Bubbling AdS geometries

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Introduction

We are interested in giving the sugra description of BPS states in $\mathcal{N}=4$ SYM, more precisely to states that are associated to chiral operators such as $Tr(X^k Y^l Z^m)$ (multi-trace operators are also allowed).

X, Y, Z are chiral superfields which rotate into each other under the action of the $SO(6)$ R-symmetry group.

J_1, J_2, J_3 are the generators of the Cartan subgroup.

When gravitational backreaction is taken into account, the turning on of J_1, J_2 and J_3 in succession breaks the isometries of the five-sphere from $SO(6)$ to $SO(4)$, $SO(2)$ and finally the identity. Combining this with the unbroken $SO(4)$ isometry of s-wave states in AdS$_5$, the natural family of backgrounds we are interested in takes the form

supersymmetries	chiral primary	isometry
1/2 BPS	$\Delta = J_1$	$S^3 \times S^3$
1/4 BPS	$\Delta = J_1 + J_2$	$S^3 \times S^1$
1/8 BPS	$\Delta = J_1 + J_2 + J_3$	S^3

1/2 BPS (LLM) solutions

The $S^3 \times S^3$-preserving LLM solutions (Lin, Lunin, Maldacena):

$$\begin{aligned} ds_{10}^2 &= g_{\mu\nu} dx^\mu dx^\nu + e^H (e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2), \\ F_{(5)} &= (1 + *_{10}) F_{(2)} \wedge \Omega_3. \end{aligned}$$

All such 1/2 BPS states are describable in terms of a single harmonic function $Z = \frac{1}{2} \tanh G$

$$\left(\partial_1^2 + \partial_2^2 + y \partial_y \frac{1}{y} \partial_y \right) Z(x_1, x_2, y) = 0.$$

The resulting ten-dimensional metric is then of the form

$$ds_{10}^2 = -h^{-2}(dt + \omega)^2 + h^2(dx_1^2 + dx_2^2 + dy^2) + y(e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2)$$

where $h^{-2} = 2y \cosh G$.

The bubbling picture arises through the observation that regularity of the metric demands that only one of the three-spheres collapses in an appropriate manner as $y \to 0$:

$$Z(x_1, x_2, y = 0) = \pm \tfrac{1}{2}.$$

A typical boundary condition on the $y = 0$ plane paints black and white regions, corresponding to either S^3 shrinking to zero size:

- the AdS vacuum is represented by a black disk;
- the plane wave solution is given by a black half-plane;
- the KK gravitons are given by small ripples on the AdS disk;
- the giant gravitons are given by $i)$ small holes inside the AdS disk or $ii)$ small droplets outside the AdS.

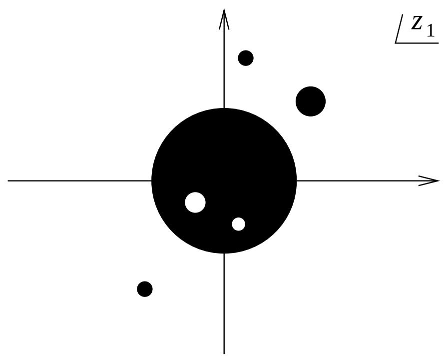

Figure 1: The LLM bubbling AdS picture

The complete form of Z may then be obtained through an appropriate Green's function solution to the Laplace eqn.

The LLM solutions admit a dual description in terms of a gauged SU(N) quantum mechanics matrix model with a harmonic oscillator potential (Berenstein; Corley, Jevicki, Ramgoolam). This is equivalent to a system of N^2 oscillators

$$H = \frac{1}{2}(a^\dagger)^i_j a^j_i + \frac{1}{2}N^2$$

An alternative description is obtained by going to the eigenvalue basis

$$L = \frac{1}{2}\dot{\lambda}_i^2 - \frac{1}{2}\lambda_i^2$$

which is accompagnied by a measure factor $\mu = \prod(\lambda_i - \lambda_j)$.

In this description, the eigenvalues are fermions.

The ground state for the N fermions corresponds to the Fermi sea, and the spectrum is given by exciting the Fermi surface. In the phase space of a single eigenvalue, the ground state is pictured as round disk. The excitations of the Fermi sea mirror the bubbling AdS LLM boundary conditions.

