Hydrodynamics and String Theory

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I will talk about hydrodynamics of non-extremal NS5 branes and computation of transport coefficients in the holographically dual string theory. This talk is mostly based on [1]. As part of the introduction and motivation I'll discuss other issues related to hydrodynamics.

Hydrodynamics

Hydrodynamics is an effective theory describing time evolution of the densities of conserved charges in the regime of long wavelengths, i.e. at a scale l such that

$$l_{micro} \ll l \ll L\,, \tag{1}$$

where l_{micro} is a characteristic scale of microscopic processes in the system (e.g. a correlation length), and L is a typical size of the system. Hydrodynamic description becomes unreliable when the inequality (1) is not satisfied. Schwarzschild black holes do not correspond to any hydrodynamic regime in a (hypothetical) holographicaly dual description because in that case $L \sim 1/T \sim l_{micro}$. We also do not expect the linearized hydrodynamic description to hold near a point of a phase transition, where the correlation length is typically large. We shall see that in the case of LST, predictions of the linearized hydrodynamics are not in contradiction with the hydrodynamic limit of the correlation functions obtained in the microscopic theory at the tree level.

To derive the dispersion relations for the shear and the sound modes, consider small deviations from equilibrium $< T_{\mu\nu} > + \tilde{T}_{\mu\nu}(t,x)$ in the stress-energy tensor of a theory in a $D+1$ dimensional Minkowski space. The equations of linearized hydrodynamics follow from the conservation law $\partial_\mu T^{\mu\nu} = 0$,

$$\begin{aligned} \partial_0 \tilde{T}^{00} + \partial_i T^{0i} &= 0\,, \\ \partial_0 T^{0i} + \partial_j \tilde{T}^{ij} &= 0\,, \end{aligned} \tag{2}$$

supplemented by the constitutive relations which express all components $\tilde{T}_{\mu\nu}$ in terms of fluctuations $\tilde{T}^{00}$, $\tilde{T}^{0i}$ of the densities of conserved charges (energy and momentum):

$$T^{00} = \epsilon + \tilde{T}^{00} \,, \tag{3}$$

$$T^{ij} = \delta^{ij}\left(P + \frac{\partial P}{\partial \epsilon}\,\tilde{T}^{00}\right) - \frac{1}{\epsilon + P}\left[\eta\left(\partial_i T^{0j} + \partial_j T^{0i} - \frac{2}{D}\delta^{ij}\partial_k T^{0k}\right) + \zeta \delta^{ij}\partial_k T^{0k}\right] \,, \tag{4}$$

where $\epsilon =< T^{00} >$, ϵ and P are the equilibrium energy density and pressure, η and ζ are the shear and bulk viscosities, respectively. Assuming the coordinate dependence of the form $\propto e^{-i\omega t + iqz}$, we find that the system (2) has two types of eigenmodes - the shear mode with the dispersion relation

$$\omega = -\frac{i\eta}{\epsilon + P}\, q^2 = -\frac{i\eta}{sT}\, q^2 \,, \tag{5}$$

and the sound mode whose dispersion relation is determined by the equation

$$\omega^2 + i\,\Gamma\,\omega\, q^2 - v_s q^2 = 0 \,, \tag{6}$$

where $v_s = \partial P/\partial \epsilon$ is the speed of sound and

$$\Gamma = \frac{1}{\epsilon + P}\left[\zeta + \left(2 - \frac{2}{D}\right)\,\eta\right] \tag{7}$$

is the damping constant. For nonvanishing speed of sound the dispersion relation is

$$\omega = \pm v_s q - \frac{i\Gamma}{2}\, q^2 + \cdots \,, \tag{8}$$

where ellipses denote terms suppressed for $q\Gamma/v_s \ll 1$. However, if $v_s = 0$, we find only one nontrivial solution,

$$\omega = -i\Gamma\, q^2 \,. \tag{9}$$

The dispersion relations for the shear and the sound wave modes appear as the poles of the retarded Green's functions of the stress-energy tensor

$$G_{\mu\nu,\rho\sigma}(\omega, q) = -i \int dt d^D x e^{-i\omega t + iqz}\, \theta(t) \langle [T_{\mu\nu}(t, \mathbf{x}), T_{\rho\sigma}(0)] \rangle \,. \tag{10}$$

where $\mathbf{x} = (x^a, z),\ a = 1, \ldots, 4$ and the spatial momentum is chosen along the z direction. In the hydrodynamic limit $\omega/T \ll 1$, $q/T \ll 1$ we expect the following:

• Each of the shear mode correlators G_{zx^a,zx^a}, G_{tx^a,tx^a}, G_{tx^a,zx^a}, where $x^a \neq z$, has a pole at ω given by (5) .

