Spanning trees on the Sierpinski gasket

Shu-Chiuan Chang (1997-2002)
Department of Physics
National Cheng Kung University
Tainan 70101, Taiwan
and Physics Division
National Center for Theoretical Science
National Taiwan University
Taipei 10617, Taiwan

and

Lung-Chi Chen
Department of Mathematics
Fu Jen Catholic University
Taipei 24205, Taiwan

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Acknowledgments

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1 Introduction

1.1 Motivations

- The enumeration of the number of spanning trees was first considered by Kirchhoff in the analysis of electric circuits more than one and a half century ago [Kirchhoff].
- It is a problem of interest in mathematics [Biggs, Welsh, Burton and Pemantle, Lyons] and physics [Temperley, Wu].
- The Tutte polynomial or the partition function of the q-state Potts model in a special limit gives the number of spanning trees [Fortuin and Kasteleyn, Wu].
- The spanning tree problem is related to network, percolation, polymer, sandpile, ...
- Consider self-similar fractal lattices which have scaling invariance.
- A well-known example of fractal is the Sierpinski gasket.

1.2 Definitions

- A graph $G(V, E)$ is defined by its vertex set V and edge set E [Harary, Biggs].
- Denote $v(G) = |V|$ as the number of vertices and $e(G) = |E|$ as the number of edges of G, respectively.
- Denote the number of edges attached to the vertex v_i as degree k_i.
- A k-regular graph is a graph that each of its vertices has the same degree k.
- A spanning subgraph $G'(V, E')$ is a subgraph of $G(V, E)$ with $v(G') = |V|$ and $E' \subseteq E$.
- A tree is a connected graph with no circuits.
- A spanning tree is a spanning subgraph of G that is a tree. Therefore, it has $v(G) - 1$ edges.
- Denote the number of spanning trees on the graph G as $N_{ST}(G)$.
- A planar graph G has a dual graph G^*.

1.3 A well known method

- The adjacency matrix $A(G)$ is an $v(G) \times v(G)$ matrix with elements

$$A(G)_{ij} = \begin{cases} 1 & \text{if vertices } v_i \text{ and } v_j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases} .$$

- The Laplacian matrix $Q(G)$ is the $v(G) \times v(G)$ matrix with elements

$$Q(G)_{ij} = k_i \delta_{ij} - A(G)_{ij} .$$

- One of the eigenvalues of $Q(G)$ is always zero.
  Denote the rest as $\lambda_i(G)$, $1 \le i \le v(G) - 1$.

- The number of spanning trees is given by [Biggs]

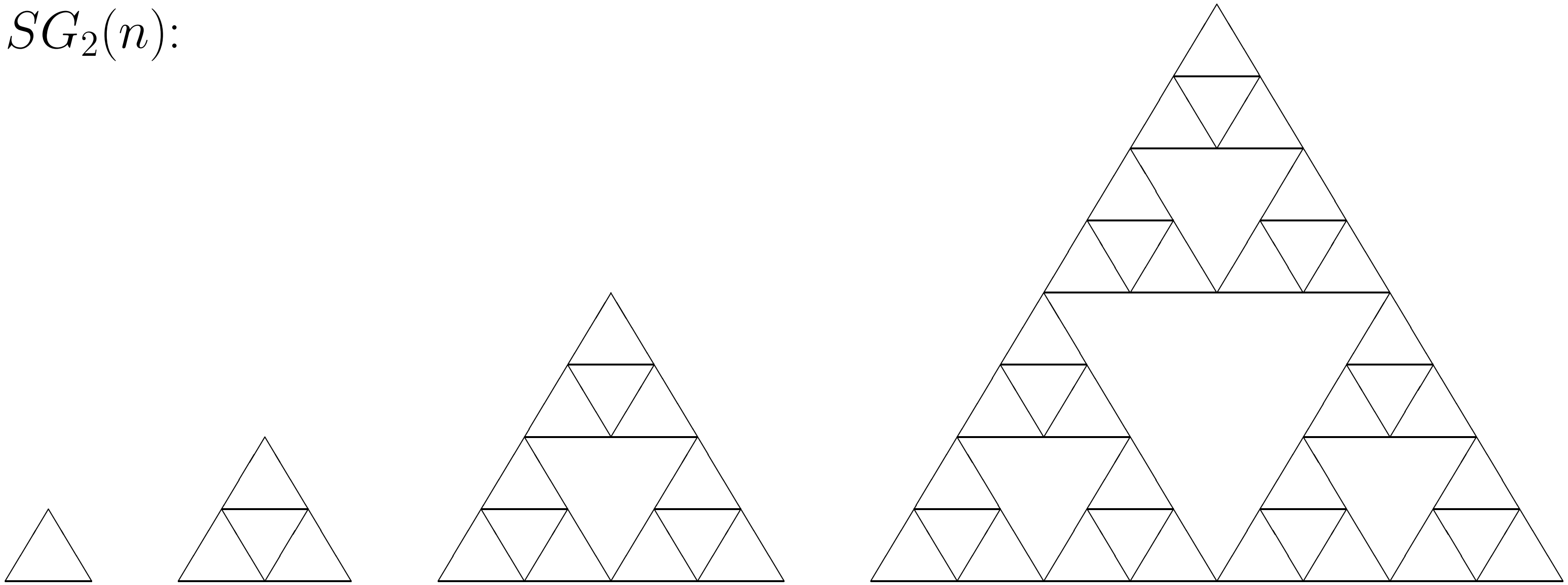
$$N_{ST}(G) = \frac{1}{v(G)} \prod_{i=1}^{n-1} \lambda_i(G) .$$

- For $N_{ST}(G)$ grows exponentially with $v(G)$ as $v(G) \to \infty$, define the asymptotic growth constant

$$z_G \equiv \lim_{v(G) \to \infty} \frac{\ln N_{ST}(G)}{v(G)} .$$

1.4 Sierpinski gasket $SG_d(n)$

- The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$:



$SG_2(0)$ $SG_2(1)$ $SG_2(2)$ $SG_2(3)$

- $SG_2(n)$ at stage $n = 0$ is an equilateral triangle.
- Stage $n + 1$ is obtained by the juxtaposition of three n-stage structures.
- $SG_d(n)$ can be built in any Euclidean dimension d.
- $SG_d(0)$ at stage $n = 0$ is a complete graph with $(d + 1)$ vertices.

