Some Applications of Integrable Models to Condensed Matter Physics

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YITP and Integrable Models

- **C.N. Yang**

  Yang-Baxter Equation, Nested Bethe ansatz, Thermodynamics of integrable models, Ground State Properties of XXZ Chain in a Magnetic Field ...

- **B.M. McCoy**

  $T = 0$ Correlation Functions in Free Fermionic Theories (Quantum Ising Chain, XY Chain), Spectrum of XYZ chain, Chiral Potts Model, ...

- **V.E. Korepin**

  Exact Scattering Matrices, $T \geq 0$ Quantum Correlation Functions in delta-function Bose gap and gapless lattice models (Hubbard model), ...

J. Perk, R. Shrock, B. Sutherland, C. Tracy, A. Tsvelik, . . .

Low-D ⟶ strong quantum fluctuations ⟶ “exotic” quantum-mechanical properties

- Confinement
- quantum number fractionalization (spin-charge separation)
- topillogical excitations
- dynamical mass generation

Recent years: strong experimental efforts to study these phenomena in actual systems that are approximately described by IM

[condensed matter systems, ultracold trapped atom gases]

Integrable Models

Example: O(3) nonlinear σ-model.

$$\mathcal{L} = \frac{1}{2g}\left[\frac{1}{v}\left(\frac{\partial \mathbf{n}}{\partial t}\right)^2 - v\left(\frac{\partial \mathbf{n}}{\partial x}\right)^2\right] , \quad \mathbf{n}\cdot\mathbf{n} = 1.$$

1. Exact Spectrum: 3 particles with $S^z = \pm 1, 0$ and

$$E = \Delta \cosh\theta , \qquad P = \frac{\Delta}{v}\sinh\theta.$$

2. Factorized Scattering Matrix

$$S^{(N)}(\theta_1, \ldots, \theta_N) \;=\; \prod_{j=2}^{N}\prod_{k=1}^{j-1} S(\theta_k, \theta_j) \ .$$

Yang-Baxter-Equation

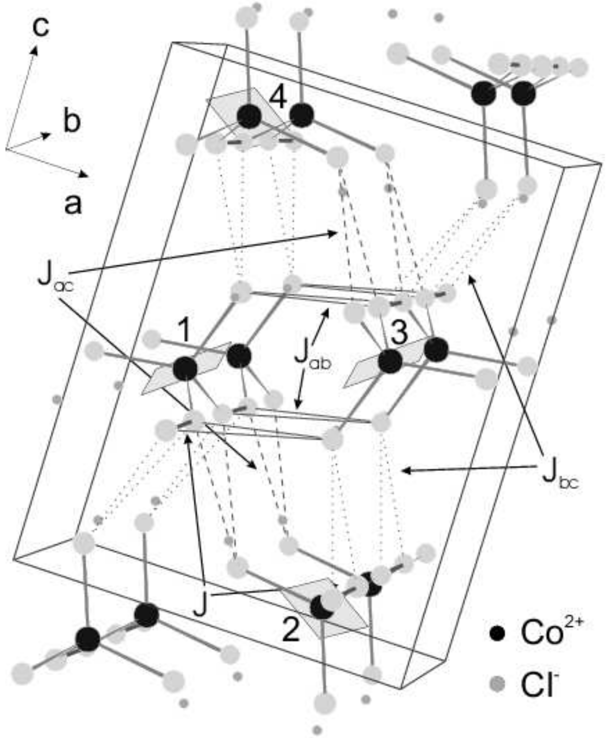
$$S_{ab}^{a'b'}(\theta_1,\theta_2)S_{a'c}^{a''b'}(\theta_1,\theta_3)S_{b'c'}^{b''c''}(\theta_2,\theta_3) = S_{bc}^{b'c'}(\theta_2,\theta_3)S_{ac'}^{a'c''}(\theta_1,\theta_3)S_{a'b'}^{a''b''}(\theta_1,\theta_2).$$

Physical Excitations scatter purely elastically.

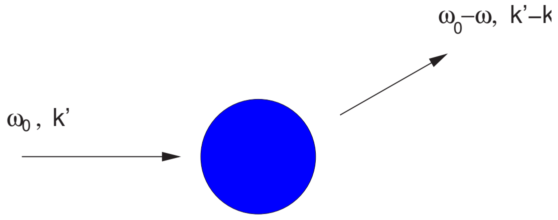
Application to Condensed Matter Systems I

Many materials display a **Separation of Energy Scales:** e.g. spatially anisotropic spin-1/2 Heisenberg magnet $H = \sum_{\langle jk \rangle} J_{jk}\, \mathbf{S}_j \cdot \mathbf{S}_k$.

