HIGHER SPINS LICHNEROWICZ ALGEBRAS & SPINNING PARTICLES

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MASSLESS HIGHER SPINS

spin s $\qquad \varphi_{\mu_1 \dots \mu_s}$ $\qquad$ TOTALLY SYMMETRIC

$$\varphi^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma \mu_5 \dots \mu_s} = 0$$ double-trace-free

gauge invariance

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \dots \mu_s)}$$

$$\xi^{\rho}{}_{\rho \mu_4 \dots \mu_s} = 0$$ trace-free

counting $(d=4)$

$$2 = \frac{4.5\dots(4+s-1)}{s!} - \frac{4.5\dots(4+s-5)}{(s-4)!}$$

$$- 2 \times \left(\frac{4.5\dots(4+s-2)}{(s-1)!} - \frac{4.5\dots(4+s-4)}{(s-3)!} \right)$$

NEED

DYNAMICS ON SPINNING SPACE

spinning particles

CALCULUS OF TENSORS w/ ARBITRARY INDEX STRUCTURE

Lichnerowicz Super Algebras

LICHNEROWICZ WAVE OPERATOR

$$\boxed{\begin{aligned}\Delta^{(s)}\varphi_{\mu_1\cdots\mu_s} \equiv\; & D^\rho D_\rho \varphi_{\mu_1\cdots\mu_s} \\ & + s(s-1)\, R_{(\mu_1}{}^{\rho}{}_{\mu_2}{}^{\sigma}\, \varphi_{\mu_3\cdots\mu_s)\rho\sigma} \\ & + s\, R_{\rho(\mu_1}\varphi_{\mu_2\cdots\mu_s)}{}^{\rho}\end{aligned}}$$

Curvature modified Laplacian.

PROPERTIES

$$\boxed{\begin{aligned} &g^{\mu\nu}\Delta^{(s)}\varphi_{\mu\nu\mu_3\ldots\mu_s} = \Delta^{(s-2)}\varphi^{\rho}{}_{\rho\mu_3\ldots\mu_s} \\ &\Delta^{(s+2)}g_{(\mu_1\mu_2}\varphi_{\mu_3\ldots\mu_s)} = g_{(\mu_1\mu_2}\Delta^{(s)}\varphi_{\mu_3\ldots\mu_s)} \\ &D^{\mu}\Delta^{(s)}\varphi_{\mu\mu_2\ldots\mu_s} \overset{*}{=} \Delta^{(s-1)}D^{\mu}\varphi_{\mu\mu_2\ldots\mu_s} \\ &\Delta^{(s+1)}D_{(\mu_1}\varphi_{\mu_2\ldots\mu_{s+1})} \overset{*}{=} D_{(\mu_1}\Delta^{(s)}\varphi_{\mu_2\ldots\mu_{s+1})} \end{aligned}}$$

* LOCALLY SYMMETRIC SPACE

CLAIM BOSONIC SUPERSYMMETRY ALGEBRA

LEFSCHETZ ALGEBRA

$N = 4$ SUSY, KÄHLER MANIFOLD

$$\partial\partial^* + \partial^*\partial = \tfrac{1}{2}\Delta_F = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

Kähler form $\quad \omega_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}} = \omega$

GRADING

$$SO(3)\left\{\begin{array}{l} N = \underbrace{p + q}_{\text{hol/anti hol. degrees}} - \dim_M \\ g : \alpha^{(p,q)} \longmapsto \omega \wedge \alpha^{(p+1,q+1)} \\ tr : \alpha^{(p,q)} \longmapsto * \alpha \wedge * \omega^{(p-1,q-1)} \end{array}\right.$$

DOUBLETS

$$(\partial, \bar\partial^*), \quad (\bar\partial, \partial^*)$$

SYMMETRIC TENSORS

= (WAVE) FUNCTIONS

DIFFERENTIAL FORMS

$$\omega = \omega_{\mu_1 \cdots \mu_k} dx^{\mu_1} \cdots dx^{\mu_k}$$

$$\boxed{dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu}$$

SYMMETRIC TENSORS

$$\varphi = \varphi_{\mu_1 \cdots \mu_s} dx^{\mu_1} \cdots dx^{\mu_s}$$

$$\boxed{dx^\mu dx^\nu = + dx^\nu dx^\mu}$$

Example

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = ds^2(dx^\mu)$$

LINE INTERVAL IS FUNCTION OF dx^μ

ALGEBRA OF DIFFERENTIALS

$$\Gamma(\odot T^*M) \ni \Phi(dx^\mu)$$ ANALYTIC

SUMS OF TENSORS
& DIFFERENT #
INDICES OK.

CALL

$$\partial_\mu = \frac{\partial}{\partial(dx^\mu)} \neq \frac{\partial}{\partial x^\mu}$$

$$\Rightarrow [\partial_\mu, dx^\nu] = \delta_\mu^\nu$$

String Theory,
Labastida
Vasiliev
→ commuting spinors

OPERATORS

$sl(2,\mathbb{R})$

$N = dx^\mu \partial_\mu$ INDEX

$tr = g^{\mu\nu}(x)\partial_\mu\partial_\nu$ TRACE

$g = g_{\mu\nu} dx^\mu dx^\nu$ METRIC

DOUBLET "SUSIES"

$div = \partial^\mu D_\mu$ DIVERGENCE

$grad = dx^\mu D_\mu$ GRADIENT

"SUPER" ALGEBRA

$$[\mathrm{div}, \mathrm{grad}] = \Delta - \tfrac{1}{4} R^{\#\#}$$

$$= \Box - \tfrac{1}{2} R^{\#\#}$$

$$\overset{*}{=} \Box - 2c$$

* CONSTANT CURVATURE

LICHNEROWICZ WAVE OPERATOR

$$\Box\, \varphi_{\mu_1\cdots\mu_s}\, dx^{\mu_1}\cdots dx^{\mu_s} = \Delta^{(s)} \varphi_{\mu_1\cdots\mu_s}\, dx^{\mu_1}\cdots dx^{\mu_s}$$

SYMMETRIC SPACES

$\Box$ IS CENTRAL

$$c = g\,\mathrm{tr} - N(N+d-2)$$

SL(2,IR) CASIMIR.

