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Wandering in superspace

From Stony Brook to Belgium

Alexander Sevrin
Vrije Universiteit Brussel
and
The International Solvay Institutes

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Memories

The CNY ITP @ Stony Brook, the Ellis Island for European theoretical physicists…

- First encounter with PvN: Trieste Spring School in 1986…
- August 1988: start of postdoc career in Stony Brook.
- Starting with ***the book***, numerous collaborations later on!
- August 1991: leaving for the West Coast…(just missed hurricane Bob but found my wife).

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Memories

Stony Brook = great time

- The Italian house
- The parties with the music department
- The numerous scientific – and sometimes not so scientific – discussions with Barry, Bill, Frank, Fred, George, Jack, Martin, Peter, Robert, Vladimir & Warren.
- Peter's Friday seminars…
- Dinners with Marie, Peter & the kids…
- Sailing on the Sound and further away…
- …

Thank you!!!

Hopefully I brought back some of the Stony Brook spirit to Belgium and passed it on to my students…

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Introduction

- A bit of Stony Brook physics: the off-shell description and geometry of supersymmetric non-linear sigma models in two dimensions…
- An almost solved problem…

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Supersymmetry

- Supersymmetry (susy) = extension of the Poincaré group (translations P & Lorentz transformations M) with a number of fermionic transformations (Q).

$$[P,P]=0$$
$$[M,P]=P$$
$$[M,M]=M$$
$$[P,Q]=0$$
$$[M,Q]=Q$$
$$\{Q,Q\}=P$$

Note: different from susy in nuclear or statistical physics.

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Supersymmetry

- Representations of the supersymmetry algebra
  - All particles in a single irrep have the same mass
  - There are an equal number of bosons and fermionic degrees of freedom in a single irrep.

    E.g. chiral multiplet in 4 dimensions: one complex scalar (2 propagating bosonic dof) and one chiral spinor (2 propagating fermionic dof).

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Supersymmetry

Simple example in d=1, scalar $\phi(\tau)$ and spinor $\psi(\tau)$.

Transformations:

$$\delta\phi = i\,\varepsilon\,\psi$$

$$\delta\psi = -\varepsilon\dot{\phi}$$

Algebra:

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]\phi = 2i\,\varepsilon_1\varepsilon_2\,\dot{\phi}$$

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]\psi = 2i\,\varepsilon_1\varepsilon_2\,\dot{\psi}$$

Invariant action:

$$\mathcal{S} = \frac{1}{2}\int d\tau\,\left(\dot{\phi}\dot{\phi} + i\,\psi\dot{\psi}\right)$$

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Superspace

- Supersymmetry as a coordinate transformation? (Salam & Strathdee)

  - Simple d=1 example, introduce superspace, one bosonic coordinate τ and one fermionic (anticommuting) coordinate $\theta,\ \theta^2 = 0$.
    Supersymmetry = translation of the fermionic coordinate?

    $$\delta\theta = -\varepsilon$$

    Transformation on the bosonic coordinate determined by supersymmetry algebra,

    $$\delta\tau = i\,\theta\,\varepsilon$$

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Superspace

– Fields become superfields. E.g. scalar superfield,

$$\Phi(\tau,\theta)=\phi(\tau)+i\,\theta\,\psi(\tau)$$

Transforms as a scalar,

$$\Phi(\tau,\theta)\to\Phi'(\tau',\theta')=\Phi(\tau,\theta)$$

which becomes explicitely,

$$\Phi'(\tau',\theta')=\Phi'(\tau+\delta\tau,\theta+\delta\theta)=\Phi'(\tau,\theta)-i\,\varepsilon\,\psi(\tau)+i\,\theta\,\varepsilon\,\dot{\phi}(\tau)$$

or in components:

$$\delta\phi=i\,\varepsilon\,\psi$$
$$\delta\psi=-\varepsilon\dot{\phi}$$

Superspace

– Introduce fermionic derivative,

$$D \equiv \frac{\partial}{\partial\theta} - i\,\theta\frac{\partial}{\partial\tau}, \qquad D^2 = -i\,\frac{\partial}{\partial\tau}$$

– Introduce fermionic integral,

$$\int d\theta\, F(\tau,\theta) \equiv D\, F(\tau,\theta)\big|_{\theta=0}$$

– Manifestly invariant action,

$$\begin{aligned}
\mathcal{S} &= \frac{i}{2}\int d\tau\, d\theta\, \dot{\Phi} D\Phi \\
&= \frac{i}{2}\int d\tau\left(-i\,\dot{\Phi}\dot{\Phi} + D\dot{\Phi}D\Phi\right)\big|_{\theta\to 0} \\
&= \frac{1}{2}\int d\tau\left(\dot{\phi}\dot{\phi} + i\,\psi\dot{\psi}\right)
\end{aligned}$$

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Superspace

– If you want to know more about superspace, read the bible:

S.J. Gates, Marcus T. Grisaru, M. Roček, W. Siegel: SUPERSPACE OR ONE THOUSAND AND ONE LESSONS IN SUPERSYMMETRY: HEP-TH 0108200.

