

String gases and chaos

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Notes by Ari Pakman

The Brandenberger-Vafa scenario

This is an old but very intriguing picture of the early universe cosmology in string theory [1]. We imagine that the early universe was spatially compact, say a T^9 torus, with a size of order of the string scale l_s , and very hot, with many degrees of freedom excited. In particular there was a network of string winding around the universe and a gas of KK modes.

Question: What is the physical evidence for this picture?

Answer: There is certainly no physical evidence, aside from the fact that if you evolve the present universe back in time the temperature increases, so it is natural to assume that there was a very hot early phase. The main argument is that this seems a generic state of string theory, with all possible degrees of freedom excited.

Now, this should be a quasi-stable state, for the following reason. The tension of the winding modes keeps the universe from expanding, but the gas of KK excitations has momentum along the various directions, and thus pushes towards expansion. If the size of the torus is of order l_s , the two forces are balanced, and this balance can remain for an arbitrary time. But eventually thermal fluctuations will drive the system out of equilibrium and some dimensions can become larger than the string scale l_s . The mechanism for this is the annihilation of two strings with opposite windings. There is always a reservoir of such strings since in a spatially compact universe the total winding number should be zero. This annihilation of winding modes diminishes their negative pressure and allows the universe to expand.

Question: What if the thermal fluctuations cause a given dimension to shrink by creation of

winding/anti-winding pairs?

Answer: It could be either way, since by T-duality a shrinking is an expansion.

We would like to know how many dimensions could decompactify in this way. The crucial observation is that two string worldsheets will generically intersect in $2+2 = 3 + 1$ dimensions, and in higher dimensions they will miss each other. This is actually a property of classical string worldsheets, but the classical picture is reliable at energies below the string scale. So this mechanism would explain the number of big dimensions in our present universe.

Comment (by C.Vafa): Mathematically this picture is related to the fact that in the classification of manifolds, the nature of the classification changes drastically below and above four dimensions. This would be a stringy worldsheet application of these ideas.

Some problems in the BV scenario

In [2] we decided to look at this scenario a littler closer, and we set out to compute the probability that different numbers of spacial dimensions get decompactified. We introduce a metric for a rectangular torus as

$$ds^2 = -dt^2 + \alpha' \sum_{i=1}^9 e^{2\lambda_i(t)} dx_i^2 \quad (1)$$

and it is convenient to define a modified dilaton $\tilde{\phi}$ as

$$\tilde{\phi} = 2\phi - \sum_{i=1}^9 \lambda_i, \quad (2)$$

which has nice properties because it is invariant under T-duality and furthermore, if we reduce the system to 1+0 dimensions, the effective coupling is

$$g_{eff} = e^{\tilde{\phi}}. \quad (3)$$

Question: Why is this the effective coupling? Is not always e^{ϕ} the coupling for string interactions?

Answer: The probability that two winding strings annihilate is

$$\text{prob.} \sim \frac{e^{2\phi}}{Vol(T^9)} e^{2\lambda_1} \sim e^{\tilde{\phi}} e^{2\lambda_1} \quad (4)$$

Now, a problem comes in from the Hamiltonian constraint of general relativity, namely, the time-time component of Einstein equations, $G_{00} = 8\pi G_N T_{00}$. In the context of cosmology this equation

is usually written as

$$H^2 = \frac{8\pi G_N}{3} \rho \quad (5)$$

where H is the Hubble parameter and ρ is the energy density of the matter. In our setting this equation becomes

$$\dot{\tilde{\phi}}^2 = \sum_{i=1}^9 \dot{\lambda}_i^2 + e^{\tilde{\phi}} \rho V, \quad (6)$$

where V is the volume of the compactified space. The right hand side is strictly positive, so $\tilde{\phi}$ evolves monotonically in time. There are two possibilities, and both are problematic for the BV scenario:

- $\dot{\tilde{\phi}} < 0$: In this case the effective coupling (3) becomes weaker in time, and the strings end up not interacting at all. So the winding/anti-winding annihilation does not take place! And one can verify that the time-scales are such that this is the final outcome regardless of the initial conditions.
- $\dot{\tilde{\phi}} > 0$: In this case the effective coupling (3) becomes strong, the 10D string regime is no longer reliable and we must consider the full M-theory. For this case we did a different analysis in [3]. We considered a gas of M2 branes in a torus. In this setting, the Hamiltonian constraint implies that the volume of the torus increases monotonically, so the gas of M2 branes becomes non-interacting since it gets diluted by the cosmic expansion (except for very fine-tuned initial conditions). So the analysis is different but the outcome is basically the same.

Chaos and the dynamics of moduli fields in the early universe

Despite the above problems with the BV scenario, it is such a nice idea that you cannot stop thinking about it. It seems we have left something out of our analysis. To illustrate the idea, let us review an interesting work by Horne and Moore [4]. They studied the effective action for the vacuum dynamics of the IIB string in 10D. We have the RR scalar a_R and the dilaton ϕ combined into

$$\tau = a_R + ie^{-\phi} \quad (7)$$

and clearly τ lives in the upper half-plane. In a spatially homogeneous setting, we have $\tau = \tau(t)$, and the equations of motion reduce to a geodesic equation for τ with the Poincaré metric

$$ds^2 = \frac{d\tau d\bar{\tau}}{(Im\tau)^2} \quad (8)$$

The geodesics are semi-circles centered on the real axis. But this is leaving out S duality! What we should really do is to fold out the semicircular geodesic into the fundamental domain of the $SL(2, \mathbb{Z})$ group acting on τ as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1 \quad (9)$$

As noted in [4], this results in a curve that ergodically fills the whole fundamental domain, by hitting the walls and bouncing back. There is only a set of measure zero in the space of trajectories which do not fill the fundamental domain, and this set includes precisely the trajectory we studied above, when we assumed $a_R = 0$.