• The scalar mode correlators $G_{x^a x^b, x^a x^b}$, where $x^a \neq z$, $a \neq b$, do not exhibit hydrodynamic poles.

• The correlators of the sound mode, $G_{tt,tt}$, $G_{zz,zz}$, $G_{tz,tz}$, all have poles at ω given by (8) , or, if $v_s = 0$, by (9) . The correlator $G_{x^a x^a, x^a x^a}$, where $x^a \neq z$, belongs to the same family, unless

$$v_s = 0\,, \quad \zeta = \frac{2}{D}\,\eta\,, \tag{11}$$

in which case the corresponding mode decouples from the sound wave mode, as follows from (4) .

Hydrodynamics and Gravity

The viscosity of ideal gas is proportional to the mean free path. Thus, it is not surprising that η/s in Yang-Mills theory diverges as t'Hooft coupling of the theory goes to zero. At strong coupling, η/s can be computed via AdS/CFT [2]. The result is consistent with (5), with $\eta/s = 1/4\pi$. This turns out to be a universal value, also valid for other non-extremal branes. This prompted [3] to conjecture that the value of η/s is bounded from below by $1/4\pi$. They checked that this inequality is satisfied for a number of substances. The conjecture received supporting evidence in [4] where α' corrections to η/s were computed and found to be positive. It would be interesting to understand how string theory modifies the picture. With this motivation, we study a system of k NS5 branes at finite temperature, which is described holographically by an exact conformal field theory.

Little String Theory

We start by reviewing the thermodynamics of LST closely following the presentation in [5]. We take the supergravity solution for the k coincident non-extremal NS5-branes in the string frame and take the limi $g_s \rightarrow 0$ keeping fixed the energy above extremality, per unit volume

$$\epsilon \equiv \frac{E}{V_5} = \frac{1}{(2\pi)^5 \alpha'^3}\,\mu, \qquad\qquad \mu \equiv \frac{r_0^2}{g_s^2 \alpha'}\,. \tag{12}$$

The Wick-rotated near-horizon Euclidean geometry is

$$ds^2 = k\alpha' \left(d\phi^2 + \tanh^2\phi\, d\tau^2 + d\Omega_3^2\right) + dx_5^2\,, \tag{13}$$

$$e^{2\Phi} = \frac{k}{\mu \cosh^2 \phi}\,. \tag{14}$$

The absence of the conical singularity at $\phi = 0$ requires τ to be 2π-periodic. The inverse temperature is equal to the circumference of the temporal circle in (13),

$$\beta_H = 2\pi\sqrt{k\alpha'}\,. \tag{15}$$

As reviewed in [5], strings propagating in the background (13), (14) are described by an exact conformal field theory. We review some details of that theory in Section 4.

In the gravity approximation the inverse temperature is constant (independent of the energy density)

$$\beta = \frac{\partial S}{\partial E} = \beta_H\,. \tag{16}$$

Therefore, the entropy is proportional to the energy,

$$S = \beta_H E\,, \tag{17}$$

and LST has Hagedorn density of states

$$\rho(E) \sim E^{\alpha} e^{\beta_H E}\left[1 + \mathcal{O}\left(\frac{1}{E}\right)\right]\,. \tag{18}$$

The coefficient α in (18) has been computed in [5] and was found to be *negative*. This has important implications for the phase structure of LST. The relation

$$\beta = \partial S(E)/\partial E \tag{19}$$

together with (18) leads to the following energy-temperature relation

$$\beta - \beta_H = \frac{\alpha}{E} + \mathcal{O}\left(\frac{1}{E^2}\right)\,. \tag{20}$$

Thus for temperatures slightly above the Hagedorn temperature the energy is given by

$$E = \frac{\alpha}{\beta - \beta_H}\left[1 + \mathcal{O}(\beta - \beta_H)\right]\,. \tag{21}$$

In this regime, one can perform consistent perturbative expansion in powers of $\beta - \beta_H$ or, equivalently, in powers of $1/E$. This is the type of expansion we will be interested in. As we review below, this corresponds to the genus expansion in the dual string theory.

