Asymptotic Freedom and the Spectral Index of String Vacua

Sergei Gukov

August 1st, 2005

Talk at the Third Simons Workshop in Mathematics and Physics
SUNY at Stony Brook, July 25 - August 26, 2005

Notes by Ari Pakman

Topological Strings

I am going to tell you about some work I have been doing with Kirill Saraikin and Cumrun Vafa, the first part of which has appeared in [1] and the second part will appear soon [2].

Since this is the first talk in this workshop about topological strings, I would like to start by introducing topological strings and describing some of their properties.

For our purposes we will be interested in studying topological strings on a Calabi-Yau 3-fold X. As in the standard string theory, the topological string deals with maps from a genus g Riemann surface Σ_g into X. This map must satisfy certain conditions, as being holomorphic, etc. The result is an object called the *topological partition function*

$$Z_{top} = \exp\left[\sum_g \lambda^{2g-2}\mathcal{F}_g\right] , \tag{1}$$

which is an expansion in genus. The *topological* nature of this theory refers to the fact that it is much simpler than the physical string theory. There are two main reasons to be interested in the topological string:

- Firstly, it computes quantities in the *physical* string theory which are protected by supersymmetry. In particular, effective actions of 4D string and field theories, *e.g.* those of Seiberg-Witten type.
- Secondly, it computes interesting topological invariants of the 3-fold X.

The quantities computed depend on some data that characterizes the manifold X. The geometry of a Calabi-Yau depends on the Kähler and the complex structure, and, correspondingly, there are two types of topological strings,

- **The A-model**: it depends only on the Kähler moduli t_I ($I = 1, \ldots, h^{1,1}$) of the Calabi-Yau, and the terms in (1) are

$$\mathcal{F}_g(X, t_I) = \sum_{\beta \in H_2(X,\mathbb{Z})} GW_g(\beta) t^\beta \,, \tag{2}$$

  where $GW_g(\beta)$ are the so-called Gromov-Witten invariants. These count the number of stable maps in class β, and direct computation of these invariants is quite involved.

- **The B-model**: This model depends on the complex structure moduli of the Calabi-Yau. Here computations are relatively simple, *e.g.* in genus zero the results are expressed in terms of the periods of the Calabi-Yau. Choosing a symplectic basis of A^I and B_J cycles in X ($I, J = 1, \ldots h^{2,1} + 1$), we can introduce

$$X^I = \int_{A^I} \Omega \qquad\qquad F_J = \int_{B_J} \Omega \,. \tag{3}$$

  where Ω is the unique holomorphic covariantly constant three-form on the Calabi-Yau. These variables are not independent; they are related by

$$F_I = \frac{\partial \mathcal{F}_0}{\partial X^I} \,, \tag{4}$$

  So, we see how the first term in the genus expansion (1) appears from relatively simple geometrical properties of X. The X^I variables actually provide a projective parametrization of the complex structure, so there are only $h^{2,1}$ independent complex coordinates. For higher genus terms in (1) the details are different but the idea is conceptually similar. For example, $\mathcal{F}_1$ is the holomorphic version of the Ray-Singer torsion. Using a recursive procedure exploiting the so-called *holomorphic anomaly*, one can obtain all the higher $\mathcal{F}_g$ coefficients starting from $\mathcal{F}_0$, as explained in [3].

Mirror symmetry, in its simplest form, says that a Calabi-Yau X has a mirror Calabi-Yau $\tilde{X}$, whose Hodge numbers are related as ($h^{1,1} = \tilde{h}^{2,1}$ $h^{2,1} = \tilde{h}^{1,1}$) and such that

$$Z^A_{top}(X, t) = Z^B_{top}(\tilde{X}, X^I) \,. \tag{5}$$

Once we have properly identified the mirror pairs X and $\tilde{X}$, it is very easy to compute the right hand side of (5), and thus obtain the Gromov-Witten invariants on the left-hand side.

