\tau_{j-1}(t_q) + \tau_{j+1}(t_q)$$

$$[\tau_k(t_q), \tau_j(t_s)] = 0$$

Particularly, for j**=2:** $x_r = y_q, y_r = \omega^2 x_q$; $\sigma_j - \sigma_j' = 0,1$;.

$\tau_2(t_q)$

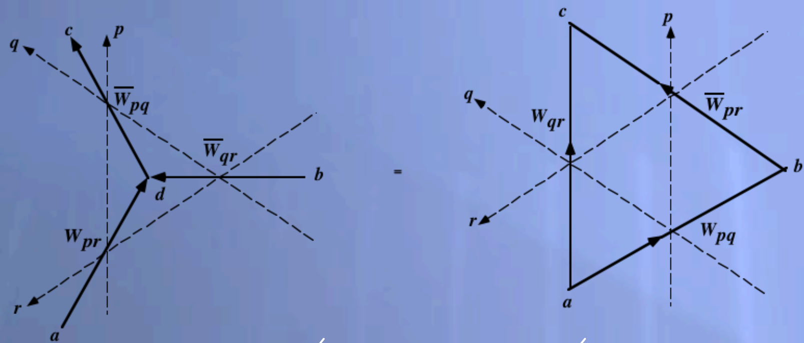


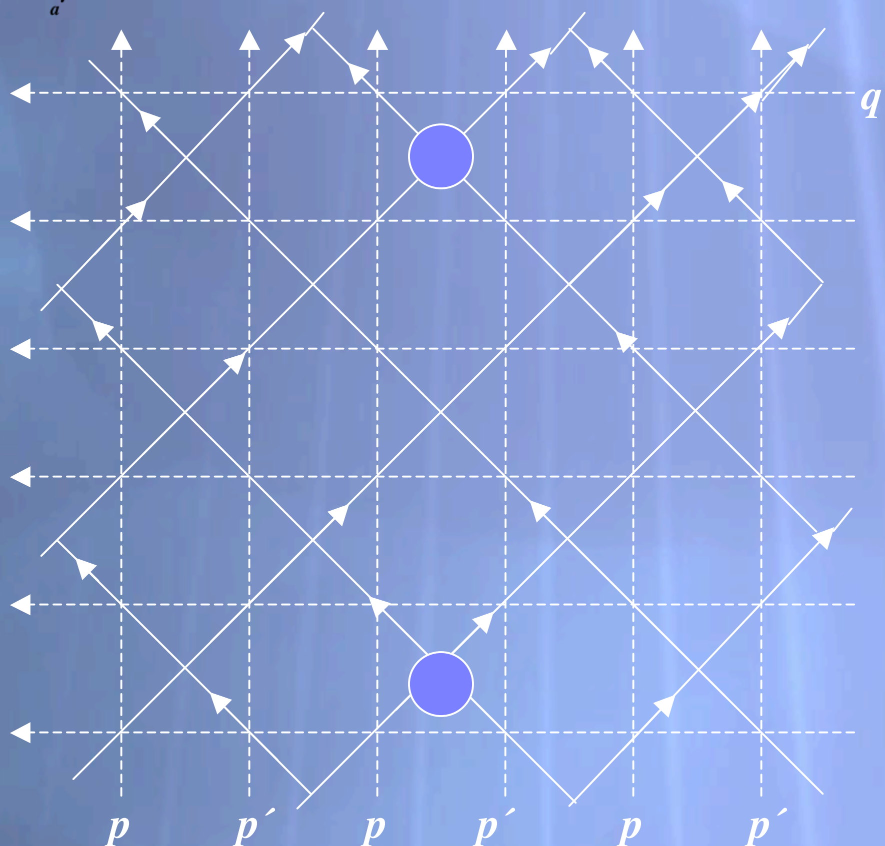
$a-d=0,1$
$b-c=0,1$
$a-b=0\ldots N-1$
$d-c=0\ \ldots N-1$

Yang Baxter Equation: $RTT'=T'TR$; **or** $RUU'=U'UR$ **where** R **is the R-matrix of the six-vertex model, and** T **or** U **is what is inside the red box, or the purple box.**