$J \gg J_{ab},\ J_{ac},\ J_{bc} \longrightarrow$ weakly coupled chains.



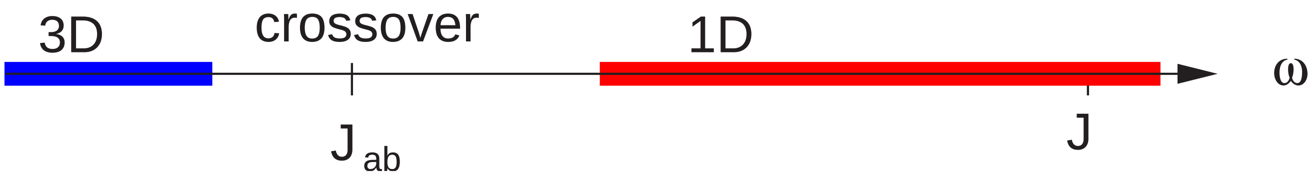
Inelastic neutron scattering experiments measure



$$\frac{d\sigma}{d\Omega} \propto S(\omega,\mathbf{k}) = \frac{1}{2\pi N}\sum_{\alpha,\beta}\left[\delta^{\alpha\beta} - \frac{k^{\alpha}k^{\beta}}{\mathbf{k}\cdot\mathbf{k}}\right]\int dt\ e^{i\omega t}\sum_{j,l} e^{i\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_l)}\langle S_j^{\alpha}(t)S_l^{\beta}(0)\rangle$$

$\omega \gg J_{ab}, J_{ac}, J_{bc} \longrightarrow$ **3D effects negligible (perturbative and small)**

Dynamical Response of Quasi-1D Materials:



Application to Condensed Matter Systems II

1D not a problem, but shouldn't integrability lead to "unphysical" behaviour?

⟶ not a (big) problem as physical observables are generally **nonlocal** with respect to the elementary excitations.

Example: $T = 0$ 2-point function in O(3) nonlinear σ-model

$$\begin{aligned} G_R(\omega, q) &= \int dtdx\ \theta(t)\ e^{i\omega t - iqx} \langle [n^a(t,x), n^a(0,0)] \rangle \\ &= \frac{Z}{\omega - \sqrt{\Delta^2 + (vq)^2} + i\epsilon} + \dots \end{aligned}$$

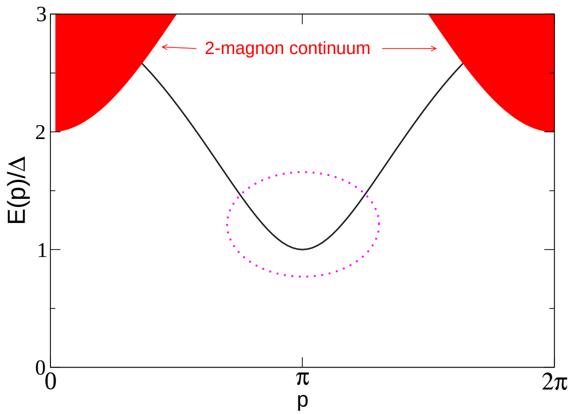
δ-function in $\mathrm{Im} G_R(\omega, q)$ ⟶ coherent propagation of a particle.

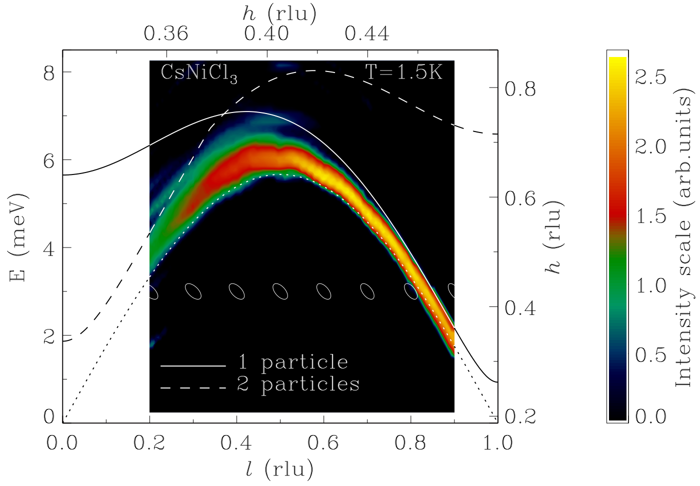
At $T > 0$ the δ-function broadens!

Reason: field n^a is **nonlocal** with respect to the elementary excitations.