WAVE FUNCTIONS?
WHAT MODEL?

DIFFERENTIAL FORMS

$\omega_{[\mu_1\cdots\mu_k]}$

SYMMETRIC TENSORS

$\varphi_{(\mu_1\cdots\mu_s)}$

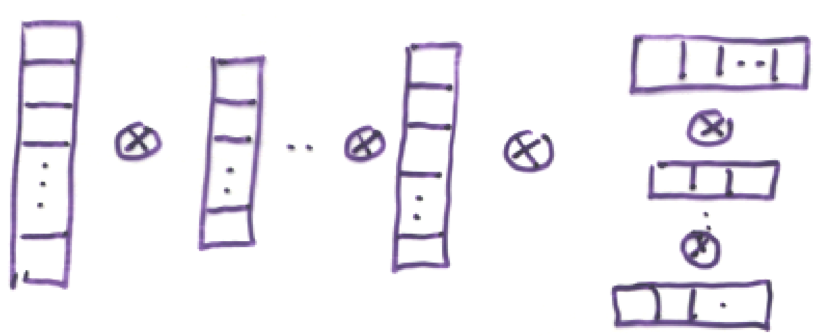
RIEMANN-LIKE TENSORS

$R_{[\mu\nu][\rho\sigma]}$

SPINOR TENSORS, GRAVITINI

$\psi^{\alpha}_{\mu_1\cdots\mu_k}$

ARBITRARY TENSORS



$\times$ spinors

SPINNING PARTICLES

$$S = \frac{1}{2}\int dt \left\{ \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu + i X^\mu_\alpha \frac{DX^\alpha_\mu}{dt} + \frac{1}{4} R_{\mu\rho}{}^{\nu\sigma} X^\mu_\alpha X^\alpha_\nu X^\rho_\beta X^\beta_\sigma \right\}$$

Energy integral for geodesic motion

Bose & Fermi spin degrees of freedom

4-fermi/box interactions

$$X^\mu_\alpha \in (\odot T^*M)^{\otimes 2p} \otimes (\Lambda M)^{\otimes 2q} \otimes S$$

DICTIONARY

$|\Psi\rangle \quad \leftrightarrow \quad$ tensor spinors

$-2H \quad \leftrightarrow \quad \Box$ Lichnerowicz wave op.

Q's $\quad \leftrightarrow \quad$ "Super Lichnerowicz Algebra"

SPINNING DEGREES OF FREEDOM

SUPER OSCILLATORS

$$\left[X^\mu_\alpha , X^\nu_\beta\right\} = g^{\mu\nu} J_{\alpha\beta} = \begin{cases} \begin{pmatrix} \begin{matrix} & -\mathbb{1}_{p\times p} \\ \mathbb{1}_{p\times p} & \end{matrix} & \\ & \begin{matrix} & \mathbb{1}_{q\times q} \\ \mathbb{1}_{q\times q} & \end{matrix} \end{pmatrix} \\ \begin{pmatrix} \begin{matrix} & -\mathbb{1}_{p\times p} \\ \mathbb{1}_{p\times p} & \end{matrix} & \\ & \begin{matrix} & \mathbb{1}_{q\times q} \\ \mathbb{1}_{q\times q} & \end{matrix} \end{pmatrix} \end{cases}$$

FOCK SPACE-CREATION OPERATORS PRODUCE TENSOR INDICES

SYMMETRIES

$J_{\alpha\beta}$ $Osp(2p|Q)$ INVARIANT

$$\delta X^\mu_\alpha = J_{\beta\alpha} \Lambda^{\gamma\beta} X^m_\gamma$$

osp(2p|Q) "Lefschetz" ALGEBRA

$$f_{\alpha\beta}$$

WORLDLINE

$$\delta x^\mu = \xi \dot{x}^\mu , \quad \delta X^\mu_\alpha = \xi \dot{X}^\mu_\alpha \qquad H$$

LICHNEROWICZ WAVE OPERATOR

$-\frac{1}{2}\Box$

"SUPER" SYMMETRY*

$$\delta x^\mu = i X^\mu_\alpha \varepsilon^\alpha$$

Osp(2p|Q) fundamental representation

$$\delta X^\mu_\alpha = \dot{x}^\mu \varepsilon_\alpha$$

covariant variation

v_α

DIFFERENTIAL OPERATORS

div

grad

d

δ

$\not{D}$

* LOCALLY SYMMETRIC SPACE (δ "fermi" variations)

QUANTUM SYMMETRY ALGEBRA

ANY RIEMANNIAN MANIFOLD

$$\left[v_\alpha , v_\beta \right\} = J_{\alpha\beta}\, \Delta - \tfrac{1}{2} X^\mu_\alpha X^\nu_\beta R^{\#}_{\mu\nu}$$

$$\left[f_{\alpha\beta} , v_\gamma \right\} = 2\, v_{[\alpha} J_{\beta]\gamma}$$

$$\left[f_{\alpha\beta} , f_{\gamma\delta} \right\} = 2\, v_{[\alpha} J_{\beta]\gamma}$$

$$\left[\Box , f_{\alpha\beta} \right\} = 0 \qquad \Box = \Delta + \tfrac{1}{4} R^{\#\#}$$

SYMMETRIC SPACES $\leftarrow$ $\left[\Box , v_\alpha \right] = 0$