- Fractal dimensionality for SG_d [Gefen and Aharony]:

$$D(SG_d) = \frac{\ln(d+1)}{\ln 2} \ .$$

- The numbers of edges and vertices for $SG_d(n)$:

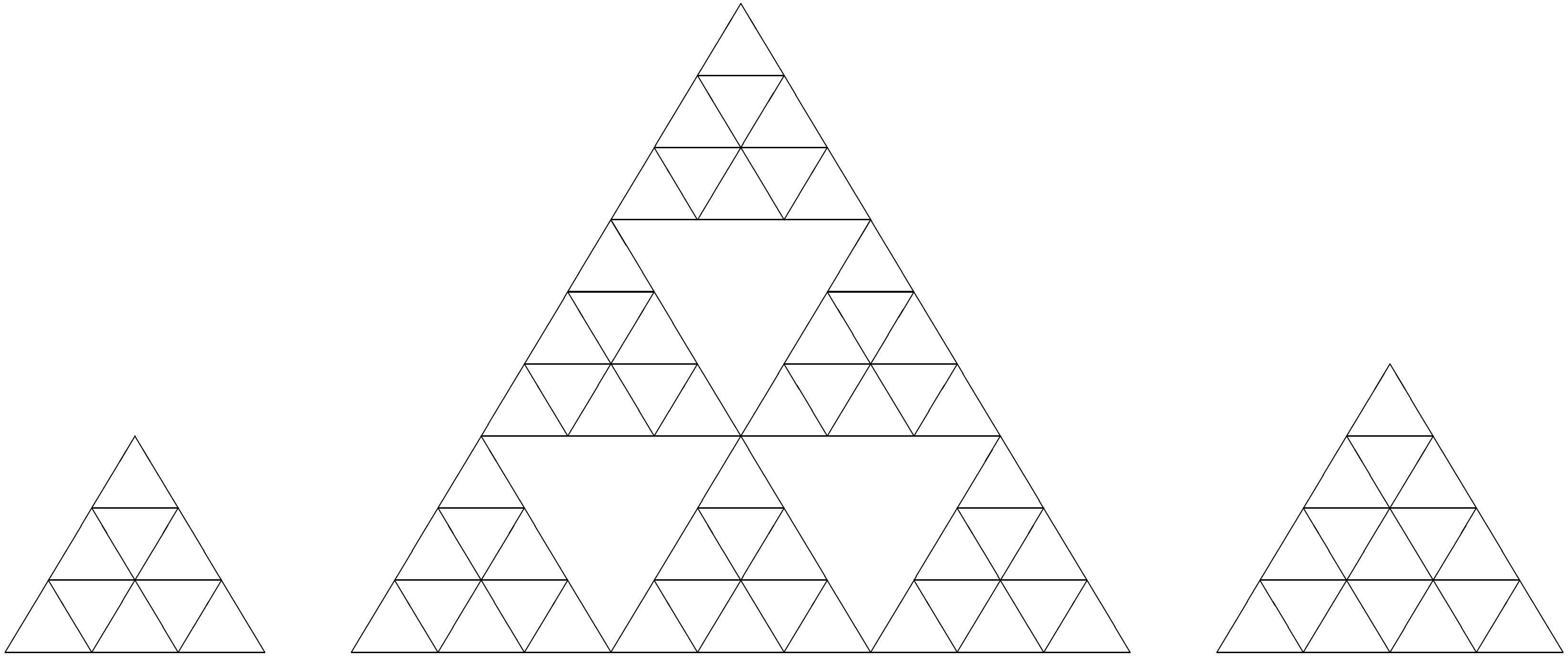
$$\begin{aligned} e(SG_d(n)) &= \binom{d+1}{2}(d+1)^n = \frac{d}{2}(d+1)^{n+1} \ , \\ v(SG_d(n)) &= \frac{d+1}{2}[(d+1)^n + 1] \ . \end{aligned}$$

- Except the $(d+1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. Therefore, SG_d is $2d$-regular in the large n limit.

1.5 Generalized Sierpinski gasket $SG_{d,b}(n)$

- The side length b which is an integer larger or equal to two [Hilfer and Blumen].
- The generalized Sierpinski gasket at stage $n+1$ is constructed with b layers of stage n hypertetrahedrons.
- The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case.

- The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$:



$SG_{2,3}(1)$ $SG_{2,3}(2)$ $SG_{2,4}(1)$

- The Hausdorff dimension for $SG_{d,b}$ [Hilfer and Blumen]:

$$D(SG_{d,b}) = \frac{\ln \binom{b+d-1}{d}}{\ln b} \ .$$

- $SG_{d,b}$ is not k-regular even in the thermodynamic limit.