Difference: $q \rightarrow \omega$, **with** $|q|<1$ **and** $|\omega|=1$

Correlation Functions





- Star-Triangle Equation
- Z-invariant models
- Order parameter depends only on the temperature variable *k*.
- Correlation function depends only on *k* and the rapidity lines sandwiched between the two spins
- Super-integrable model: $x_{p'}=y_p$ and $y_{p'}=x_p$,

Superintegrable Chiral Potts Model

- Superintegrable: $x_{p'}=y_p$ and $y_{p'}=x_{p'}$

$$\tau_2(t)v_Q = \boxed{[(1-t)^L + \omega^Q(1-\omega t)^L]}\, v_Q; \quad Q \in Z$$

$$\mathrm{T}(x_q,y_q)\mathrm{T}(y_q,\omega^j x_q)v_Q = P(t_q^N)v_Q, \quad j=0\cdots N-1$$

where $t=t_q=x_qy_q$

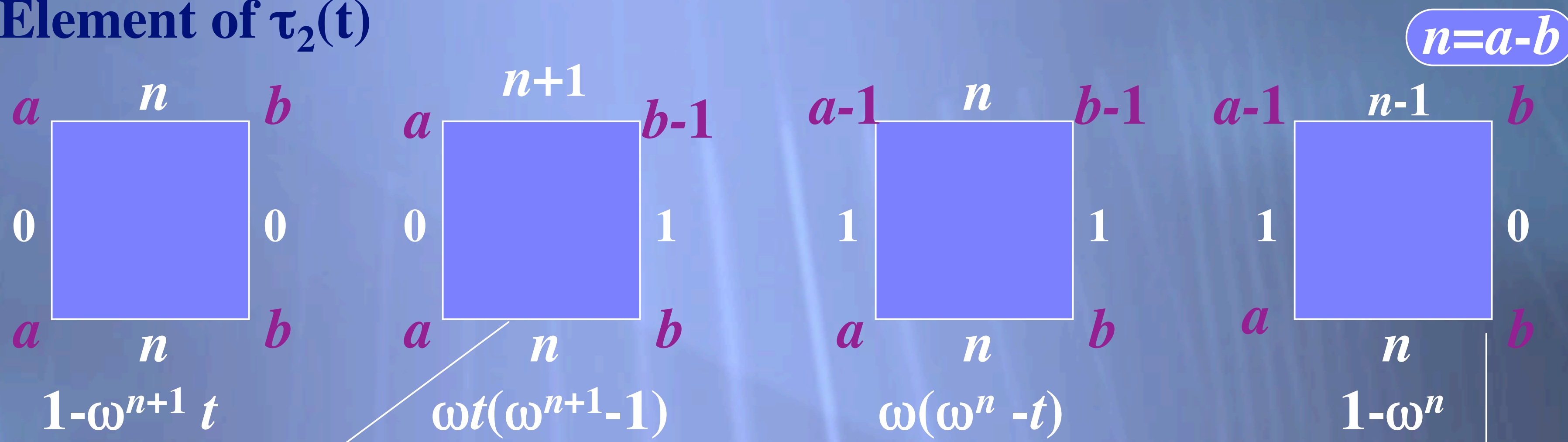
$$P(t^N) = \sum_{n=0}^{N-1} \omega^{nQ} \frac{(1-t^N)^L}{(1-\omega^n t)^L}$$

Drinfeld Polynomial

How many independent eigenvectors have this as eigenvalue? For N=3, and *L* small, direct calculations show the eigenspace is degenerate only if *L* is a multiple of *N*, and the degeneracy is 2^r with $r=(N-1)L/N$ instead of 2^L.

Nilpotent Raising and Lowering Operators

Element of $\tau_2(t)$



$|\sigma_1\ldots\sigma_L\rangle \rightarrow |n_1\ldots n_L\rangle$: $n_j=\sigma_j-\sigma_{j+1}$; $\sigma_{L+1}=\sigma_1 \rightarrow n_1+\ldots+n_L=0 \pmod{N}$

Lowering Operator $e=X^{-1}(1-Z)$, $e|n\rangle=|n-1\rangle$; $e|0\rangle=0$; $e^N=0$

$$Z_{ab}=\omega^{a-1}\delta_{a,b},\quad X_{ab}=\delta_{a,b+1},\quad ZX=\omega XZ$$