Integrable Quantum Field Theories

$$H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} \ , \quad \mathbf{S}_j^2 = S(S+1).$$





I. A. Zaliznyak et. al. Phys. Rev. Lett. 87, 017202 (2001).

$\Delta \ll J \longrightarrow$ continuum description applies: $O(3)$ nonlinear σ-model

$$\mathcal{L} = \frac{1}{2g}\left[\frac{1}{v}\left(\frac{\partial \mathbf{n}}{\partial t}\right)^2 - v\left(\frac{\partial \mathbf{n}}{\partial x}\right)^2\right] \ , \quad \mathbf{n}\cdot\mathbf{n} = 1.$$

$$\mathbf{S}_j \approx S(-1)^{ja_0}\mathbf{n}(x) + \frac{1}{g}\mathbf{n}(x) \times \frac{\partial \mathbf{n}(x)}{\partial t}.$$

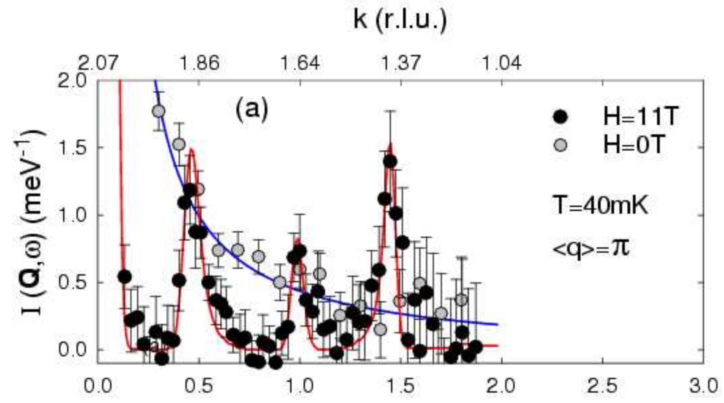
Correlation Functions of the Lattice Model $\longrightarrow$ Correlation Functions of the QFT:

$$S(\omega, k=\frac{\pi}{a_0}+q) \propto \sum_{\alpha,\beta}\left[\delta^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2}\right]\int dt\ dx\ e^{i\omega t-iqx}e^{iq(R_j-R_l)}\langle n^\alpha(t,x)n^\beta(0,0)\rangle$$

Restrictions: $\omega \ll J$ and $|qa_0| \ll \pi$.

Similarly: $S(\omega,q)$ with $|qa_0| \ll \pi$. $(n^a \longrightarrow \epsilon_{abc}n^b\frac{\partial n^c}{\partial t})$.

Example: quantum sine-Gordon model in $CuCl_2 \cdot 2((CD_3)_2SO)$



How to calculate $T=0$ correlation functions?

Form Factor Bootstrap Approach

Karowski and Weisz '78, **Smirnov** '93, Lukyanov '95,...

Faddeev-Zamolodchikov algebra:

$$\begin{aligned} A_a(\theta_1)A_b(\theta_2) &= S_{ab}^{a'b'}(\theta_1-\theta_2)A_{b'}(\theta_2)A_{a'}(\theta_1) \\ A_a^\dagger(\theta_1)A_b^\dagger(\theta_2) &= S_{ab}^{a'b'}(\theta_1-\theta_2)A_{b'}^\dagger(\theta_2)A_{a'}^\dagger(\theta_1) \\ A_a^\dagger(\theta_1)A_b(\theta_2) &= 2\pi\delta_{ab}\delta(\theta_1-\theta_2)+S_{ba'}^{b'a}(\theta_1-\theta_2)A_{b'}(\theta_1)A_{a'}^\dagger(\theta_1). \end{aligned}$$

Ground State:

$$A_a(\theta)|0\rangle = 0\ .$$

Scattering States:

$$|\theta_1,\ldots,\theta_n\rangle_{a_1\ldots a_n} = A_{a_1}^\dagger(\theta_1)\ldots A_{a_n}^\dagger(\theta_n)|0\rangle$$

$$E_n = \sum_{k=1}^{n}\Delta\cosh\theta_n\ , \qquad P_n = \sum_{k=1}^{n}\frac{\Delta}{v}\sinh\theta_n.$$

Form Factors:

$$f^{\mathcal{O}}(\theta_n,\ldots,\theta_1)_{a_n\ldots a_1} = \langle 0|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_n\rangle_{a_1\ldots a_n}$$

Idea: analytic properties of S-matrix ⟶ analytic properties of form factors ⟶ form factors.