1.6 Upper bound

- For a k-regular graph G_k, a general upper bound is $z_{G_k} \leq \ln k$.
- For a k-regular graph G_k with $k \geq 3$, a stronger upper bound for $N_{ST}(G_k)$ is given by [McKay, Chung and Yau]

$$N_{ST}(G_k) \leq \left(\frac{2 \ln v(G_k)}{v(G_k) k \ln k} \right) (c_k)^{v(G_k)}$$

  where

$$c_k = \frac{(k-1)^{k-1}}{[k(k-2)]^{\frac{k}{2}-1}} \ .$$

- The corresponding upper bound for z_{G_k} is

$$z_{G_k} \leq \ln(c_k) \ ,$$

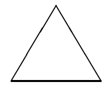
  so that $z_{SG_d} \leq \ln(c_{2d})$.
- Define the ratio

$$r_{SG_d} = \frac{z_{SG_d}}{\ln(c_{2d})} \ .$$

2 The number of spanning trees on $SG_2(n)$

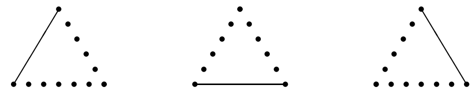
2.1 Definitions

- Define $f_2(n) \equiv N_{ST}(SG_2(n))$ as the number of spanning trees.



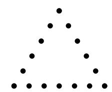
$f_2(n)$

- Define $ga_2(n)$, $gb_2(n)$, $gc_2(n)$ as the number of spanning subgraphs with two trees such that one of the outmost vertices belongs to one tree and the other two outmost vertices belong to the other tree.



$ga_2(n)$ $gb_2(n)$ $gc_2(n)$

- Define $h_2(n)$ as the number of spanning subgraphs with three trees such that each of the outmost vertices belongs to a different tree.

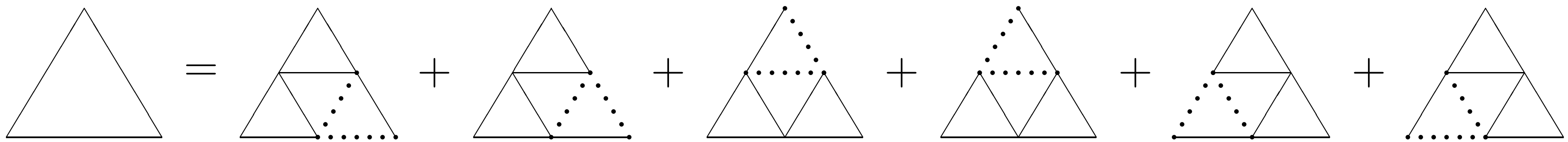


$h_2(n)$

- Because of rotation symmetry, define $g_2(n) \equiv ga_2(n) = gb_2(n) = gc_2(n)$.
- $h_2(n)$ is the number of spanning trees on $SG_2(n)$ with the three outmost vertices identified.
- The initial values at stage 0 are $f_2(0) = 3$, $g_2(0) = 1$, $h_2(0) = 1$.

2.2 Recursion relations

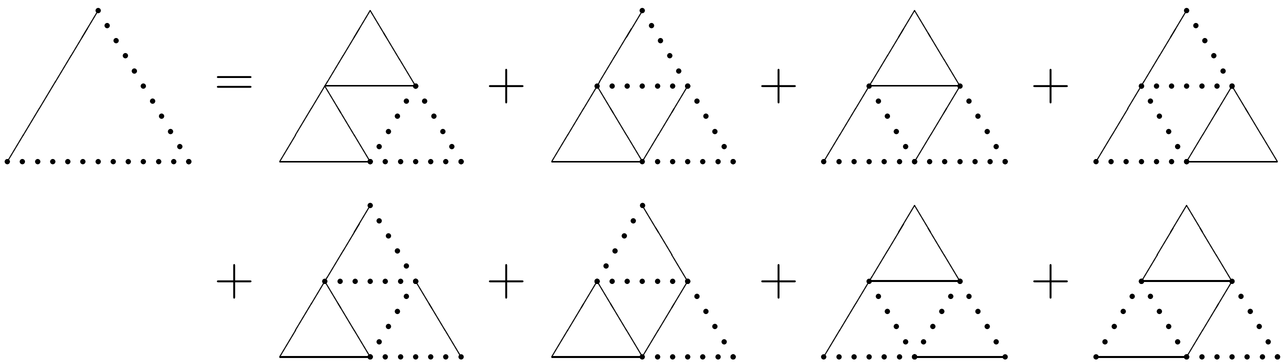
- Illustration for the expression of $f_2(n+1)$:



- Recursion relation for any non-negative integer n:

$$f_2(n+1) = 2f_2^2(n)[ga_2(n) + gb_2(n) + gc_2(n)] = 6f_2^2(n)g_2(n) \ .$$

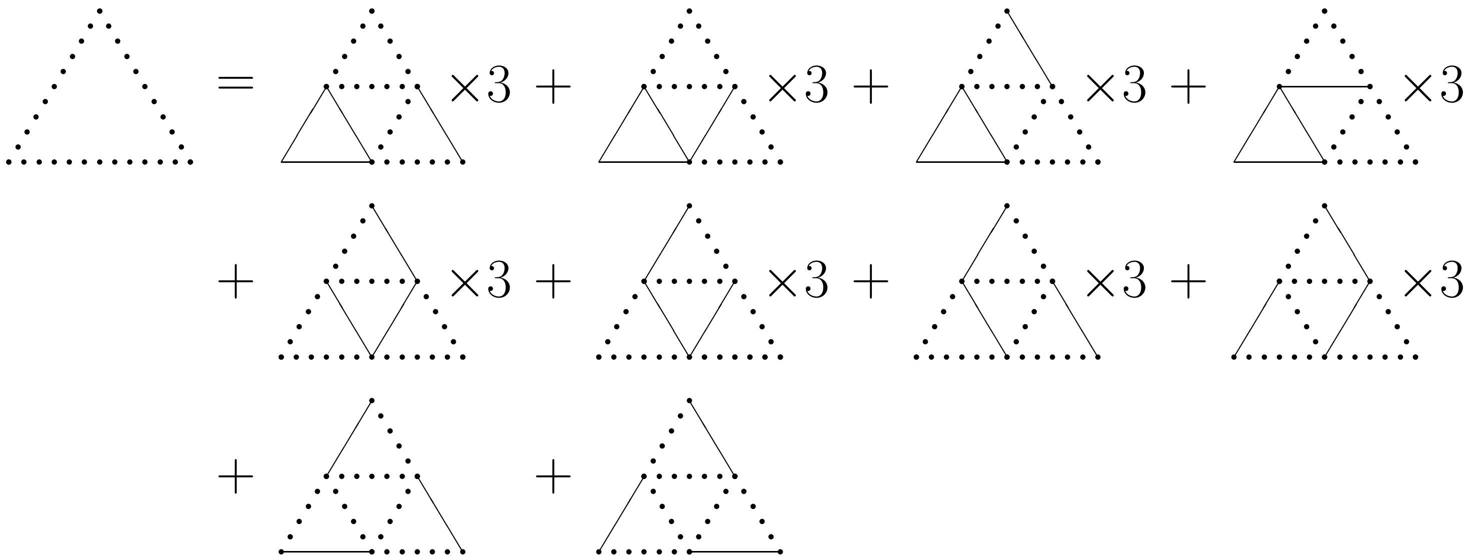
- Illustration for the expression of $ga_2(n+1)$:



