A. Rebhan Lect 2 Sept. 22

Have found

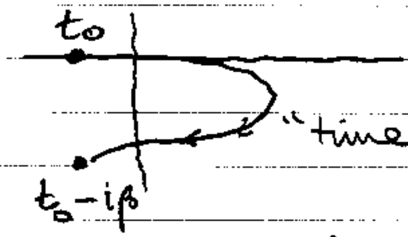
$$\langle T_c \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{Z} \operatorname{Tr}\left[e^{-\beta H} T_c \cdots \right]$$

Generating functional

$$Z[j] = \mathcal{N} \int_{\phi(t_0) = \phi(t_0 - i\beta)} \mathcal{D}\phi \, \exp i \int_{C:\, t_0 \to t_0 - i\beta} d^4x \left(\mathcal{L}(\phi) + j(x)\phi(x) \right)$$

continuous path in the parameter

T_c is always ordering along the path



"time along path" τ

Generalizes time ordering (Schwinger, Mahathappa)

Always choose path such that exponent is decreasing

Perturbation theory from source representation & Wick's thm.

$$\langle T_c \, e^{i \int d^4x \, j\phi} \rangle_0 = \exp\left[-\frac{1}{2} \int_C d^4x \int_C d^4y \cdot j(x) D_C^{(0)}(x - x') j(x') \right]$$

free

along path

Wick-Bloch-DeDominicis

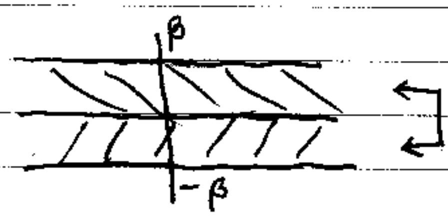
Int$^{\underline{n}}$ from usual $\exp\left(\frac{i}{\hbar} V \left(\frac{1}{i} \frac{\delta}{\delta j} \right) \right)$

All the differences from standard cases are in the propagators

General properties

$$D_c(t,t') = D_c(t-t')$$
w/ $\mathrm{Im}\, t, t' \in (0, -i\beta)$

$t-t'$ plane



defined on strips w/ possible discontinuity

For path w/ monotonically decreasing Im t:

$$D_c(t-t') = \langle T_c\, \phi(t)\phi(t')\rangle$$
$$= \theta(-\mathrm{Im}(t-t'))\, \overbrace{\langle \phi(t)\phi(t')\rangle}^{\equiv D^>}$$
$$+\, \theta(\mathrm{Im}(t-t'))\, \underbrace{\langle \phi(t')\phi(t)\rangle}_{\equiv D^<}$$

Now

$$D^>(t) = \langle \phi(t)\, \phi(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left[\{e^{-\beta H}\phi(t)\, e^{\beta H}\}e^{-\beta H}\phi(0)\right]$$

$$\{\cdots\} = \phi(t+i\beta)$$

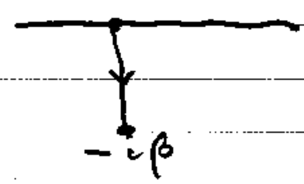
$$= \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\phi(0)\, \phi(t+i\beta)\right]$$

$$= D^<(t+i\beta)$$

KMS property: $\left\{ \begin{array}{l} \mathrm{Im}\, t < 0 \\ D_c(t) = D_c(t+i\beta) \end{array}\right.$

Special Cases

• Matsubara



Periodic of finite range of (imag.) time

$\downarrow$

discrete frequencies

sometimes written as
$$\begin{cases} \Delta(\tau) = D_c(t=-i\tau) \\ \text{KMS} \Rightarrow \quad \Delta(\tau) = \Delta(\tau-\beta)\ ; \quad \tau > 0 \end{cases}$$

$$D(t) = \underbrace{\frac{1}{-i\beta}}_{\text{length of the interval}} \sum \tilde{D}(z_n)\, e^{-iz_n t}$$

w/ $\tilde{D}(z_n) = \int_0^{-i\beta} dt\, D(t)\, e^{iz_n t}$

In Fourier space $\quad i\tilde{D} = \left.\dfrac{1}{-k_0^2 + \vec{k}^2 + m^2}\right|_{k_0 \to z_n}$

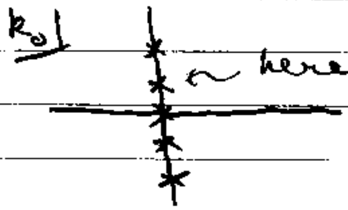
$\Rightarrow \quad z_n = 2\pi i n T$ (bosons)