Now, consider the series (1) for the B model (if mirror symmetry holds, then the same is valid for the A model). Remember that the X^I coordinates are projective coordinates. One can show that a term

$$\lambda^{2-2g}\mathcal{F}_g(X^I) \tag{6}$$

is homogeneous of degree $2-2g$ under rescalings of X^I. More precisely, it is a section of a line bundle whose first Chern class is $2-2g$. Put differently, it is a section of a bundle $\mathcal{L}^{\otimes(2-2g)}$, where $\mathcal{L}$ is a line bundle on the complex structure moduli space, where the holomorphic 3-form Ω takes values in. From (6) we see that the overall rescaling of the projective coordinates X^I, is equivalent to the rescaling of the coupling constant λ. So, one can keep all the $h^{2,1}+1$ variables X^I, which altogether parametrize the complex structure moduli as well as the coupling constant of the theory.

From topological partition function to the wave function of the universe

The next crucial point about the topological partition function is that it is not really a function, but a wave function! This was realized first in [3], and further clarified by Witten in [4]. It has *both holomorphic and anti-holomorphic* dependence on the complex structure moduli, which is captured by the so-called *holomorphic anomaly equation* of [3]. In the B model, it has the following origin. Quantization of the topological string can be viewed as a quantization of the space $H^3(X,\mathbb{R})$ with the natural symplectic structure. This requires a choice of *polarization*, that is a separation of the variables into "canonical coordinates" and "canonical momenta". Each polarization gives a different topological partition function Z_{top}. In fact, under a change of polarization, Z_{top} transforms as a wave function,

$$\psi_{top} = Z_{top}. \tag{7}$$

For example, if we interchange a pair of coordinate and momentum variables in the polarization, the corresponding Z_{top}'s are related to each other by a Fourier transform, as expected for a wave function, and as verified explicitly in some examples. This is a nontrivial and restrictive property, which one might hope to use in order to solve the topological string on a compact Calabi-Yau.

Question: Does this "background dependence" have a physical meaning?
Answer: Yes. For example, when compatifying to 4D in a Calabi-Yau, the effective action will depend on the choice of vacuum around which we expand.

Comment (by C.Vafa): A "mini" version of the holomorphic anomaly can be seen in an N=2 theory written in terms of a prepotential. If you change from magnetic to electric variables, the prepotential changes by a Legendre transformation, which is a *classical* version of the Fourier transform.

- Yes, precisely as there are several dual forms of writing a prepotential, there are many ways of writing Z_{top} . In terms of the periods (3), one has $2(h^{2,1}+1)$ coordinates $ReX^I, ImX^I, ReF_J, ImF_J$ and one selects coordinate-momentum pairs among them.

The interpretation of the topological partition function as a wave function is a very deep notion and certainly not fully understood yet. An important progress in this direction was made by Ooguri, Vafa and Verlinde (OVV) in [6]. They argued that

> *the topological partition function Z_{top} on a given Calabi-Yau CY_3 is the minisuperspace Hartle-Hawking wavefunction for flux compactifications of Type II strings on $M^9 = S^1 \times S^2 \times CY_3$.*

The statement holds both for A and B theories (related by mirror symmetry). What is the minisuperspace Hartle-Hawking (HH) wave function? There has been a lot of progress in understanding string theory in static spacetimes, but we are very bad at explaining how our universe evolved from certain initial conditions to its present state. Hartle and Hawking proposed in [5] that our universe evolved from an initial universe of zero size, in the far past Euclidean time. This "Nothing" universe is connected to our present universe by a gravitational instanton, which has our present universe M as a spacelike boundary. The wavefunction of the universe is a functional of the 3D metric in our present time,

$$\Psi_{HH}(g_M) = \sum_{W:\partial W = M} \int [\mathcal{D}g_W]\, e^{-S_{Euclidean}[g_W]}\,, \tag{8}$$

where the path integration is done over all previous topologies W and histories g_W. Invariance of this path integral under reparametrizations of the time direction transverse to M imposes on $\Psi(g_M)$ a differential equation called the Wheeler-DeWitt equation.