- Recursion relation for any non-negative integer n:

$$\begin{aligned} ga_2(n+1) &= f_2^2(n)h_2(n) + 3f_2(n)ga_2^2(n) + 2f_2(n)ga_2(n)gc_2(n) + 2f_2(n)ga_2(n)gb_2(n) \\ &= f_2^2(n)h_2(n) + 7f_2(n)g_2^2(n) \ . \end{aligned}$$

- Illustration for the expression of $h_2(n+1)$:



- Recursion relation for any non-negative integer n:

$$
\begin{aligned}
h_2(n+1) &= 4f_2(n)h_2(n)[ga_2(n)+gb_2(n)+gc_2(n)] \\
&\quad +2gc_2(n)ga_2(n)[gc_2(n)+ga_2(n)]+2ga_2(n)gb_2(n)[ga_2(n)+gb_2(n)] \\
&\quad +2gb_2(n)gc_2(n)[gb_2(n)+gc_2(n)]+2ga_2(n)gb_2(n)gc_2(n) \\
&= 12f_2(n)g_2(n)h_2(n)+14g_2^3(n) \ .
\end{aligned}
$$

2.3 Results

- Solutions:

$$
\begin{aligned}
f_2(n) &= 2^{\alpha_2(n)}\ 3^{\beta_2(n)}\ 5^{\gamma_2(n)}\ , \\
g_2(n) &= 2^{\alpha_2(n)}\ 3^{\beta_2(n)-n-1}\ 5^{\gamma_2(n)+n}\ , \\
h_2(n) &= 2^{\alpha_2(n)}\ 3^{\beta_2(n)-2n-1}\ 5^{\gamma_2(n)+2n}\ .
\end{aligned}
$$

- The exponents are

$$\alpha_2(n) = \frac{1}{2}(3^n - 1)\ , \qquad \beta_2(n) = \frac{1}{4}(3^{n+1} + 2n + 1)\ , \qquad \gamma_2(n) = \frac{1}{4}(3^n - 2n - 1)\ .$$

- The numbers of edges and vertices for $SG_2(n)$:

$$e(SG_2(n)) = 3^{n+1}\ , \qquad v(SG_2(n)) = \frac{3}{2}(3^n + 1)\ .$$

- The asymptotic growth constant for SG_2:

$$z_{SG_2} = \frac{1}{3}\ln 2 + \frac{1}{2}\ln 3 + \frac{1}{6}\ln 5 \simeq 1.048594856...$$

2.4 Corollaries

- The number of spanning trees is the same for the dual:

$$N_{ST}(SG_2^*(n)) = N_{ST}(SG_2(n)) \ .$$

- Because SG_2 is 4-regular in the large n limit, $z_{SG_2^*} = z_{SG_2}$ [Chang and Wang].
- Denote the non-zero eigenvalues of the Laplacian matrix $Q(SG_2(n))$ as $\lambda(SG_2(n))_i$ for $1 \le i \le v(SG_2(n)) - 1$.

$$\prod_{i=1}^{v(SG_2(n))-1} \lambda(SG_2(n))_i = v(SG_2(n)) f_2(n) \ .$$

- $Q(SG_2(n))$ does not look simple to diagonalize. For examples,

$$Q(SG_2(1)) = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$Q(SG_2(2)) = \begin{pmatrix}
4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 4 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}$$

3 The number of spanning trees on $SG_{2,b}(n)$ with $b = 3, 4$

3.1 The number of spanning trees on $SG_{2,3}(n)$

- The numbers of edges and vertices for $SG_{2,3}(n)$:

$$e(SG_{2,3}(n)) = 3 \times 6^n \ , \qquad v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5} \ .$$

- There are $(6^n - 1)/5$ vertices of $SG_{2,3}(n)$ with degree six, $6(6^n - 1)/5$ vertices with degree four, and the three outmost vertices have degree two.

- Define $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$ as before. The initial values are $f_{2,3}(0) = 3$, $g_{2,3}(0) = 1$, $h_{2,3}(0) = 1$.

