Testing Dijkgraaf-Vafa conjecture at higher genus

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Topological strings have been a source of insights and a powerful tool to study many areas of mathematics and physics. One such example is their physical applications to supersymmetric gauge theories in four dimensions. Through the technique of geometric engineering [1], a large class of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theory can be embedded in local Calabi-Yau compactifications of type II string theory. The low energy effective holomorphic quantities such as prepotential and superpotential in the gauge theories and their gravitational corrections can be computed by topological string theory on the local Calabi-Yau geometries.

In [2], Dijkgraaf and Vafa proposed a remarkable relation between B-model topological string on a non-compact geometry and a corresponding matrix model. The geometry has a local description as a hypersurface in $\mathbb{C}^4$

$$vw = W'(x)^2 + f(x) + y^2 \tag{1}$$

where x, y, v, w are coordinates of $\mathbb{C}^4$, $W(x)$ is a polynomial of degree $n+1$, and $f(x)$ is polynomial of degree $n-1$ that splits the n double zeros of $W'(x)^2$. The corresponding matrix model is a simple bosonic matrix model with the matrix potential $W(x)$. Further generalization of this conjecture to the case of Calabi-Yau geometry with ADE type singularities can also be made [3].

The works of Dijkgraaf and Vafa has sparked exciting developments by relating the topological strings and matrix models to four-dimensional supersymmetric gauge theory. Through these relations, the effective superpotential in a large class of $\mathcal{N} = 1$ gauge theory can be computed exactly by the planar diagrams of the corresponding matrix models [4]. Later it was shown that the relation between gauge theory and matrix model can be derived within the context of gauge theory without

the use of topological strings [5, 6]. Following the proposal of Dijkgraaf and Vafa, there have also been many other developments, e.g. using matrix models to study phases of gauge theory [7].

In light of these important implications it is very interesting to further explore the Dijkgraaf-Vafa conjecture. In [2] it is already shown by Dijkgraaf and Vafa that the special geometry relation and the Picard-Fuchs equation that determine the tree level (genus zero) topological string amplitude on the geometry (1) arise from the planar diagrams of the corresponding matrix model. The planar loop equation of the matrix model gives the spectral curve of the local geometry. It is conjectured that higher genus topological string amplitudes should also be computed by higher genus diagrams in the matrix model. These higher genus amplitudes compute gravitional corrections of holomorphic quantities in related four-dimensional supersymmetric gauge theory. This has been verified in the case of genus one in [8, 9].

In this paper we take the first steps to test the Dijkgraaf-Vafa conjecture at genus great than one. To do this it is necessary to use the holomorphic anomaly equation developed in [10]. Previous studies of Dijkgraaf-Vafa conjecture at lower genus do not need to use the propogators in Kodaira-Spencer theory, which will be the new and key components in our case to solve the holomorphic anomaly equation at higher genus. The holomorphic anomaly equations express the anti-holomorphic dependence of higher genus topological string amplitudes in terms of lower genus amplitudes, and can be integrated to express higher genus amplitudes recursively in terms of lower genus amplitudes plus a holomorphic ambiguity that arises as integration constant. Therefore the topological string amplitudes can be computed recursively genus by genus once we determine the holomorphic ambiguity. We study the holomorphic anomaly equations in detail and fix the holomorphic ambiguity by using some initial data of the matrix models for the local Calabi-Yau geometry (1). Once the holomorphic ambiguity is fixed, we then make highly non-trivial checks of the equivalence between the topplogical string amplitudes with the matrix model calculations.

We should also mention that the Dijkgraaf-Vafa correspondence between topological strings and matrix models is an example of the large N duality that underlies the AdS/CFT correspondence [11]. In this example the techniques in topological string theory such as holomorphic anomaly equation are well developed, such that we can do the string theory computations exactly. We hope the lessons we learn from these examples will be useful to the study of AdS/CFT duality.