The 1/4 BPS solutions

The 1/4 BPS solutions have an $S^3 \times S^1$ isometry:

$$ds_{10}^2 = -h^{-2}(dt+\omega)^2 + h^2[2(Z+\tfrac{1}{2})^{-1}\partial_i\partial_{\bar{j}}K dz^i dz^{\bar{j}} + dy^2] + y[e^G d\Omega_3^2 + e^{-G}(d\psi + \mathcal{A})^2],$$

where

$$h^{-2} = 2y\cosh G, \qquad Z = \tfrac{1}{2}\tanh G.$$

The LLM function Z is related to the Kahler potential

$$Z = -\frac{1}{2}y\partial_y \frac{1}{y}\partial_y K(z_i, \bar{z}_{\bar{i}}; y),$$

and furthermore the Kähler metric must satisfy a Monge-Ampère-type equation:

$$\log\det h_{i\bar{j}} = \log(Z+\frac{1}{2}) + n\eta\log y + \frac{1}{y}(2-n\eta)\partial_y K + D(z_i, \bar{z}_{\bar{j}}),$$

where $D(z_i, \bar{z}_{\bar{j}})$ arises as an integration constant. The Ricci form on the base must satisfy

$$\mathcal{R} = i\partial\bar{\partial}\log\det h_{i\bar{j}} = i\left(2i\eta\mathcal{F} + (2-n\eta)\frac{1}{y}\partial\bar{\partial}\partial_y K + \partial\bar{\partial}\log(Z+\frac{1}{2})\right),$$

where $\mathcal{F} = d\mathcal{A}$. As a consequence, $\partial\bar{\partial}D = 2i\eta\mathcal{F}$ and ultimately

$$(1+*_4)\partial\bar{\partial}D = \frac{4}{y^2}(1-n\eta)\partial\bar{\partial}K.$$

The LLM solutions

The only solutions which are obtained when considering a decomposable base $K = K_1(z_1, \bar{z}_1, y) + K_2(z_2, \bar{z}_2, y)$, with $Z = Z(z_1, \bar{z}_1, y), D = D(z_2, \bar{z}_2, y)$ and taking the U(1) charge $2 = n\eta$, are the LLM solutions.

The simplifications that follow from our assumptions allow to factorize the Monge-Ampère equation

$$\partial_1 \partial_{\bar{1}} K_1 = \frac{1}{2}(Z + \frac{1}{2})$$

together with a Liouville equation for D:

$$\partial_2 \partial_{\bar{2}} D + 2e^D = 0.$$

So now the 4d base reads

$$ds_4^2 = (Z + \frac{1}{2}) dz_1 \bar{d}z_1 + y^2 ds^2(CP^1)$$

corresponding to the LLM embeddings.

1/8 BPS solutions

These solutions have only an S^3 isometry. Following susy analysis we arrive at:

$$ds^2_{10} = -y^2(dt+\omega)^2 + \frac{1}{y^2} h_{ij} dx^i dx^j + y^2 d\Omega_3^2$$

$$F_{(5)} = F_{(2)} \wedge \omega_{(3)} + \frac{1}{y^3} *_7 F_{(2)}.$$

The 6d base is Kahler with the Kahler potential constrained by

$$\Box_6 R = -\tfrac{1}{8} R_{ij} R^{ij} + \tfrac{1}{2} R^2.$$

The one-form ω is related to the Ricci form

$$\mathcal{R} = i R_{i\bar{j}} dz^i \wedge d\bar{z}^j = 2\eta d\omega.$$

In addition, $F_{(2)}$ and the scalar curvature R satisfy

$$F = d[y^4(dt+\omega)] - 2\eta J, \qquad R = -\frac{8}{y^4},$$

where R is the scalar curvature of h_{ij} (also Kim, Gava et al.).

Example: $AdS_5 \times S^5$

$$ds^2_{10} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\Omega_5^2,$$

Identifying the S^3 in AdS$_5$ with the S^3 of the 1/8 BPS ansatz yields $y = \sinh\rho$. After introducing the complex coordinates

$$\begin{aligned} z_1 &= r\cos\theta_1 e^{i\phi_1}, \\ z_2 &= r\sin\theta_1\cos\theta_2 e^{i\phi_2}, \qquad z_3 = r\sin\theta_1\sin\theta_2 e^{i\phi_3} \end{aligned}$$

and shifting $\phi_i = \psi_i - t$ we obtain

$$ds^2_7 = -\sinh^2\rho(dt+\omega)^2 + \sinh^{-2}\rho\left(\sinh^2\rho\, d\rho^2 - \cosh^2\rho\frac{dr^2}{r^2} + \sinh^2\rho\frac{|dz_i|^2}{r^2} + \frac{|\bar{z}_i dz_i|^2}{r^4}\right),$$

To eliminate the original ρ coordinate, we define $r = \cosh\rho$. The resulting 6d base metric

$$ds^2_6 = (|z_i|^2 - 1)\frac{|dz_i|^2}{|z_i|^2} + \frac{|\bar{z}_i dz_i|^2}{(|z_i|^2)^2},$$

is Kähler, with K given by

$$K = \tfrac{1}{2}|z_i|^2 - \tfrac{1}{2}\log(|z_i|^2).$$

The complex coordinates z_i cover the space completely, and are furthermore restricted to the region $|z_i|^2 \geq 1$, since $r = \cosh(\rho) \geq 1$.