- Recursion relations for any non-negative integer n:

$$f_{2,3}(n+1) = 18 f_{2,3}^4(n) g_{2,3}(n) h_{2,3}(n) + 142 f_{2,3}^3(n) g_{2,3}^3(n) \ ,$$

$$g_{2,3}(n+1) = 2 f_{2,3}^4(n) h_{2,3}^2(n) + 77 f_{2,3}^3(n) g_{2,3}^2(n) h_{2,3}(n) + 171 f_{2,3}^2(n) g_{2,3}^4(n) \ ,$$

$$h_{2,3}(n+1) = 60 f_{2,3}^3(n) g_{2,3}(n) h_{2,3}^2(n) + 564 f_{2,3}^2(n) g_{2,3}^3(n) h_{2,3}(n) + 468 f_{2,3}(n) g_{2,3}^5(n) \ .$$

- Solutions:

$$f_{2,3}(n) = 2^{\alpha_{2,3}(n)} \; 3^{\beta_{2,3}(n)} \; 5^{\gamma_{2,3}(n)} \; 7^{\delta_{2,3}(n)} \; ,$$

$$g_{2,3}(n) = 2^{\alpha_{2,3}(n)} \; 3^{\beta_{2,3}(n)+n-1} \; 5^{\gamma_{2,3}(n)+n} \; 7^{\delta_{2,3}(n)-n} \; ,$$

$$h_{2,3}(n) = 2^{\alpha_{2,3}(n)} \; 3^{\beta_{2,3}(n)+2n-1} \; 5^{\gamma_{2,3}(n)+2n} \; 7^{\delta_{2,3}(n)-2n} \; .$$

- The exponents are

$$\alpha_{2,3}(n) = \frac{2}{5}(6^n - 1) \; , \qquad \beta_{2,3}(n) = \frac{1}{25}(13 \times 6^n - 15n + 12) \; ,$$

$$\gamma_{2,3}(n) = \frac{1}{25}(3 \times 6^n - 15n - 3) \; , \quad \delta_{2,3}(n) = \frac{1}{25}(7 \times 6^n + 15n - 7) \; .$$

- The asymptotic growth constant for $SG_{2,3}$:

$$z_{SG_{2,3}} = \frac{2}{7}\ln 2 + \frac{13}{35}\ln 3 + \frac{3}{35}\ln 5 + \frac{1}{5}\ln 7 \simeq 1.133231895...$$

3.2 The number of spanning trees on $SG_{2,4}(n)$

- The numbers of edges and vertices for $SG_{2,4}(n)$:

$$e(SG_{2,4}(n)) = 3 \times 10^n \ , \qquad v(SG_{2,4}(n)) = \frac{4 \times 10^n + 5}{3} \ .$$

- There are $(10^n - 1)/3$ vertices of $SG_{2,4}(n)$ with degree six, $(10^n - 1)$ vertices with degree four, and the three outmost vertices have degree two.

- Define $f_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$ as before. The initial values are again $f_{2,4}(0) = 3$, $g_{2,4}(0) = 1$, $h_{2,4}(0) = 1$.

- Recursion relations for any non-negative integer n:

$$\begin{aligned} f_{2,4}(n+1) &= 2f_{2,4}^7(n)h_{2,4}^3(n) + 516f_{2,4}^6(n)g_{2,4}^2(n)h_{2,4}^2(n) \\ &\quad +5856f_{2,4}^5(n)g_{2,4}^4(n)h_{2,4}(n) + 11354f_{2,4}^4(n)g_{2,4}^6(n) \ , \\ g_{2,4}(n+1) &= 82f_{2,4}^6(n)g_{2,4}(n)h_{2,4}^3(n) + 2786f_{2,4}^5(n)g_{2,4}^3(n)h_{2,4}^2(n) \\ &\quad +14480f_{2,4}^4(n)g_{2,4}^5(n)h_{2,4}(n) + 13732f_{2,4}^3(n)g_{2,4}^7(n) \ , \end{aligned}$$

$$h_{2,4}(n+1) = 20f_{2,4}^6(n)h_{2,4}^4(n) + 2388f_{2,4}^5(n)g_{2,4}^2(n)h_{2,4}^3(n) + 30948f_{2,4}^4(n)g_{2,4}^4(n)h_{2,4}^2(n)$$
$$+83234f_{2,4}^3(n)g_{2,4}^6(n)h_{2,4}(n) + 42210f_{2,4}^2(n)g_{2,4}^8(n) \ .$$

- Solutions:

$$f_{2,4}(n) = 2^{\alpha_{2,4}(n)}\ 3^{\beta_{2,4}(n)}\ 5^{\gamma_{2,4}(n)}\ 41^{\delta_{2,4}(n)}\ 103^{\epsilon_{2,4}(n)} \ ,$$
$$g_{2,4}(n) = 2^{\alpha_{2,4}(n)}\ 3^{\beta_{2,4}(n)-1}\ 5^{\gamma_{2,4}(n)}\ 41^{\delta_{2,4}(n)-n}\ 103^{\epsilon_{2,4}(n)+n} \ ,$$
$$h_{2,4}(n) = 2^{\alpha_{2,4}(n)}\ 3^{\beta_{2,4}(n)-1}\ 5^{\gamma_{2,4}(n)}\ 41^{\delta_{2,4}(n)-2n}\ 103^{\epsilon_{2,4}(n)+2n} \ .$$

- The exponents are

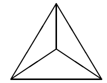
$$\alpha_{2,4}(n) = \frac{2}{9}(10^n - 1) \ , \qquad \beta_{2,4}(n) = \frac{1}{3}(10^n + 2) \ , \qquad \gamma_{2,4}(n) = \frac{1}{9}(10^n - 1) \ ,$$
$$\delta_{2,4}(n) = \frac{2}{27}(2 \times 10^n + 9n - 2) \ , \qquad \epsilon_{2,4}(n) = \frac{2}{27}(10^n - 9n - 1) \ .$$

- The asymptotic growth constant for $SG_{2,4}$:

$$z_{SG_{2,4}} = \frac{1}{6}\ln 2 + \frac{1}{4}\ln 3 + \frac{1}{12}\ln 5 + \frac{1}{9}\ln(41) + \frac{1}{18}\ln(103) \simeq 1.194401490...$$

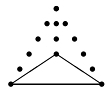
4 The number of spanning trees on $SG_d(n)$ with $d = 3, 4$

4.1 The number of spanning trees on $SG_3(n)$



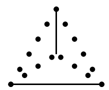
$f_3(n)$

- Define $f_3(n) \equiv N_{ST}(SG_3(n))$ as the number of spanning trees.
- Define $g_3(n)$ as the number of spanning subgraphs with two trees such that one of the outmost vertices belongs to one tree and the other three outmost vertices belong to the other tree.