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Strings

- The position of a string is described by

$$X^a(\sigma^0,\sigma^1), \qquad \sigma^0 \in \mathbb{R}, \quad \sigma^1 \in [0,2\pi], \quad a \in \{1,\cdots,D\}$$

$$X^a(\sigma^0,\sigma^1+2\pi) = X^a(\sigma^0,\sigma^1)$$

- Geometry target manifold characterized by metric and closed 3-form.

- Supersymmetric non-linear sigma-models in two dimensions have deep roots in Stony Brook…

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Strings

- Luis Alvarez-Gaume, Daniel Z. Freedman, GEOMETRICAL STRUCTURE AND ULTRAVIOLET FINITENESS IN THE SUPERSYMMETRIC SIGMA MODEL, CMP 80:443,1981, cited 397 times.
- S.J. Gates, Jr., C.M. Hull, M. Roček, TWISTED MULTIPLETS AND NEW SUPERSYMMETRIC NONLINEAR SIGMA MODELS, NPB 248:157,1984, cited 402 times.
- T. Buscher, U. Lindström, M. Roček, NEW SUPERSYMMETRIC SIGMA MODELS WITH WESS-ZUMINO TERMS, PLB 202:94,1988, cited 47 times.
- M. Roček, K. Schoutens, A. Sevrin, OFF-SHELL WZW MODELS IN EXTENDED SUPERSPACE, PLB 265:303,1991, cited 86 times.
- Ulf Lindström, Martin Roček, Rikard von Unge, Maxim Zabzine, GENERALIZED KAHLER MANIFOLDS AND OFF-SHELL SUPERSYMMETRY, CMP 269:833-849,2007; [HEP-TH 0512164], cited 24 times.
- Ulf Lindström, Martin Roček, Rikard von Unge, Maxim Zabzine, LINEARIZING GENERALIZED KAHLER GEOMETRY, JHEP 0704:061,2007, [HEP-TH 0702126], cited 3 times.

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Susy sigma-models

- Supersymmetrize the sigma-model
  - Introduce fermions through N=(1,1) susy transformations:

$$\delta X^a = i\varepsilon^+\psi^a_+ + i\varepsilon^-\psi^a_-,$$
$$\delta\psi^a_+ = -\varepsilon^+\partial_{\ddagger}X^a + \dots,$$
$$\delta\psi^a_- = -\varepsilon^-\partial_{=}X^a + \dots.$$

  - Construct invariant action and complete transformation rules

$$\mathcal{S} = -\frac{1}{2}\int d^2\sigma\Big(\big(G_{ab}+B_{ab}\big)\partial_{\ddagger}X^a\partial_{=}X^b + i\,g_{ab}\,\psi^a_+\nabla^{(+)}_{=}\psi^b_+ + i\,g_{ab}\,\psi^a_-\nabla^{(-)}_{\ddagger}\psi^b_- + \frac{1}{2}R^{(-)}_{abcd}\psi^a_-\psi^b_-\psi^c_+\psi^d_+\Big)$$

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Susy sigma-models

– Transformation rules:

$$\delta X^a = i\varepsilon^+ \psi_+^a + i\varepsilon^- \psi_-^a,$$
$$\delta \psi_+^a = -\varepsilon^+ \partial_{\ddagger} X^a + i\,\varepsilon^-\,\Gamma^{\;a}_{(+)bc}\psi_+^b \psi_-^c,$$
$$\delta \psi_-^a = -\varepsilon^- \partial_{=} X^a + i\,\varepsilon^+\,\Gamma^{\;a}_{(-)bc}\psi_-^b \psi_+^c.$$

– Algebra closes *on-shell*:

$$\begin{aligned}[\delta(\varepsilon_1^+), \delta(\varepsilon_2^+)]\psi_-^a \quad &= \quad 2i\,\varepsilon_1^+ \varepsilon_2^+\,\partial_{\ddagger}\psi_-^a \\ &\quad -2i\,\varepsilon_1^+ \varepsilon_2^+\,\left(\nabla_{\ddagger}\psi_-^a - \frac{i}{2} R^a_{(-)bcd}\psi_-^b \psi_+^c \psi_+^d\right)\end{aligned}$$