This nice proposal has the drawbacks of its generality, since in practice we cannot perform any of these sums or path integrals. The way to make sense of the Hartle-Hawking proposal, as noticed in [5], is to reduce the path integral to a very limited set of modes, say, one or two parameters. This is called the *minisuperspace* aproximation. For example, in [5] the three dimensional spatial slice M was taken to be an S^3, and the free parameter that changed in Euclidean time was its radius (which gets interpreted as the cosmological constant). In the OVV proposal, the reduced set of modes are those captured by the topological string, namely Kähler moduli or complex moduli of the Calabi-Yau (plus the fluxes turned on in the background).

The OVV proposal is very interesting since it gives us a fully string theoretical approach to cosmology through a very specific realization of the Hartle-Hawking wave function. Moreover, since Z_{top} is a sum over all the genera (see (1)), we get all the string theory corrections to the minisuperspace wave function.

Comment (by S.Vandoren): It would be interesting to see if from the stringy-corrected Hartle-

Hawking wave function one can read corrections to the Wheeler-DeWitt equations, which many people have been looking for.

Once we have the wave function Ψ_{HH} of our Universe, we can start computing correlation functions, as in quantum mechanics, of the type

$$\langle \Psi_{HH} | \mathcal{O}_1 \ldots \mathcal{O}_n | \Psi_{HH} \rangle \, . \tag{9}$$

I will describe to you two applications of this idea.

A toy model of string cosmology

This follows closely the original idea of Hartle-Hawking. The questions we want to ask are about the density perturbations of the early universe, which can now be measured by various experiments, notably the W-Map, which is releasing wonderful data. These measurements probe how the metric in the early universe was distorted. So we want to compute correlators of the metric perturbations $\delta g_{\mu\nu}$. Since this is a tensor, it is usual to define

$$\rho(x) = \delta g^{\mu}_{\mu}(x) \, . \tag{10}$$

The two-point correlators of ρ can be measured. They turn out to be extremely small, and behave as

$$\langle \rho_{\vec{k}} \, \rho_{-\vec{k}} \rangle \sim |\vec{k}|^{-3} \tag{11}$$

in a momentum basis. The standard explanation of this perturbations is the theory of inflation. One can also think of an alternative cosmological model where these scalar fluctuations are encoded in the Hartle-Hawking wave function. Computations have been done using the ordinary Hartle-Hawking wave function, without much success (see [7]). But now that we have the full stringy Hartle-Hawking wave function, it is natural to ask if string theory can yield the right behavior.

Since this is a toy model, we will assume our universe is originally maximally symmetric: a three-sphere S^3. The only way to confine a three sphere in our $M^9 = S^1 \times S^2 \times CY_3$ is to have $S^3 \subset CY_3$. Contrary to the usual case where we compactify and make the Calabi-Yau small, in this case the S^3 inside the Calabi-Yau is big since it is our universe. The variations of the moduli of this Calabi-Yau lead to variations of the metric of this S^3. Taking for concreteness the B model, the moduli is given by variations of the complex structure, or equivalently, the holomorphic three form. Writing the latter as

$$\Omega = \Omega_0 e^{\phi} \, , \tag{12}$$

with Ω_0 a reference 3-form, the field ϕ gets associated with the volume of S^3, and it is natural to compute correlators of $\delta\phi$. A specific realization of the embedding $S^3 \subset CY_3$ is given by the *deformed conifold*, which is a Calabi-Yau given by T^*S^3. It can be expressed as an hypersurface in $\mathbb{C}^4$,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu\,, \qquad\qquad \mu \in \mathbb{C}\,, \tag{13}$$

where $|\mu|$ is roughly the radius of S^3. Now, the good news is that this model can be solved exactly, because the topological B model of the deformed conifold is equivalent to the $c = 1$ string theory at the self-dual radius [8].