$z_n = 2\pi i (n + \frac{1}{2}) T$ (antiperiodic)

with chemical potential $z_n \to z_n + \mu$

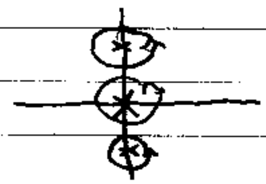
Generally want to compute sums like

$$T\sum_{n=-\infty}^{\infty} F(k_0 = 2\pi i n T)$$



← here are poles of $\frac{\beta}{e^{\beta k_0}-1}$ (w residue 1)

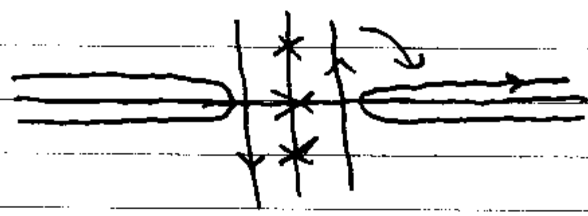
$\Downarrow$



if F is regular on Im axis

$$= \frac{T\beta}{2\pi i}\int_C dk_0\, F(k_0)\,\frac{1}{e^{\beta k_0}-1}$$

$\Downarrow$ assuming $\left\{\begin{array}{l} F \text{ singular only on real axis} \\ \text{sufficiently small @ } \infty \end{array}\right.$



$$= \frac{}{2\pi i}\int dk_0\, F(k_0)\,\underbrace{\frac{1}{e^{\beta k_0}-1}}_{f(k_0)}\quad ; \quad n(k_0) \equiv f(|k_0|)$$

Re $k_0 > 0$ weight Disc F w/ Bose-Einstein distribution

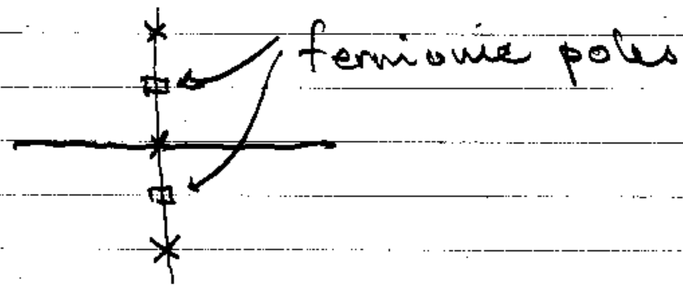
Re $k_0 > 0$

$$\Rightarrow f(-k_0) = \frac{1}{e^{-\beta k_0}-1} = -1 - n(k_0)$$

← remains as $T \to 0$!

fermionic Matsubara

$$f \to -\beta \frac{1}{e^{\beta k_0}+1} \quad \text{(shifts the poles)}$$



$$\Rightarrow \quad T \sum_{\substack{\text{fermic}\\ \text{at } T}} = 2\frac{T}{2} \sum_{\substack{\text{bosonic}\\ \text{at } T/2}} - T \sum_{\substack{\text{bos}\\ T}}$$

replacement

so can get fermionic sums from bosonic sums

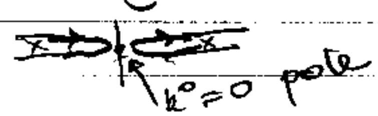
i.e. use identity
$$-\frac{1}{e^{\beta_0 k_0}+1} = \frac{2}{e^{2\beta_0 k_0}-1} - \frac{1}{e^{\beta k_0}-1}$$

Propagator as a bosonic sum

$$\mathcal{D}(t=-i\tau) = T \sum_n \underbrace{i\tilde{\mathcal{D}}}_{\frac{1}{\omega_n^2+E_k^2}} e^{-i\omega_n \tau}$$

integral form: $\omega_n \to k_0$

$$= \frac{1}{2\pi i} \int_{C'} dk_0 \, e^{k_0|\tau|} \underbrace{\frac{1}{-k_0^2+E_k^2}}_{\frac{-1}{(k_0-E_k)(k_0+E_k)}} \frac{1}{e^{\beta k_0}-1}$$



$$= \frac{1}{2E_k} e^{E_k|\tau|} n(E_k) - \frac{1}{2E_k} e^{-E_k|\tau|} (1+n(E_k))$$

This is the causal Matsubara propagator ("mixed" time/wave number representation)

From Propagator to PT

$$\mathcal{L}_I = -\lambda\phi^4 \qquad \mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