Eq. (18) implies that the free energy $\mathcal{F}$ of LST is determined by

$$-\beta\mathcal{F} = S - \beta E \simeq -(\alpha + 1)\log(\beta - \beta_H) \simeq (\alpha + 1)\log E\,. \tag{22}$$

In the second equality we used Eq. (21). The leading term in the free energy, which is proportional to energy, vanishes due to Eq. (17). The string theory partition function is related to the free energy of LST via

$$Z_{string} = -\beta \mathcal{F} \,. \tag{23}$$

Genus zero string partition function is proportional to energy,

$$e^{-2\Phi_0} Z_0 = \frac{\mu}{k} Z_0 \sim \frac{\epsilon}{k} Z_0 \,, \tag{24}$$

but, as explained in [5], Z_0 vanishes. Hence, to compute the first non-trivial term in the free energy one must compute the string partition function on the torus. This partition function is proportional to $\log E$. The computation was done in [5], where the coefficient α was found to be

$$\alpha = -1 - a_1 V_5 \,, \tag{25}$$

where a_1 is a positive number which scales as $(k\alpha')^{-5/2}$ [5]. From (22) it follows that the pressure $P = -\partial \mathcal{F}/\partial V_5 \sim a_1 \log E$, and thus the speed of sound,

$$v_s = \sqrt{\frac{\partial P}{\partial E}} \sim \frac{1}{\sqrt{E}} \,, \tag{26}$$

vanishes at $T = T_H$.

Computing two-point function of $T_{\mu\nu}$

We consider a system of k non-extremal NS5 branes. The Euclidean version of the near horizon geometry defines an exact superconformal field theory $\mathbf{R}^5 \times \frac{SL(2)}{U(1)} \times SU(2)$. We denote by X^μ coordinates on $\mathbf{R}^5$ and by ψ_μ their superpartners. Far from the tip of the cigar, the background has an asymptotic form of a cylinder with linear dilaton. Both ϕ and τ have their fermion superpartners ψ_ϕ and ψ_τ. The central charge of the cigar theory is $c_{SL(2)/U(1)} = 3 + 6/k$, so that the total central charge is $15/2 + 3 + 6/k + 9/2 - 6/k = 15$.

Below we focus on the quantities which are holomophic on the worldsheet (there are similar expressions for their antiholomorphic counterparts). The asymptotic expressions for the generators of the $\mathcal{N} = 2$ worldsheet superconformal algebra are

$$G^+ = i\psi\partial X^* + iQ\partial\psi, \qquad G^- = i\psi^*\partial X + iQ\partial\psi^*, \qquad J = \psi\psi^* + iQ\partial\tau \,, \tag{27}$$

where $\psi = (\psi_\phi + i\psi_\tau)/\sqrt{2}$ and $Q = \sqrt{2/k}$. The important set of observables in the $SL(2)/U(1)$ model consists of Virasoro primaries V_{jm} with the conformal dimension and the $U(1)_R$ charge given respectively by

$$\Delta[V_{jm}] = -\frac{j(j+1)}{k} + \frac{m^2}{k}, \qquad q = \frac{2m}{k}\,. \tag{28}$$

The asymptotic behavior of V_{jm} is

$$V_{jm} \cong \frac{e^{imQ\tau}e^{jQ\phi}}{2j+1}\,. \tag{29}$$

This allows us to compute the action of superconformal generators on V_{jm}:

$$G^+_{-\frac{1}{2}}V_{jm} \cong -iQ(j+m)\psi V_{jm}, \qquad G^-_{-\frac{1}{2}}V_{jm} \cong -iQ(j-m)\psi^* V_{jm}\,. \tag{30}$$

Now we briefly describe the computation of the two-point function of the stress-energy tensor (10). For more details, see [1]. String theory computation is performed in Eucledian space, making ω quantized in the units of temperature. To recover the Lorenzian version of the correlator, we must perform analytic continuation to imaginary frequencies.

Consider polarization that is longitudinal on the boundary. The vertex operator has the following asymptotic form

$$V^l = c\bar{c}e^{-\varphi-\bar{\varphi}}\xi^{za}\left[(\psi_z + A\psi_\phi)\bar{\psi}_a + \psi_a(\bar{\psi}_z + A\bar{\psi}_\phi)\right]e^{iqz}V_{jm\bar{m}}\,. \tag{31}$$

