Anton Rebhan Thermal Field Theory I
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Thermal Field Theory
(incl. Chemical Potential)

QFT + SM
From non-rel. → rel.
1960 Silin 1960, Fradkin 1965 (both Lebedev)
EW 1972 Kirzhnitz & Linde (applications to early universe: EW phase transitions)

QCD {
1975 Collins & Perry: TFT in QCD → pert. methods at high enough $T, \mu \gg \Lambda_{QCD}$ (augmented by statistical methods)
developed ⟼ IR problems ↓
1990: Braaten & Pisarski
HTL (Hard Thermal Loops)
[later HDL: D = dense]
}

mid 90's: EW theory sphaleron enhancement of B {production/violation} by high-T. By late 90's M_{Higgs} ⟼ not in Std. Model, but perhaps w/SUSY

Main current interest is in QCD

Quantum Statistical Mechanics

Grand canonical ensemble

↳ particle number

pure states ⟼ density operator

$$\rho = \sum_n p_n |n\rangle\langle n| \text{, generalizes } \rho_{pure} = |\psi\rangle\langle\psi|$$

$$\sum p_n = 1$$

statistical average: thermal expectation

$$\langle A \rangle = \mathrm{Tr}[\rho A]$$
$$= \sum_n p_n \langle n | A | n \rangle$$

entropy

$$S = \langle -\ln\rho \rangle = -\sum_n p_n \ln p_n$$

conserved quantities

$$E = \langle H \rangle, \quad N = \langle N \rangle$$ mean values prescribed

$$[H, N] = 0$$

Max. entropy generalized to
minimize $\sum p_n \ln p_n + \lambda(\sum p_n - 1) + \alpha(\sum p_n N - N)$

$$+ \beta(\sum p_n E_n - E)$$

$$0 = \frac{\partial}{\partial p_k} \longmapsto \quad 0 = \log p_k + 1 + \lambda + \alpha n_k + \beta E_k$$

$$\Rightarrow \quad p_k = e^{-1-\lambda-\alpha n_k - \beta E_k}$$

$$\Rightarrow \hat{\rho} = \underbrace{e^{-1-\lambda}}_{\substack{\text{normalize to} \\ [\mathrm{Tr}(e^{-\alpha\hat{N}-\beta\hat{H}})]^{-1} =: Z^{-1}(\beta,\alpha)}} e^{-\alpha\hat{N}-\beta\hat{H}}$$

Z: partition function

$$Z = \mathrm{Tr}\left[e^{-\alpha N-\beta H}\right]$$

$$\frac{\partial \ln Z}{\partial \alpha} = -\langle N\rangle$$

$$\frac{\partial \ln Z}{\partial \beta} = -\langle H\rangle = -E$$

$$\ln Z \equiv \beta \underbrace{Vp}_{-\Omega}$$

V volume
p thermodyn. pressure
Ω grand canonical thermodyn. potential

Entropy

$$S = \langle -\log\hat{\rho}\rangle = \ln Z + \langle \beta H + \alpha N\rangle$$

Alternate notation: $\alpha = -\mu\beta$

μ: chemical potential

$$\beta = \frac{1}{k_B T} \underset{\text{units}}{\Longrightarrow} \frac{1}{T}$$

Then $\hat{\rho} = \frac{1}{Z} e^{-\beta\underbrace{(H-\mu N)}_{\bar{H}\text{: shifted } H}}$

$$\Rightarrow S = \underbrace{\ln Z}_{\beta V p} + \beta E - \beta\mu N$$

$$\Rightarrow \quad TS = \underbrace{Vp}_{-\Omega} + E - \mu N$$

$$\Omega = \underbrace{E - TS}_{F \text{ Helmholtz free en.}} - \mu N$$

Issue of relativistic invariance

heat bath: $u^\mu = \delta^\mu{}_0$ preferred rest frame

can generalize

$$\hat{H} = u_\mu \hat{P}^\mu$$

$$\hat{N} = u_\mu \hat{J}^\mu$$

sometimes: $\beta_\mu = \beta u_\mu$

then

$$Z = \mathrm{Tr}\left[\exp \int d\Sigma_\mu \left[-\beta_\nu \hat{T}^{\mu\nu} - \alpha \hat{j}^{\mu}\right.\right.$$