Perturbations to the sphere can be obtained by adding to the right hand side of (13) a general term $\epsilon(x_i)$. We will focus on $\epsilon(x_i) = \epsilon(x, y)$ deformations of the form

$$xy - zw = \mu + \epsilon(x, y) = \mu + \sum_{n>1} (t_n x^n - t_{-n} y^n)\ , \tag{14}$$

where x, y, z, w are simple linear combinations of the x_i's. These perturbations correspond roughly to different spherical armonics in S^3, and in the $c = 1$ theory they correspond to tachyon momentum modes, whose correlators we know how to compute from matrix models. The number of possible deformations here is infinite, since this is a non-compact Calabi-Yau. Finally, one can check that the relation with the size of the S^3 ϕ in (12) is

$$\phi = \partial_p \epsilon(x, y)\,, \tag{15}$$

where $x = p^{1/2}e^{i\theta}$, $y = p^{1/2}e^{-i\theta}$. The end result for the two-point function of the scalar ρ in (10) is

$$\langle \rho_{\vec{k}}\, \rho_{-\vec{k}} \rangle \sim g_s^2 |\vec{k}|\,, \tag{16}$$

which (unfortunately!) differs from the observed cosmological behavior (see (11)).

Note that we are not considering in (14) deformations depending on z, w. This would correspond to tachyon winding modes, whose correlators we do not know yet how to compute with full generality.

A toy model of string phenomenology

Unlike the application we have just considered, in the more conventional scenario of Calabi-Yau compactifications our world lies in the uncompactified dimensions. Ignoring the fact that in the OVV proposal we are instructed to compactify three additional dimensions to $S^1 \times S^2$, we can think of $|\psi_{top}|^2$ as a probability density in the space of string vacua, parametrized by the moduli of the Calabi-Yau.

The critical points of this probability density are peaked in the so-called *attractor points* of the Calabi-Yau. These are points in the moduli space of the Calabi-Yau satisfying

$$Im a_i^p - \tau_{ij} Im a_j = 0\,, \tag{17}$$

where

$$a_p^i = \frac{\partial F(a)}{\partial a_i} \qquad\qquad \tau_{ij} \frac{\partial^2 F(a)}{\partial a_i \partial a_j} \, , \tag{18}$$

and F is the prepotential coming from the topological string partition function.

The attractor equations (17) can be obtained from a variational principle, keeping one of the periods of the holomorphic 3-form fixed,

$$X^0 = \int_{A^0} \Omega = fixed \sim \frac{1}{g_s} \, . \tag{19}$$

This constraint makes sense, since otherwise Ω is defined only projectively. Note that the attractor equations are highly nonlinear, but they are not differential equations. Moreover, it can be shown that their solutions are generically isolated points.

If one wants the attractor points to be not just critical but maxima or minima, one gets the additional constraint $Ima_i = Ima_i^p = 0$, namely, all the periods are real, or more generically, can be chosen to have the same phase. Physically this means that BPS bound states of branes become marginal. In fact, it is the maximal allowed set of marginal bound states.

Finally, an interesting result we obtained is that the second derivative of the probability $|\psi_{top}|^2$ with respect to the moduli is correlated with the beta function of the low energy effective theory. Namely, at the maximum points vector multiplets become massless and one has a theory with negative beta functions, *i.e.* an asymptotically free theory.

References

[1] S. Gukov, K. Saraikin and C. Vafa, "A stringy wave function for an S**3 cosmology," arXiv:hep-th/0505204.

[2] S. Gukov, K. Saraikin and C. Vafa, "The Entropic Principle and Asymptotic Freedom," hep-th/0509109.

[3] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[4] E. Witten, "Quantum background independence in string theory," arXiv:hep-th/9306122.

[5] J. B. Hartle and S. W. Hawking, "Wave Function Of The Universe," Phys. Rev. D **28**, 2960 (1983).

[6] H. Ooguri, C. Vafa and E. P. Verlinde, "Hartle-Hawking wave-function for flux compactifications," arXiv:hep-th/0502211.

[7] S. W. Hawking and N. Turok, "Open inflation without false vacua," Phys. Lett. B **425**, 25 (1998) [arXiv:hep-th/9802030].

[8] D. Ghoshal and C. Vafa, "C = 1 string as the topological theory of the conifold," Nucl. Phys. B **453**, 121 (1995) [arXiv:hep-th/9506122].