Self-energy $\frac{1}{2}$ 

Δm^2

$$\Delta m^2 = \frac{4!\,\lambda}{2}\, T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(2\pi n T)^2 + k^2 + m^2}$$

same contour trick

$$= 12\lambda \int \frac{d^3k}{(2\pi)^3} \underbrace{\frac{1}{2\sqrt{k^2+m^2}}}_{2E_k} \Big(\underbrace{2n(\sqrt{k^2+m^2})}_{T\text{-dep't}} + \underbrace{1}_{T=0} \Big)$$

Concentrate on the T-dependent portion

How to do the integral?

$$T \gg m: \quad 12\lambda \int_0^\infty \frac{k^2\,dk}{2\pi^2} \frac{1}{\cancel{k}} \frac{1}{e^{k/T}-1}$$

$$= 12\lambda T^2 \underbrace{\int_0^\infty \frac{dx\,x}{2\pi^2} \frac{1}{e^x - 1}}_{\frac{1}{2\pi^2}\zeta(2)\Gamma(2)} \; ; \quad \zeta(2) = \pi^2/6$$

$$= \lambda T^2$$

T-dependent mass

relevant formula

$$\int_0^\infty dk\, k^{s-1} \frac{1}{e^k - 1} = \Gamma(s)\zeta(s)$$

Thermal mass:

The simplest example of a "hard thermal loop" (HTL) ~ (coupling) T
dominated by k≫T

Such masses can occur as well in gauge theory

Aside: energy-momentum tensor remains traceless

Thermal mass as due to scattering with particles in the heat bath

Can undo SSB ("negative m^2" can be shifted positive)

Trick from Arnold & Espinoza (1993)
- transparency -

High-T expansions without pain

(Arnold + Espinosa 1993)

e.g.

$$I = T \sum\!\!\!\!\!\!\int \frac{1}{\omega_n^2 + k^2 + m^2}$$

$(\omega_n = 2\pi n T)$

- dim. reg. $\int \frac{d^d k}{(2\pi)^d}$, $d = 3 - 2\varepsilon$
- separate off $n=0$ mode
- $\sum\limits_{n \neq 0}$ last of all

n = 0 mode:

$$T \int \frac{d^{3-2\varepsilon} k}{(2\pi)^{3-2\varepsilon}} \frac{1}{k^2 + m^2} \underset{(\varepsilon \to 0)}{=} \frac{T (m^2)^{\frac{1}{2}} \overbrace{\Gamma(-\tfrac{1}{2})}^{-2\sqrt{\pi}}}{(4\pi)^{3/2}} = -\frac{Tm}{4\pi}$$

$(1 - \frac{d}{2} \to -\frac{1}{2})$

$n \neq 0$:

$$T \sum_{n \neq 0} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2 + m^2} =$$

$$= 2T \sum_{n=1}^{\infty} \int \frac{d^d k}{(2\pi)^d} \sum_{\ell=0}^{\infty} \frac{(-m^2)^\ell}{(\omega_n^2 + k^2)^{\ell+1}}$$

$$= 2T \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} (-m^2)^\ell \cdot \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\ell + 1 - \frac{d}{2})}{\Gamma(\ell+1)} (2\pi n T)^{d - 2\ell - 2}$$

$$= \frac{2T (2\pi T)^{d-2}}{(4\pi)^{d/2}} \sum_{\ell=0}^{\infty} (-1)^\ell \left(\frac{m}{2\pi T}\right)^\ell \frac{\Gamma(\ell + 1 - \frac{d}{2})}{\Gamma(\ell+1)} \zeta(2\ell + 2 - d) \times \mu^{2\varepsilon}$$

$$\zeta(-1) = -\frac{1}{12}, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta(1 + 2\varepsilon) = \frac{1}{2\varepsilon} + \gamma_E, \quad \zeta(2) = \frac{\pi^2}{6},$$

$$\to I = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{8\pi^2}\left(\frac{1}{2\varepsilon} + \ln\frac{\bar{\mu}}{4\pi T} + \gamma_E\right) + \cdots$$

also:

$$P = \frac{\log Z}{\beta V} = -\frac{1}{2} T \sum_n \int \frac{d^d k}{(2\pi)^d} \log(\omega_n^2 + k^2 + m^2)$$

$$= -\frac{1}{2} \int_0^{m^2} dm'^2 \, I(m'^2) \underbrace{- \frac{1}{2} T \sum\!\!\!\!\!\!\int \log(\omega_n^2 + k^2)}_{\pi^2 T^4/90}$$

$$P = \frac{\pi^2 T^4}{90} - \frac{m^2 T^2}{24} + \frac{m^3 T}{12\pi} + \frac{m^4}{32\pi^2}\left(\frac{1}{2\varepsilon} + \ln\frac{\bar{\mu}}{4\pi T} + \gamma_E\right) + \cdots$$