Form Factor Axioms

$f^{\mathcal{O}}(\theta_1, \ldots, \theta_n)_{a_1 \ldots a_n}$ are meromorphic functions of θ_n in the strip $0 \leq \mathrm{Im}(\theta_n) < 2\pi$ that fulfil

1. **Scattering Axiom**

$$\begin{aligned} f^{\mathcal{O}}_{a_1,\cdots,a_{i+1},a_i,\cdots,a_n}(\theta_1,\cdots,\theta_{i+1},\theta_i,\cdots,\theta_n) \quad &= \\ S^{a'_i,a'_{i+1}}_{a_i a_{i+1}}(\theta_i-\theta_{i+1}) f^{\mathcal{O}}_{a_1,\cdots,a'_i,a'_{i+1},\cdots,a_n}(\theta_1,\cdots,\theta_i,\theta_{i+1},\cdots,\theta_n). \end{aligned}$$

2. **Periodicity Axiom:** $f^{\mathcal{O}}_{a_1,\cdots,a_n}(\theta_1,\cdots,\theta_n) = f^{\mathcal{O}}_{a_n,a_1,\cdots,a_{n-1}}(\theta_n - 2\pi i, \theta_1, \cdots, \theta_{n-1}).$

3. **Annihilation Pole Axiom**

$$\begin{aligned} i\ \mathrm{res}_{\theta_n=\theta_{n-1}+\pi i} f^{\mathcal{O}}(\theta_1,\cdots,\theta_n)_{a_1,\cdots,a_n} \quad &= \quad f^{\mathcal{O}}(\theta_1,\cdots,\theta_{n-2})_{a'_1,\cdots,a'_{n-2}} \delta_{a_n a'_{n-1}} \\ \times\Bigg(\delta^{a'_1}_{a_1}\delta^{a'_2}_{a_2}\cdots\delta^{a'_{n-2}}_{a_{n-2}}\delta^{a'_{n-1}}_{a_{n-1}} - S^{a'_{n-1}a'_1}_{\tau_1 a_1}(\theta_n-\theta_1) S^{\tau_1 a'_2}_{\tau_2 a_2}(\theta_n-\theta_2)\cdots S^{\tau_{n-3}a'_{n-2}}_{a_{n-1}a_{n-2}}(\theta_{n-1}-\theta_{n-2})\Bigg). \end{aligned}$$

4. **Lorentz Invariance:** $f^{\mathcal{O}}_{a_1\cdots a_n}(\theta_1+\alpha,\cdots,\theta_n+\alpha) = e^{s\alpha} f^{\mathcal{O}}_{a_1\cdots a_n}(\theta_1,\cdots,\theta_n).$

$T = 0$ Dynamical Correlation Functions

$$\langle 0|\mathcal{O}(t,x)\ \mathcal{O}^\dagger(0,0)|0\rangle \ = \ \sum_{n=0}^{\infty}\sum_{\{a_j\}}\int \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} e^{-i\Delta\sum_{j=1}^{n} t\cosh\theta_j - x/v\sinh\theta_j} \times \ \left|\langle 0|n^a(0,0)|\theta_1,\ldots,\theta_n\rangle_{a_1\ldots a_n}\right|^2$$

Need Fourier transform for applications ⟶

- first few terms $n = 1, \ldots 4$ give very accurate (1% level) results at $\omega < 10\Delta$.
- large $\omega \gg \Delta$ ⟶ need many terms in expansion ⟶ use RG improved PT instead.

⟶ **Accurate Results for all ω.**

Recent Developments: $T > 0$ Dynamical Correlation Functions in theories with nontrivial S-matrix

$$\frac{\mathrm{tr}\left[e^{-\beta H} n^a(x,t)\ n^a(0)\right]}{\mathcal{Z}} = \frac{1}{\mathcal{Z}}\sum_{n,m=0}^{\infty}\sum_{a_j,b_k}\int \frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}\,\frac{d\theta'_1\dots d\theta'_m}{m!(2\pi)^m}\, e^{\Delta\sum_{j=1}^{m}(it-\beta)\cosh\theta_j - ix/v\sinh\theta_j}$$

$$\times\quad e^{-i\Delta\sum_{l=1}^{m} t\cosh\theta'_l - x/v\sinh\theta'_l}\ \left|{}_{a_n\dots a_1}\langle\theta_n,\dots,\theta_1|n^a(0,0)|\theta'_1,\dots,\theta'_m\rangle_{b_1\dots b_m}\right|^2$$

Partition Function: $\mathcal{Z} = \sum_{n=0}^{\infty}\sum_{a_1,\dots,a_n}\int \frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}\ {}_{a_n\dots a_1}\langle\theta_n,\dots,\theta_1|\theta_1,\dots,\theta_n\rangle_{a_1\dots a_n}$

Problems: (1) $\mathcal{Z}$ and FFs ill-defined as we work with infinite volume states
${}_a\langle\theta|\theta'\rangle_b = \delta(\theta-\theta')\delta_{ab}$ (2) Need to sum infinite number of terms in the spectral rep.