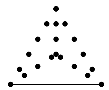
$g_3(n)$

- Define $h_3(n)$ as the number of spanning subgraphs with two trees such that two of the outmost vertices belong to one tree and the other two outmost vertices belong to the other tree.



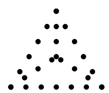
$h_3(n)$

- Define $p_3(n)$ as the number of spanning subgraphs with three trees such that two of the outmost vertices belong to one tree and the other two outmost vertices separately belong to the other trees.



$p_3(n)$

- Define $q_3(n)$ as the number of spanning subgraphs with four trees such that each of the outmost vertices belongs to a different tree.



$q_3(n)$

- There are four equivalent $g_3(n)$, three equivalent $h_3(n)$, and six equivalent $p_3(n)$.

- $q_3(n)$ is the number of spanning trees on $SG_3(n)$ with the four outmost vertices identified.
- The initial values at stage 0 are $f_3(0) = 16$, $g_3(0) = 3$, $h_3(0) = 1$, $p_3(0) = 1$, $q_3(0) = 1$.
- Define $gh_3(n) \equiv g_3(n) + h_3(n)$.
- Recursion relations for any non-negative integer n:

$$f_3(n+1) = 72 f_3^2(n) gh_3(n) p_3(n) + 56 f_3(n) gh_3^3(n) \ ,$$

$$\begin{aligned} gh_3(n+1) &= 6 f_3^2(n) gh_3(n) q_3(n) + 26 f_3^2(n) p_3^2(n) \\ &\quad + 120 f_3(n) gh_3^2(n) p_3(n) + 22 gh_3^4(n) \ , \end{aligned}$$

$$\begin{aligned} p_3(n+1) &= 6 f_3^2(n) p_3(n) q_3(n) + 14 f_3(n) gh_3^2(n) q_3(n) \\ &\quad + 120 f_3(n) gh_3(n) p_3^2(n) + 88 gh_3^3(n) p_3(n) \ , \end{aligned}$$

$$\begin{aligned} q_3(n+1) &= 144 f_3(n) gh_3(n) p_3(n) q_3(n) + 208 f_3(n) p_3^3(n) \\ &\quad + 56 gh_3^3(n) q_3(n) + 720 gh_3^2(n) p_3^2(n) \ . \end{aligned}$$

- Solutions:

$$
\begin{aligned}
f_3(n) &= 2^{\alpha_3(n)}\ 3^{\beta_3(n)} , \\
gh_3(n) &= 2^{\alpha_3(n)-n-2}\ 3^{\beta_3(n)+n} , \\
p_3(n) &= 2^{\alpha_3(n)-2n-4}\ 3^{\beta_3(n)+2n} , \\
q_3(n) &= 2^{\alpha_3(n)-3n-4}\ 3^{\beta_3(n)+3n} .
\end{aligned}
$$

- The exponents are

$$\alpha_3(n) = 4^{n+1} + n , \qquad \beta_3(n) = \frac{1}{3}(4^n - 3n - 1) .$$

- The numbers of edges and vertices for $SG_3(n)$:

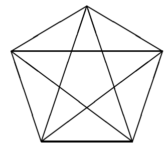
$$e(SG_3(n)) = 6 \times 4^n , \qquad v(SG_3(n)) = 2(4^n + 1) .$$

- The asymptotic growth constant for SG_3:

$$z_{SG_3} = 2\ln 2 + \frac{1}{6}\ln 3 \simeq 1.569396409...$$

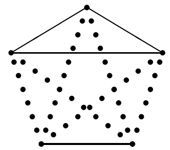
4.2 The number of spanning trees on $SG_4(n)$

- Define $f_4(n) \equiv N_{ST}(SG_4(n))$ as the number of spanning trees.



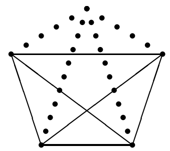
$f_4(n)$

- Define $g_4(n)$ as the number of spanning subgraphs with two trees such that two of the outmost vertices belong to one tree and the other three outmost vertices belong to the other tree.



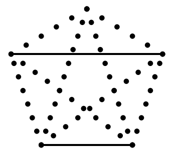
$g_4(n)$

- Define $h_4(n)$ as the number of spanning subgraphs with two trees such that one of the outmost vertices belong to one tree and the other four outmost vertices belong to the other tree.



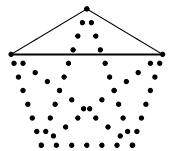
$h_4(n)$

- Define $p_4(n)$ as the number of spanning subgraphs with three trees such that one of the outmost vertices belong to one tree, two of the other outmost vertices belong to another tree and the rest two outmost vertices belong to the other tree.



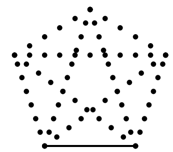
$p_4(n)$

- Define $q_4(n)$ as the number of spanning subgraphs with three trees such that three of the outmost vertices belong to one tree and the other two outmost vertices separately belong to the other trees.



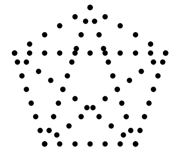
$q_4(n)$

- Define $r_4(n)$ as the number of spanning subgraphs with four trees such that two of the outmost vertices belong to one tree and the other three outmost vertices separately belong to the other trees.