Raising Operator $f=(1-Z)X$, $f|n\rangle=|n+1\rangle$; $f|N-1\rangle=0$; $f^N=0$

$$u_j(t)=\begin{bmatrix} 1-\omega t Z_j & -\omega t f_j \\ e_j & \omega(Z_j-t) \end{bmatrix}$$

$$Z_j=1\otimes\ldots\otimes Z\otimes 1\ldots\otimes 1$$
$$X_j=1\otimes\ldots\otimes X\otimes 1\ldots\otimes 1$$

Relevant Operators

$$U(t) = u_1(t)u_2(t)\ \ldots\ u_L(t)$$

$$R(t_q/t_r)U(t_q)U(t_r) = U(t_r)U(t_q)\ R(t_q/t_r)$$

$$U(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$$

$$\tau_2(t) = \mathrm{Tr}\ U(t) = A(t) + D(t)$$

$$A(t) = \sum_{n=0}^{L} A_n t^n, \qquad B(t) = \sum_{n=0}^{L} B_n t^n,$$

$$C(t) = \sum_{n=0}^{L} C_n t^n, \qquad D(t) = \sum_{n=0}^{L} D_n t^n,$$

$$A_0 = D_L = 1,\ B_0 = C_L = 0 \qquad B_L = \sum_{J=1}^{L}\prod_{m=1}^{j-1} Z_m f_j, \qquad C_0 = \sum_{J=1}^{L} \omega^{j-1}\prod_{m=1}^{j-1} Z_m e_j,$$

$$A_L = D_0 \omega^{-L} = \prod_{m=1}^{L} Z_m \qquad B_1 = \sum_{J=1}^{L} \omega^{L-j} f_j \prod_{m=j+1}^{L} Z_m, \quad C_{L-1} = \sum_{J=1}^{L} e_j \prod_{m=j+1}^{L} Z_m,$$

$$B_j^{(N)} = B_j^{N}/[N]!,\ j=1,..,L; \quad C_n^{(N)} = C_n^{N}/[N]!,\ n=0,..,L-1;$$

$$[n] = 1 + \ldots + \omega^{n-1}, \qquad [n]! = [n]\ldots[1]$$

Highest Weight Vector

$$\left[\tau_2(t),B_L^{(N)}\right]=(\omega-1)B(t)B_L^{(N-1)}(A_L-1),$$
$$\left[\tau_2(t),B_1^{(N)}\right]=(1-\omega^{-1})t^{-1}B(t)B_1^{(N-1)}(1-D_0),$$
$$\left[\tau_2(t),C_0^{(N)}\right]=(\omega-1)C(t)C_0^{(N-1)}(D_0-1),$$
$$\left[\tau_2(t),C_{L-1}^{(N)}\right]=(\omega-1)\omega t C(t)C_{L-1}^{(N-1)}(A_L-1),$$

$$A_L=\prod_{m=1}^{L}Z_m\rightarrow\prod_{m=1}^{L}\omega^{n_m}=1,$$
$$D_0=\omega^L\prod_{m=1}^{L}Z_m\rightarrow\omega^L\rightarrow 1,$$

if $L=pN$

Highest and Lowest Weight Vectors : $x_i^+|\Omega\rangle=0$; $x_i^-|\bar{\Omega}\rangle=0$

$x_0^+=C_0^{(N)}/(1-\omega)^N$, $x_{-1}^+=C_{L-1}^{(N)}/(1-\omega)^N$, $|\Omega\rangle=|0,\ldots,0\rangle$,
$x_0^-=B_L^{(N)}/(1-\omega)^N$, $x_1^-=B_1^{(N)}/(1-\omega)^N$, $|\bar{\Omega}\rangle=|-1,\ldots,-1\rangle$,

$$x_0^-|\Omega\rangle=x_0^-|0,\cdots,0\rangle=\sum_{n_1+\cdots n_L=N}|n_1,\cdots,n_L\rangle,$$

$n_m=0,\ldots,N-1$, $m=1\ldots L$

$$(x_0^-)^{(r)}|\Omega\rangle=\sum_{n_1+\cdots n_L=rN}|n_1,\cdots,n_L\rangle=|N-1,\cdots,N-1\rangle$$

$rN=(N-1)L$

Quantum group $L(sl_2)$

Generators

$x_0^+, x_0^-, x_{-1}^+, x_1^-, h_0 = [x_0^+, x_0^-] = [x_{-1}^+, x_1^-], h_1 = [x_0^+, x_1^-], h_{-1} = [x_{-1}^+, x_0^-]$