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Susy sigma-models

– How to find model independent representation? Solution: auxiliary fields! Off-shell degrees of freedom: D bosonic and 2 D fermionic, add D bosonic auxiliary fields F^a to balance:

$$
\begin{aligned}
\delta X^a &= i\varepsilon^+\psi^a_+ + i\varepsilon^-\psi^a_- ,\\
\delta\psi^a_+ &= -\varepsilon^+\partial_{\ddagger}X^a - \varepsilon^- F^a ,\\
\delta\psi^a_- &= -\varepsilon^-\partial_{=}X^a + \varepsilon^+ F^a ,\\
\delta F^a &= -i\varepsilon^+\partial_{\ddagger}\psi^a_- + i\varepsilon^-\partial_{=}\psi^a_+ .
\end{aligned}
$$

– Off-shell closure:

$$
\begin{aligned}
[\delta(\varepsilon_1^+),\delta(\varepsilon_2^+)] &= 2i\,\varepsilon_1^+\varepsilon_2^+\,\partial_{\ddagger} ,\\
[\delta(\varepsilon_1^-),\delta(\varepsilon_2^-)] &= 2i\,\varepsilon_1^-\varepsilon_2^-\,\partial_{=} ,\\
[\delta(\varepsilon^+),\delta(\varepsilon^-)] &= 0.
\end{aligned}
$$

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Susy sigma-models

- The invariant action,

$$\mathcal{S} = -\frac{1}{2}\int d^2\sigma\,\Big(\big(G_{ab}+B_{ab}\big)\partial_{\ddagger}X^a\partial_{=}X^b + i\,G_{ab}\,\psi^a_+\nabla^{(+)}_{=}\psi^b_+ + i\,G_{ab}\,\psi^a_-\nabla^{(-)}_{\ddagger}\psi^b_- + \frac{1}{2}R^{(-)}_{abcd}\psi^a_-\psi^b_-\psi^c_+\psi^d_+ + \big(F^a - i\Gamma^{\;a}_{(-)cd}\psi^c_-\psi^d_+\big)G_{ab}\big(F^b - i\Gamma^{\;b}_{(-)ef}\psi^e_-\psi^f_+\big)\Big).$$

- Eom for auxiliary field:

$$F^a = i\,\Gamma^{\;a}_{(-)bc}\psi^b_-\psi^c_+$$

  Implementing this in action and transformation rules gives back original situation.
- Finding auxilairy fields is a notoriously difficult problem (Stony Brook has a long tradition…)!

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Susy sigma-models

- N=(1,1) superspace
  - Introduce fermionic coordinates, θ^+ and θ^-,

$$\theta^+\,\theta^+ = \theta^-\,\theta^- = 0$$
$$\theta^+\,\theta^- = -\,\theta^-\,\theta^+$$

  - Introduce superfields:

$$\mathbb{X}^a(\sigma^{\ddagger},\sigma^{=},\theta^+,\theta^-) = X^a + i\,\theta^+\,\psi^a_+ + i\,\theta^-\,\psi^a_- + i\,\theta^+\theta^-\,F^a$$

  - Action:

$$\mathcal{S} = -\tfrac{1}{2}\int d^2\sigma\,d^2\,\theta\,\Big(G_{ab} + B_{ab}\Big)\,D_+\,\mathbb{X}^a\,D_-\,\mathbb{X}^b$$

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N=(2,2) sigma-models

- Introduction
  - Passing from the bosonic non-linear sigma-model to the N=(1,1) supersymmetric model did not give rise to any extra conditions. The geometry is still fully characterized by a manifold, a metric G_{ab} and a closed 3-form $T_{abc} = -\frac{1}{2}\big(\partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}\big)$.
  - Description of type II superstrings requires *two* left-handed and *two* right-handed supersymmetries (⇒ space-time susy):

$$N = (1,1) \Rightarrow N = (2,2)$$

Here: compactified sector $\Longrightarrow$ Euclidean signature

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N=(2,2) sigma-models

- More supersymmetry
  - Dimensional analysis $\Rightarrow$ extra susy has the form:

$$\delta\,\mathbb{X}^a = \varepsilon^+ J^a_{+\,b}(\mathbb{X}) D_+ \mathbb{X}^b + \varepsilon^- J^a_{-\,b}(\mathbb{X}) D_- \mathbb{X}^b$$

  - The transformation rules satisfy on-shell susy algebra $\Leftrightarrow$
    $\rightarrow$ almost complex structure: ${J_+}^2 = {J_-}^2 = -\mathbf{1}$
    $\rightarrow$ integrability: $N[J_+, J_+] = N[J_-, J_-] = 0$
    Nijenhuis tensor:

$$N^a{}_{bc}[A,B] \equiv A^d{}_{[b} B^a{}_{c],d} + A^a{}_d B^d{}_{[b,c]} + A \leftrightarrow B$$

  - The susy algebra closes off-shell as well $\Leftrightarrow$

$$[J_+, J_-] = 0$$

$$N[J_+, J_-] = 0$$

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N=(2,2) sigma-models

– The algebra works, what about invariance? Action invariant $\Leftrightarrow$

$\rightarrow$ metric is hermitean: $J_{\pm}{}^{c}{}_{a}\, J_{\pm}{}^{d}{}_{b}\, G_{cd} = G_{ab}$
$\rightarrow$ complex structures are covariantly constant: $\nabla^{(\pm)}_{c}\, J_{\pm}{}^{a}{}_{b} = 0$

– Conclusion: passing from N=(1,1) to N=(2,2), a lot of extra data/requirements, bihermitean complex manifold with covariantly constant complex structures. What is the geometry?