$r_4(n)$

- Define $s_4(n)$ as the number of spanning subgraphs with five trees such that each of the outmost vertices belongs to a different tree.



$s_4(n)$

- There are ten equivalent $g_4(n)$, five equivalent $h_4(n)$, fifteen equivalent $p_4(n)$, ten equivalent $q_4(n)$ and ten equivalent $r_4(n)$.

- $s_4(n)$ is the number of spanning trees on $SG_4(n)$ with the five outmost vertices identified.

- The initial values at stage 0 are $f_4(0) = 125$, $g_4(0) = 3$, $h_4(0) = 16$, $p_4(0) = 1$, $q_4(0) = 3$, $r_4(0) = 1$, $s_4(0) = 1$.

- Define $gh_4(n) \equiv 3g_4(n) + h_4(n)$ and $pq_4(n) \equiv 2p_4(n) + q_4(n)$.

- Recursion relations for any non-negative integer n:

$$\begin{aligned} f_4(n+1) &= 1440 f_4^2(n) gh_4(n) pq_4(n) r_4(n) + 520 f_4^2(n) pq_4^3(n) + 1120 f_4(n) gh_4^3(n) r_4(n) \\ &\quad + 3600 f_4(n) gh_4^2(n) pq_4^2(n) + 1320 gh_4^4(n) pq_4(n) \ , \end{aligned}$$

$$
\begin{aligned}
gh_4(n+1) \;=\;& 72f_4^2(n)gh_4(n)pq_4(n)s_4(n) + 378f_4^2(n)gh_4(n)r_4^2(n) + 816f_4^2(n)pq_4^2(n)r_4(n) \\
&+56f_4(n)gh_4^3(n)s_4(n) + 3756f_4(n)gh_4^2(n)pq_4(n)r_4(n) \\
&+2360f_4(n)gh_4(n)pq_4^3(n) + 688gh_4^4(n)r_4(n) + 2562gh_4^3(n)pq_4^2(n) \;,
\end{aligned}
$$

$$
\begin{aligned}
pq_4(n+1) \;=\;& 48f_4^2(n)gh_4(n)r_4(n)s_4(n) + 52f_4^2(n)pq_4^2(n)s_4(n) + 544f_4^2(n)pq_4(n)r_4^2(n) \\
&+240f_4(n)gh_4^2(n)pq_4(n)s_4(n) + 1252f_4(n)gh_4^2(n)r_4^2(n) \\
&+4720f_4(n)gh_4(n)pq_4^2(n)r_4(n) + 724f_4(n)pq_4^4(n) + 44gh_4^4(n)s_4(n) \\
&+3416gh_4^3(n)pq_4(n)r_4(n) + 3104gh_4^2(n)pq_4^3(n) \;,
\end{aligned}
$$

$$
\begin{aligned}
r_4(n+1) \;=\;& 72f_4^2(n)pq_4(n)r_4(n)s_4(n) + 126f_4^2(n)r_4^3(n) + 168f_4(n)gh_4^2(n)r_4(n)s_4(n) \\
&+360f_4(n)gh_4(n)pq_4^2(n)s_4(n) + 3756f_4(n)gh_4(n)pq_4(n)r_4^2(n) \\
&+2360f_4(n)pq_4^3(n)r_4(n) + 264gh_4^3(n)pq_4(n)s_4(n) + 1376gh_4^3(n)r_4^2(n) \\
+\;& 7686gh_4^2(n)pq_4^2(n)r_4(n) + 2328gh_4(n)pq_4^4(n) \;,
\end{aligned}
$$

$$
\begin{aligned}
s_4(n+1) \;=\;& 2880f_4(n)gh_4(n)pq_4(n)r_4(n)s_4(n) + 5040f_4(n)gh_4(n)r_4^3(n) \\
&+1040f_4(n)pq_4^3(n)s_4(n) + 16320f_4(n)pq_4^2(n)r_4^2(n) + 1120gh_4^3(n)r_4(n)s_4(n) \\
&+3600gh_4^2(n)pq_4^2(n)s_4(n) + 37560gh_4^2(n)pq_4(n)r_4^2(n) \\
&+47200gh_4(n)pq_4^3(n)r_4(n) + 4344pq_4^5(n) \;.
\end{aligned}
$$

- Solutions:

$$\begin{aligned} f_4(n) &= 2^{\alpha_4(n)}\ 5^{\beta_4(n)}\ 7^{\gamma_4(n)} , \\ gh_4(n) &= 2^{\alpha_4(n)}\ 5^{\beta_4(n)-n-1}\ 7^{\gamma_4(n)+n} , \\ pq_4(n) &= 2^{\alpha_4(n)}\ 5^{\beta_4(n)-2n-2}\ 7^{\gamma_4(n)+2n} , \\ r_4(n) &= 2^{\alpha_4(n)}\ 5^{\beta_4(n)-3n-3}\ 7^{\gamma_4(n)+3n} , \\ s_4(n) &= 2^{\alpha_4(n)}\ 5^{\beta_4(n)-4n-3}\ 7^{\gamma_4(n)+4n} . \end{aligned}$$

- The exponents are

$$\alpha_4(n) = \frac{3}{2}(5^n - 1) , \qquad \beta_4(n) = \frac{3}{8}(5^{n+1} + 4n + 3) , \qquad \gamma_4(n) = \frac{3}{8}(5^n - 4n - 1) .$$

- The numbers of edges and vertices for $SG_4(n)$:

$$e(SG_4(n)) = 2 \times 5^{n+1} , \qquad v(SG_4(n)) = \frac{5}{2}(5^n + 1) .$$

- The asymptotic growth constant for SG_4:

$$z_{SG_4} = \frac{3}{5}\ln 2 + \frac{3}{4}\ln 5 + \frac{3}{20}\ln 7 \simeq 1.914853265...$$