Properties $x_n^{\pm} = \pm \frac{1}{2}[h_0, x_n^{\pm}], \quad [[[x_0^{\pm}, x_{\pm 1}^{\mp}], x_{\pm 1}^{\mp}], x_{\pm 1}^{\mp}] = 0$

Serre Relation

Proven Identities

$h_0 |\Omega\rangle = [x_0^+, x_0^-] |\Omega\rangle = [x_{-1}^+, x_1^-] |\Omega\rangle = -r|\Omega\rangle, \quad r=(N-1)L/N$

$x_i^- |\Omega\rangle = 1/2[h_0, x_i^-] |\Omega\rangle, \quad x_{-i}^+ |\bar{\Omega}\rangle = -1/2[h_0, x_{-i}^+] |\bar{\Omega}\rangle, \quad i=0,1$

$[[[x_0^+, x_1^-], x_1^-], x_1^-] (x_1^-)^n |\Omega\rangle = 0, \quad [[[x_{-1}^+, x_0^-], x_0^-], x_0^-] (x_0^-)^n |\Omega\rangle = 0,$

$[[[x_0^-, x_{-1}^+], x_{-1}^+], x_{-1}^+] (x_{-1}^+)^n |\bar{\Omega}\rangle = 0, [[[x_1^-, x_0^+], x_0^+], x_0^+] (x_0^+)^n |\bar{\Omega}\rangle = 0.$

$(x_0^+)^{(n)}(x_1^-)^{(n)} |\Omega\rangle = (x_{-1}^+)^{(n)}(x_0^-)^{(n)} |\Omega\rangle = \lambda_n |\Omega\rangle$

Drinfeld Polynomial

$$\lambda_n = \sum_{m=0}^{n} (-1)^m \binom{L}{m} \frac{(L)_{nN-mN}}{(nN-mN)!}, \quad P(z^N) = N \sum_{m=0}^{r} \lambda_m z^{mN} = \sum_{n=0}^{N-1} \frac{(1-z^N)^L}{(1-z\omega^n)^L}$$

Independent Eigenvectors

$$h_1 = [x_0^+, x_1^-], \quad h_{-1} = [x_{-1}^+, x_0^-],$$

$$x_{n+1}^- = \tfrac{1}{2}[h_1, x_n^-], \quad x_{-n-1}^- = \tfrac{1}{2}[h_{-1}, x_{-n}^-]$$

$x_1^- |\bar{\Omega}\rangle = 0$

$$(x_1^-)^{(r)} |\Omega\rangle = |\bar{\Omega}\rangle = |N\text{-}1, \ldots, N\text{-}1\rangle, \quad (x_1^-)^{(n)} |\Omega\rangle = 0, \quad n > r, \quad \text{(A)}$$

$$x_n^- |\Omega\rangle = (x_0^+)^{(n-1)} (x_1^-)^{(n)} |\Omega\rangle - \sum_{j=1}^{n-1} x_j^- \lambda_{n-j} |\Omega\rangle \qquad \text{(B)}$$

For $n>r$, $x_n^- |\Omega\rangle$ is a linear combination of $x_j^- |\Omega\rangle$ $j=1\ldots r$

$|\Omega\rangle$ | $x_j^- |\Omega\rangle, \; 1 \le j \le r$ | … | $x_{n_1}^- x_{n_2}^- \cdots x_{n_j}^- |\Omega\rangle, 1 \le n_1 < n_2 \cdots < n_j \le r;$ … | $|\bar{\Omega}\rangle$

$(x_0^+)^{(j-1)} (x_1^-)^{(j)} |\Omega\rangle$ — r

$(x_0^+)^{(n_1-1)} (x_1^-)^{(n_1)} \cdots (x_0^+)^{(n_j-1)} (x_1^-)^{(n_j)} |\Omega\rangle$ — $\binom{r}{j}$

$$1 + \binom{r}{1} + \cdots + \binom{r}{j} \cdots + \binom{r}{r} = 2^r$$

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Behold, how good and how pleasant it is for brethren to dwell together in unity!…
It is like the dew of Hermon descending upon the mountains of Zion: for there the Lord commanded the blessing--life forevermore.   Psalm 133