N=(2,2) superspace

– Introduce 4 fermionic coordinates, $\theta^{\pm}$ and $\hat{\theta}^{\pm}$. Introduce 4 fermionic derivatives as well, $D_{\pm}$ and $\hat{D}_{\pm}$,

$$D_+{}^2 = -i\,\partial_{\ddagger},\; D_-{}^2 = -i\,\partial_{=}\,,\; \hat{D}_+{}^2 = -i\,\partial_{\ddagger},\; \hat{D}_-{}^2 = -i\,\partial_{=}$$

All other anti-commutators vanish.

– Integration measure, $\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}$, is dimensionless. So lagrange density is a function of scalar (super)fields → geometry characterized by a potential! How do we get dynamics?

– Answer: impose constraints on the superfields! Basic superfields: chiral, twisted chiral and semi-chiral superfields.

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N=(2,2) superspace

– Note: $\ker\,[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)$

– General solution:

$$\ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \mathrm{coker}[J_+, J_-]$$

Chiral superfields

Twisted-chiral superfields

Semi-chiral superfields

(Lindstrom, Roček, Von Unge, Zabzine; Troost, AS; Bogaerts, AS, van der Loo, Van Gils; Ivonov, Kim, Roček; …)

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N=(2,2) superspace

– Some examples:

- Only chiral fields: all Kähler manifolds where one chooses $J_+ = J_-$.
- Chiral and twisted-chiral: WZW model on $SU(2) \times U(1) \cong S^3 \times S^1$ (Roček, Schoutens, AS)
- Semi-chirals only: hyper-Kähler where $J_+ = J_1$, $J_- = J_2$ and $\{J_1, J_2\} = 0$ (AS, Troost; Lindström, Roček, von Unge, Zabzine)
- Mixed cases: $SU(2) \times SU(2)$, (Ivanov, Kim, Roček; AS, Troost) + all WZW. (Roček, Schoutens, AS)
- ….

N=(2,2) superspace

– Chiral and twisted chiral fields are the basic ingredients, semi-chiral fields can be obtained through quotient construction involving only chiral and twisted chiral fields (Lindstrom, Roček, von Unge, Zabzine)^2.

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Outlook

- NS-NS fluxes are perhaps the simplest to study. Geometric framework is more or less ready… General case strikingly similar to Kahler case, a lot to explore…
- Extension to case with boundaries (D-branes), work in progress (AS, Staessens, Wijns):
  - Boundary at either

$$\sigma = 0,\ \theta^+ - \theta^- = \hat{\theta}^+ - \hat{\theta}^- = 0$$ (B-type boundary conditions)

or

$$\sigma = 0,\ \theta^+ - \theta^- = \hat{\theta}^+ + \hat{\theta}^- = 0$$ (A-type boundary conditions)

(Ooguri, Oz, Yin)

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Outlook

- Super Poincaré invariance broken ⇒ N=(2,2) → N=2 supersymmetry ⇒ N=2 boundary superspace.
- Choice of boundary conditions = choice of superfield content. E.g.:
  - A-branes on Kähler manifold = B boundary conditions with only twisted chiral superfields.
  - B-branes on Kähler manifold = B boundary conditions with only chiral superfields.
- Starting point (Lindström, Roček, van Nieuwenhuizen; Koerber, Nevens, AS):

$$\mathcal{S} = \int d^2\sigma d^2\theta\, D' \hat{D}'\, V(X, \bar{X}) + i \int d\tau d^2\theta\, W(X, \bar{X})$$

+ boundary conditions (non-trivial for twisted chiral).

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Outlook

- In presence of isometries, chiral ↔ twisted chiral duality persists!!! ⇒ mirror symmetry and more…
- Previous ⇒ important tool for the explicit construction of lagrangian and coisotropic D-brane configurations (model building (Font, Ibanez, Marchesano)). Subtle and intricate (Kapustin)!

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Outlook

- Without boundaries: fully controlled, with boundaries: probably fully under control as well.
- If RR-backgrounds could only be incorporated… (Berkovits).
- Dilaton ↔ N=(2,2) or N=2 supergravity.
- Alpha' corrections
- Extremely nice example of a situation where off-shell susy is fully understood! A lot of physics and mathematics results are still waiting!

HAPPY BIRTHDAY CNY ITP!