5 The number of spanning trees on $SG_d(n)$ for general d

- Conjecture for $SG_d(n)$ with arbitrary dimension d (consistent with previous results):

$$N_{ST}(SG_d(n)) = 2^{\alpha_d(n)}\ (d+1)^{\beta_d(n)}\ (d+3)^{\gamma_d(n)}\ .$$

- The exponents are positive integers when d is a positive integer and n is a non-negative integer:

$$\begin{aligned}
\alpha_d(n) &= \frac{d-1}{2}[(d+1)^n - 1]\ , \\
\beta_d(n) &= \frac{d-1}{2d}[(d+1)^{n+1} + dn + d - 1]\ , \\
\gamma_d(n) &= \frac{d-1}{2d}[(d+1)^n - dn - 1]\ .
\end{aligned}$$

- The asymptotic growth constant for $SG_d(n)$:

$$z_{SG_d} = \frac{d-1}{d(d+1)}[d\ln 2 + (d+1)\ln(d+1) + \ln(d+3)]\ .$$

- For $SG_d(0)$ at stage $n=0$, $N_{ST}(SG_d(0)) = (d+1)^{d-1}$.

6 Discussion and summary

- Compared with the values $z_{\mathcal{L}_d}$ for d-dimensional hypercubic lattice $\mathcal{L}_d$ which also has degree $k = 2d$ [Shrock and Wu, Felker and Lyons], $z_{SG_d} < z_{\mathcal{L}_d}$ for all $d \geq 2$ indicates that SG_d is less densely connected than $\mathcal{L}_d$.
- z_{SG_d} approaches to $\ln(2d)$ from below when $d \to \infty$.
- We present the numbers of spanning trees on the Sierpinski gaskets $SG_2(n)$, $SG_3(n)$ and $SG_4(n)$.
- We present the numbers of spanning trees on the generalized Sierpinski gaskets $SG_{2,3}(n)$ and $SG_{2,4}(n)$.
- We conjecture the numbers of spanning trees on the Sierpinski gasket $SG_d(n)$ with arbitrary dimension d.
- The asymptotic growth constants of the numbers of spanning trees on the Sierpinski gasket have simple expressions.

Table 1: Numerical values of z_{SG_d}, $\ln(c_{2d})$, r_{SG_d}, and comparison with $z_{\mathcal{L}_d}$.

d	D	k	z_{SG_d}	$\ln(c_{2d})$	r_{SG_d}	$z_{\mathcal{L}_d}$	$\ln(2d)$
2	1.585	4	1.048594857	1.216395324	0.862051042	1.166243616	1.386294361
3	2	6	1.569396409	1.691081901	0.928042816	1.673389303	1.791759469
4	2.322	8	1.914853265	2.007768011	0.953722370	1.999707645	2.079441542
5	2.585	10	2.172764568	2.246914657	0.966999152	2.242488060	2.302585093
6	2.807	12	2.378271274	2.439389287	0.974945363	2.436626962	2.484906650
7	3	14	2.548944395	2.600557771	0.980152960	2.598676304	2.639057330
8	3.170	16	2.694814686	2.739230654	0.983785240	2.737867664	2.772588722
9	3.322	18	2.822140640	2.860943008	0.986437211	2.859910142	2.890371758
10	3.459	20	2.935085659	2.969404321	0.988442577	2.968594484	2.995732274

References

[1] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, *Ann. Phys. Chem.* **72** (1847) 497.

[2] N. L. Biggs, *Algebraic Graph Theory* (2nd ed., Cambridge Univ. Press, Cambridge, 1993).

[3] D. J. A. Welsh, *Complexity: Knots, Colourings, and Counting*, London Math. Soc. Lect. Note Ser. 186 (Cambridge University Press, Cambridge, 1993).

[4] R. Burton and R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, *Ann. Probab.* **21** (1993) 1329.

[5] R. Lyons, Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.* **14** (2005) 491.

[6] H. N. V. Temperley, in *Combinatorics: Proc. Combinatorial Mathematics* (Oxford, 1972), 356.

[7] F. Y. Wu, Number of spanning trees on a lattice, *J. Phys. A: Math. Gen.* **10** (1977) L113.

[8] C. M. Fortuin and P. W. Kasteleyn, On the random-cluster model I. Introduction and relation to other models, *Physica* **57** (1972) 536.

[9] F. Y. Wu, The Potts model, *Rev. Mod. Phys.* **54** (1982) 235.

[10] F. Harary, *Graph Theory* (Addison-Wesley, New York, 1969).

[11] Y. Gefen and A. Aharony, Solvable fractal family, and its possible relation to the backbone at percolation, *Phys. Rev. Lett.* **47** (1981) 1771.

[12] R. Hilfer and A. Blumen, Renormalisation on Sierpinski-type fractals, *J. Phys. A: Math. Gen.* **17** (1984) L537.

[13] B. McKay, Spanning trees in regular graphs, *Europ. J. Combin.* **4** (1983) 149.

[14] F. Chung and S.-T. Yau, Coverings, heat kernels and spanning trees, *J. Combin.* **6** (1999) 163.

[15] S.-C. Chang and W. Wang, Spanning trees on lattices and integral identities, *J. Phys. A: Math. Gen.* **39** (2006) 10263.

[16] R. Shrock and F. Y. Wu, Spanning trees on graphs and lattices in d dimensions, *J. Phys. A: Math. Gen.* **33** (2000) 3881.

[17] J. L. Felker and R. Lyons, High-precision entropy values for spanning trees in lattices, *J. Phys. A: Math. Gen.* **36** (2003) 8361.