It seems that the ergodicity follows from two key properties: the metric (8) has negative curvature and the fundamental domain has finite volume. The negative curvature makes two close geodesics diverge, but due to the finite volume they have nowhere to go, so they end up filling the whole space. This is interesting in relation to Cumrun's observation of two days ago, that all the moduli spaces coming from string theory have finite volume. It seems that they also come with non-positive curvature, a property which has no justification from a purely field-theoretic point of view.

Two points should be stressed before drawing any conclusion from the above result. Firstly, the geodesic motion corresponds to the dilaton in the Einstein frame, but our original interest was on the string frame. Secondly, this analysis only considered the vacuum, and we would like to include matter.

An example: Einstein gravity compactified in T^n

The above idea of restricting the dynamics of the moduli fields to the fundamental domain of a discrete group action on the moduli space, leads to consider a simple example: let us compactify $(n+1)$ -dimensional pure gravity on an n -torus.¹ To embed this in string theory, we can think that we are considering a subsector of M -theory on an $n = 10$ torus.

Let us then consider a spatially homogeneous metric written as

$$ds^2 = -dt^2 + V(t)^{2/n} g_{ij}(t) dx^i dx^j, \quad (10)$$

with $|g_{ij}| = 1$. We have factored out the total volume $V(t)$. The moduli space $\mathcal{M}$ for this family of metrics turns out to be

$$\mathcal{M} = \mathbb{R}^+ \times \mathcal{M}_n, \quad (11)$$

with

$$\mathcal{M}_n = SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R}) / SO(n, \mathbb{R}). \quad (12)$$

¹The results in this section were obtained in collaboration with Nada Petrovic.

The first factor in (11) corresponds the values of V , and $\mathcal{M}_n$ is the moduli space of g_{ij} : the symmetric $n \times n$ unimodular matrices are parametrized by the coset $SL(n, \mathbb{R})/SO(n, \mathbb{R})$, and we must further mod out by the group $SL(n, \mathbb{Z})$ of large diffeomorphism of T^n . The volume of $\mathcal{M}_n$ has been calculated by Minkowski in an amazing computation (he was interested in this for number-theoretical reasons). The volume is [5]

$$Vol(\mathcal{M}_n) = \frac{n2^{n-1}\zeta(2) \dots \zeta(n)}{Vol(S^1) \dots Vol(S^{n-1})} \quad (13)$$

Also, the space $\mathcal{M}_n$ has semi-definite negative curvature, so this is another instance of the phenomenon we mentioned above.

The Einstein action for the metric (10), boils down to a sigma model action for the space $\mathcal{M}_n$, plus a term for the volume,

$$\int d^{n+1}x \sqrt{-G} R \sim \int dt \left[Tr(g^{-1} \dot{g} g^{-1} \dot{g}) - \frac{\dot{V}^2}{V^2} \right]. \quad (14)$$

The trajectories of g_{ij} ergodically fill the moduli space also in this case. By bouncing around the space $\mathcal{M}_n$, one can ask what is the probability of arriving to a region in $\mathcal{M}_n$ of the type

$$T^n \rightarrow T^{n_1} \times T^{n_2} \quad (15)$$

where $T^{n_{1,2}}$ have radii $R_{1,2}$, and $R_1 \gg R_2$. Due to the ergodicity of the motion, this probability is proportional to the volume in moduli space of such configurations. This has been probably computed by mathematicians, but there is a nice physical way of obtaining the result. We dimensionally reduce the theory to the big torus. So we get a metric on the big torus, a metric on the small torus and a collection of KK gauge fields. The latter correspond geometrically to the ways the small torus is tilted with respect to the big torus. Then the probability of a decomposition like (15) is

$$\text{prob.} \sim \frac{Vol(\mathcal{M}_{n_1}) \times Vol(\mathcal{M}_{n_2}) \times Vol(\text{flat KK fields})}{Vol(\mathcal{M}_n)} \quad (16)$$

$$\sim \left(\frac{R_2}{R_1} \right)^{n_1 n_2} \quad (17)$$

where we have used Minkowski's result (13) and the fact that the volume of flat connections on a torus is the volume of the dual torus. So the probability is small when the radii of the two tori are different. It is minimized when $n_1 = n_2$. In other words, configurations with bigger difference between n_1 and n_2 are more probable. This scenario is potentially relevant for cosmology. The hope is that when we add string interactions, the probability gets peaked around $n_1 = 3$, explaining thus our universe. Moreover, one expects that once this most probable configuration is realized, new dynamic effects appear that stick the universe to $n_1 = 3$, and ergodicity is dynamically lost.

Comment (by A.Klemm): In CY compactifications the curvature of the moduli space comes automatically negative. This follows from

$$e^K = \int \Omega \wedge \bar{\Omega} \quad (18)$$

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K \quad (19)$$

$$R_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} \log \det g \quad (20)$$

and from this one shows that $\det R_{i\bar{j}}$ is negative. See works by Tian (1986) and Todorov.

References

- [1] R. H. Brandenberger and C. Vafa, “Superstrings In The Early Universe,” Nucl. Phys. B **316**, 391 (1989).
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- [4] J. H. Horne and G. W. Moore, “Chaotic coupling constants,” Nucl. Phys. B **432**, 109 (1994) [arXiv:hep-th/9403058].
- [5] This follows from theorem 10.2 in W. Scharlau and H. Opolka, “From Fermat to Minkowski: Lectures on the theory of numbers and its historical development” (Springer-Verlag, 1985).