Relation to the C deformations in supersymmetric gauge theory [12, 13, 14]: It has been long known that topological string amplitudes on Calabi-Yau compute certain gravitational F-terms in the low energy effective action of four-dimensional $\mathcal{N} = 2$ supergravity from type II string theory compactified on the Calabi-Yau. Specifically, suppose $F_g(S_i)$ is the genus g topological string amplitudes, then there are F-terms in the low energy $\mathcal{N} = 2$ supergravity action

$$\int d^4x \int d^4\theta F_g(S_i)(\mathcal{W}^{\alpha\beta}\mathcal{W}_{\alpha\beta})^g \tag{2}$$

here $\mathcal{W}_{\alpha\beta}$ is the $\mathcal{N} = 2$ graviphoton superfield and the S_i are the $\mathcal{N} = 2$ vector multiplets coming

from the moduli of Calabi-Yau manifold. After integrating over the $\mathcal{N} = 2$ superspace these terms become the coupling of self-dual part of the Ricci tensor R_+ and graviphoton field strength F_+.

Recently, [12, 13, 14] considered breaking the $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetric gauge theory by turning on flux. This gives rise to the following two terms in the $\mathcal{N} = 1$ superspace

$$\Gamma_1 = g \int d^4x \int d^2\theta \mathcal{W}_{\alpha\beta\gamma} \mathcal{W}^{\alpha\beta\gamma} (F_{\delta\xi} F^{\delta\xi})^{g-1} F_g(S_i), \tag{3}$$

$$\Gamma_2 = \int d^4x \int d^2\theta (F_{\alpha\beta} F^{\alpha\beta})^g N_i \frac{\partial F_g(S_i)}{\partial S_i} \tag{4}$$

where $\mathcal{W}_{\alpha\beta\gamma}$ are now $\mathcal{N} = 1$ gravitino multiplet, and S_i are the $\mathcal{N} = 1$ glueball chiral superfields coming from the original $\mathcal{N} = 2$ vector multiplets. The graviphoton field $F_{\alpha\beta}$ can be treated as background field in the $\mathcal{N} = 1$ theory. In [12] a C-deformation is introduced to deform the anti-commutation relation of gluino ψ_α to the followings

$$\{\psi_\alpha, \psi_\beta\} = 2F_{\alpha\beta} \tag{5}$$

It is shown that the effect of turning on the graviphoton background $F_{\alpha\beta}$ can be captured by this C-deformation, and the F_g in the second contribution Γ_2 (4) are computed by matrix models at genus g. We can also see that the first term Γ_1 in (3) contributes at genus $g = 1$ even when there is no graviphoton background $F_{\alpha\beta}$. It is shown in [13] that this genus one contribution is also computed by matrix model genus one diagrams. There are also some other types of gravitational corrections besides (3,4) of the form $W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} S^n$ from planar diagrams, which become trivial after the extremization of the glue ball superfield S [14]. Our results confirm these very interesting ideas in [12, 13, 14], and we provide a first direct tests of the connection between topological strings and matrix models without using the superspace techniques in related gauge theory.

Relation to loop equations in matrix model [15, 16]: In the planar limit the matrix model free energy and resolvent can be exactly solved by loop equations. An iteration procedure is introduced to systematically compute higher genus free energy and resolvent of the matrix models for one-cut solution in [15] and generalized to multi-cut solution in [16]. The iteration equation is obtained by doing $1/N$ expansion in the loop equations, and looks tantalizingly similar to the holomorphic anomaly equation in topological strings. However, the matrix models here are assumed to be Hermitian matrices in our perturbative calculations, so it is not quite clear what the anti-holomorphic dependence in topological strings should correspond to in the matrix models. Also the iteration procedure quickly becomes very complicated and it is hard to integrate for the free energy, especially for the multi-cut solutions we consider here. It would be very interesting to carry out the iterations in details for matrix models at higher genus, since this can in principle automatically fix the holomorphic ambiguity in B-model calculations, as shown for genus one in [8]. Fixing the holomorphic ambiguity is one of the main difficulties for topological string calculations in many other models. We hope further studies will clarify these issues and provide valuable lessons in fixing holomorphic ambiguity in more general models.

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