Moreover, since

$$y^2 = \sinh^2 \rho = |z_i|^2 - 1,$$

then y naturally parameterizes the radial direction in $\mathbb{C}^3$ starting from the unit five-sphere on outward.

So the AdS$_5 \times S^5$ vacuum corresponds to removing a round ball from the Kähler base.

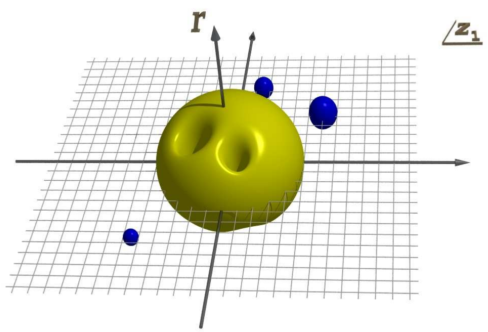

Figure 2: An LLM configuration shown as droplets in the 6d base. Here $r^2 = |z_2|^2 + |z_3|^2$, and the additional S^3 directions are suppressed. The physical space is comprised of the region outside of the droplets only.

Mini-summary

- We performed the $S^3 \to S^3 \times S^1 \to S^3 \times S^1 \times CP^1$ chain of reductions.
- After susy analysis, which involved solving algebraic and differential equations (derived from the Killing spinor equation) for Killing spinor bilinears we found that sugra solutions which have preserve 1/8, 1/4 , 1/2 susy. These solutions have a Killing vector which is time-like (or null).
- The 1/8 solutions are time-fibered over a 6d Kahler base; the Kahler potential obeys a non-linear differential equation.
- The 1/4 BPS solutions are built on a 4d Kahler base. The Kahler potential obeys a non-linear Monge-Ampère type equation. There is a special coordinate y which measures the volume of the $S^3 \times S^1$ factors.
- The 1/2 BPS solutions are built on a 2d base which is flat. There is a special y coordinate which measures the volume of the $S^3 \times S^3$ factors. The sugra solution is fully known in terms of the solutions Z of a linear (Laplace) differential equation. The regular solutions obey a certain boundary condition ($Z \to \pm\frac{1}{2}$) which lead to the bubbling AdS picture.

Regularity analysis and the droplet picture

- What to expect for 1/4 and 1/8 solutions:
- The N=4 SYM sector, when truncated to the X, Y, Z chiral superfield sector can still be described by a gauged quantum mechanics matrix model.

$$L = \frac{1}{2}(D_t\phi_i^2 - \phi_i^2) - \frac{1}{4}\sum_{i,j}[\phi_i, \phi_j]^2$$

  However, the description in terms of free fermions is no longer possible, and the bosons have non-trivial interactions (Berenstein).
- By going to the eigenvalue basis, the eigenvalues have log-repulsive interactions and confined by a harmonic oscillator potential.
- The ground state configuration for 1/4 BPS states corresponds to having the eigenvalues distributed on an S^3, whereas in the 1/8 BPS case, the eigenvalues lie on a S^5 (Berenstein).
- There is a natural map between the space of matrix eigenvalues and the n complex-dimensional base spaces of the $1/2^n$ BPS sugra solutions.
- The AdS ground state corresponds to an S^{2n-1} surface which delineates the the regions where the S^3 shrinks to zero size from those where another S^3, or the S^1 collapse. Lastly, in the 1/8 BPS case, we found that the space ends at the S^5.

A generic 1/8 BPS regularity analysis

The 1/8 BPS configurations, whose full ten-dimensional metric is of the form

$$ds_{10}^2 = -y^2(dt+\omega)^2 + \frac{2}{y^2}\partial_i\partial_{\bar{j}}K dz^i d\bar{z}^{\bar{j}} + y^2 d\Omega_3^2,$$

have a potentially singular locus at $y=0$.