$$T^{\mu\nu} = (E + P) u^\mu u^\nu - \eta^{\mu\nu} P$$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1_{3\times 3} \end{pmatrix}$$

$$Z = \mathrm{Tr}\, e^{-\beta \bar{H}} \qquad \bar{H} = H - \mu_i N_i$$

(suppress N_i terms)

$$= \sum_n e^{-\beta E_n}$$

$$= \int dq \langle q | e^{-\beta H} | q \rangle \qquad [q]\text{-coordinates}$$

Recall:
Path integrals:

$$\langle q'' | e^{-iH(t''-t)} | q' \rangle$$

$$= \mathcal{N} \int_{\substack{q(t') = q' \\ q(t'') = q''}} \mathcal{D}q(t)\, \exp \frac{i}{\hbar} \int_{t'}^{t''} dt\, L(q, \dot{q})$$

Make contact by identifying

$$\begin{cases} -\frac{i}{\hbar}(t''-t') = -\beta \\ q'' = q' = q \end{cases}$$

Path integral representation

$$Z = \mathcal{N} \int dq \int_{q(t') = q(t'') = q} \mathcal{D}q(t)\, \exp \frac{i}{\hbar} \int_0^{-i\beta\hbar} dt\, L$$

$t' = 0$, $-\frac{it''}{\hbar} = -\beta$

imaginary time

$t = -i\tau$

exponent becomes

$$\exp \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \, \underbrace{L}\,(t = -i\tau)$$

$$L = -L_E(\tau)$$

$q(t) \longrightarrow \phi(\vec{x}, t) \qquad L_E = \frac{1}{2}\left(\frac{\partial \phi}{\partial \tau}\right)^2 + (\vec{\nabla}\phi)^2 + V(\phi)$

$\phi \mapsto q$ for QM   for F.T.

Combine

$$\int dq \int_{q(t') = q(t'') = q} \mathcal{D} q(t) \longmapsto \int_{\text{periodic}} \mathcal{D} q$$

same general steps work for fermions: Grassman variables with <u>anti</u>-periodic boundary conditions

Perturbation theory:
Generating functionals

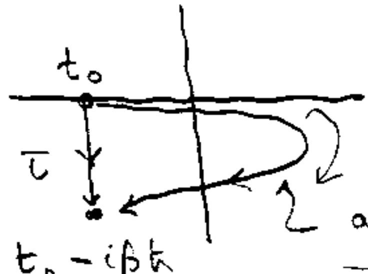
$$Z[j] = N \int \mathcal{D}\phi \, e^{i \int_C d^4x \, (\mathcal{L}(x) + j(x)\phi(x))}$$

C path $t_0 \to t_0 - i\beta\hbar$

Then

$$\langle T \phi(x_1) \cdots \phi(x_n) \rangle_0 = (-i)^n \frac{\delta Z(j)}{\delta j(x_1) \cdots j(x_n)}$$

Time contour



"contour time"

alternate paths possible

Then TO $\rightarrow$ ordering on $\mathcal{C}$

$$T \longrightarrow T_C$$

Keep $\mathcal{C}$ in LHP: otherwise get

$$e^{-\frac{i}{\hbar} H (\mathrm{Re}\,\Delta t)} \, e^{H \,\mathrm{Im}\,\Delta t}$$

with $\mathrm{Im}\,\Delta t > 0$, lose convergence

Straight line: 'Matsubara Contour'

Next time: - some details for Feynman integrals Matsubara contour

- limit $t_0 \rightarrow -\infty$, then back

$\Rightarrow$ Schwinger/Keldysh