To avoid this singularity, the ten-dimensional metric must take the form

$$ds_{10}^2 = -y^2(dt+\omega)^2 + \frac{1}{y^2}\left(y^2dy^2 + y^2d\Sigma_4^2 + \mathcal{N}_\psi^2(d\psi+A)^2\right) + y^2 d\Omega_3^2, \qquad y \ll 1.$$

Assume that

$$y^2 \equiv F(r_1^2, r_2^2, r_3^2).$$

Next we look for a Kähler potential which will give, in the region near $y=0$, a metric of the form

$$ds_6^2 = y^2dy^2 + y^2d\Sigma_4^2 + \mathcal{N}_\psi^2(d\psi+A)^2.$$

Such a Kähler potential is

$$K(z^i, z^{\bar{j}}) = \frac{1}{4}y^4 + \mathcal{O}(y^6),$$

Any potentially singular behavior as $y \to 0$ would come from the following two-dimensional part of the ten-dimensional metric

$$ds_2^2 = -y^2(dt + \omega)^2 + \frac{1}{y^2 f_y^2}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)^2.$$

Recall that the one-form ω is determined by the Kähler potential of the six-dimensional base

$$2d\omega = \mathcal{R},$$

where $\mathcal{R}$ is the Ricci form of the base. Then, in a local patch

$$\omega = \omega_i dz^i + \bar{\omega}_{\bar{j}} d\bar{z}^{\bar{j}}, \qquad \omega_i = -\frac{i\eta}{8}\partial_i \log(\det h_{mn}), \qquad \bar{\omega}_{\bar{j}} = (\omega_j)^*.$$

Since $\det h_{mn} = \mathcal{O}(y^8)$ in the coordinate system of $\{r_i, \phi_i\}$, then to leading order in y

$$\omega = \frac{1}{y^2 f_y}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3) + \mathcal{O}(y^0),$$

and the potentially singular terms cancel.

To summarize, we have investigated the region of the 1/8 BPS solutions near $y = 0$. Assuming a toric base, we have seen that the $y = 0$ locus is a five-dimensional surface Σ_5 specified by

$$F(r_1^2, r_2^2, r_3^2) = 0.$$

Furthermore, the y coordinate is orthogonal to Σ_5.

The complete ten-dimensional solution is generated by choosing an arbitrary smooth (generally disconnected) five-dimensional surface embedded in the six-dimensional Kähler base. Then the ten-dimensional solution will be non-singular provided that, in the vicinity of the Σ_5 surface,

$$K = \frac{1}{2} y^4 + \mathcal{O}(y^6)$$

The full Kähler potential is obtained by evolving the approximate $K = y^4/4 + \mathcal{O}(y^6)$ according to

$$\Box_6 R = -R_{mn} R^{mn} + \frac{1}{2} R^2,$$

We verified this equation and that $R = -\frac{8}{y^4}$ to leading order in y.

Conclusions: A universal bubbling AdS picture

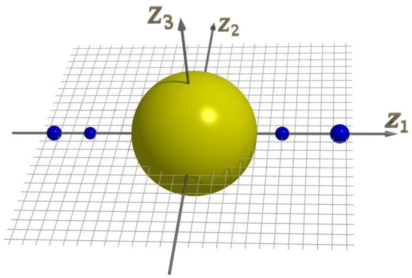

Figure 3: Schematic picture of a 1/2 BPS configuration. These 1/2 BPS configurations always preserve an $\tilde{S}^3$ invariance corresponding to rotations in the z_2-z_3 planes.

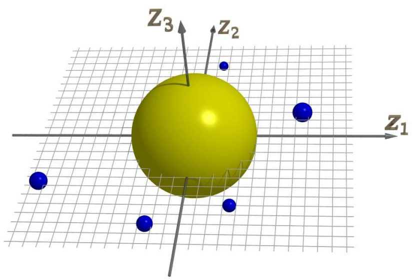

Figure 4: Picture of a 1/4 BPS configuration. The configuration is symmetric under S^1 rotations in the z_3 plane.

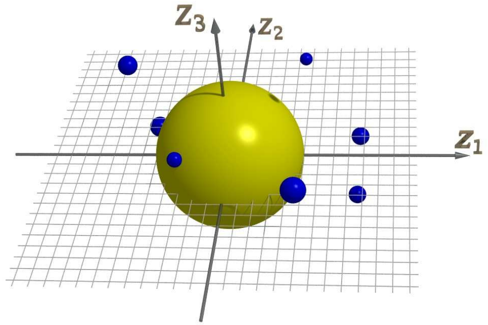

Figure 5: Schematic picture of a 1/8 BPS configuration. In general, 1/8 BPS droplets may have any topology and geometry allowed by regularity.