For a moment we will concentrate on the holomorphic part of the vertex operator,

$$(\psi_z + A\psi_\phi)e^{iqz}V_{jm}\,. \tag{32}$$

We must also require (31) to be BRST-invariant, which allows one to express j in terms of m and q and also fixes A

$$A = -\frac{\sqrt{2}q(j^2-m^2)}{\frac{4m^2}{k} - 2jq^2}\,. \tag{33}$$

The result for the two-point function corresponding to the shear mode is

$$G_{x^az,x^az}(\mathbf{q},\mathbf{w}) \sim \frac{\mathbf{w}^2\left(\mathbf{w}^2 + 2j\mathbf{q}^2 - \frac{\mathbf{q}^4}{4}\right)}{\left(\mathbf{w}^2 + j\mathbf{q}^2\right)^2}\frac{\Gamma\left(1-\frac{2j+1}{k}\right)\Gamma\left(-2j-1\right)\Gamma^2\left(1+i\frac{\mathbf{w}}{2}+j\right)}{\Gamma\left(\frac{2j+1}{k}\right)\Gamma\left(2j+2\right)\Gamma^2\left(-i\frac{\mathbf{w}}{2}-j\right)}\,, \tag{34}$$

where

$$j = -\frac{1}{2} + \frac{\sqrt{1-\mathbf{w}^2+\mathbf{q}^2}}{2}\,. \tag{35}$$

and $\mathbf{w} = w/2\pi T, \mathbf{q} = q/2\pi T$. The Green's function for the sound mode is computed in a similar manner. One simply needs to notice that both holomorphic and antiholomorphic parts of the vertex operator take the form of (32). The result for the Green's function is then

$$G_{zz,zz}(\mathbf{q},\mathbf{w}) \sim \left[\frac{\mathbf{w}^2\left(\mathbf{w}^2+2j\mathbf{q}^2-\frac{\mathbf{q}^4}{4}\right)}{\left(\mathbf{w}^2+j\mathbf{q}^2\right)^2}\right]^2 \frac{\Gamma\left(1-\frac{2j+1}{k}\right)\Gamma\left(-2j-1\right)\Gamma^2\left(1+i\frac{\mathbf{w}}{2}+j\right)}{\Gamma\left(\frac{2j+1}{k}\right)\Gamma\left(2j+2\right)\Gamma^2\left(-i\frac{\mathbf{w}}{2}-j\right)}\,. \tag{36}$$

We will be mostly interested in the poles of the Green's functions which correspond to the excitations without a gap, i.e. the hydrodynamic poles with the property $\mathbf{w}\to 0$ as $\mathbf{q}\to 0$. In this limit $j\to 0$. Consider first the shear mode [eq. (34)]. A possible source of poles is the denominator $\left(\mathbf{w}^2+j\mathbf{q}^2\right)^2$. The equation

$$\mathbf{w}^2 + j\mathbf{q}^2 = 0 \tag{37}$$

has a simple solution $\mathbf{w}^2 = -\mathbf{q}^4/4$, $j = \mathbf{q}^2/4$. Hence the denominator appears to contribute a double pole at

$$\mathbf{w} = \pm i\frac{\mathbf{q}^2}{2}\,. \tag{38}$$

However, the numerator in the first factor has a simple zero at (38)

$$\left(\mathbf{w}^2+2j\mathbf{q}^2-\frac{\mathbf{q}^4}{4}\right) = 0 \quad \text{for} \quad \mathbf{w}^2 = -\frac{\mathbf{q}^4}{4}\,. \tag{39}$$

Hence the first factor in (34) contributes only a single pole at $\mathbf{w}$ given by (38). One of these poles is cancelled by a zero coming from $\Gamma^{-2}\left(-i\frac{\mathbf{w}}{2}-j\right)$. Indeed, (38) with a plus sign is a solution of

$$-i\frac{\mathbf{w}}{2} - j = 0\,. \tag{40}$$

Therefore we are left with a single hydrodynamic pole at $\mathbf{w} = -i\mathbf{q}^2/2$.

To summarize, the retarded Green's function for the shear mode has the form

$$G_{x^a z, x^a z} \sim \frac{1}{(\mathbf{w}+\mathbf{q})(\mathbf{w}-\mathbf{q})(i\mathbf{w}-\frac{\mathbf{q}^2}{2})}\,, \tag{41}$$

where we only exhibit the structure of poles which correspond to excitations without a gap. In addition to the poles that correspond to the propagating modes, there is a single hydrodynamic pole at

$$\omega = \frac{q^2}{4\pi T}\,. \tag{42}$$

Comparing with Eq. (5) we find $\eta/s = 1/4\pi$.

Turning to the correlators in the sound channel, we observe that the difference between Eq. (34) and Eq. (36) is that in Eq. (36) the prefactor is squared. We immediately conclude that in the hydrodynamic regime the correlator $G_{zz,zz}$ has the form

$$G_{zz,zz} \sim \frac{1}{(\mathbf{w}+\mathbf{q})(\mathbf{w}-\mathbf{q})(i\mathbf{w}-\frac{\mathbf{q}^2}{2})^2}\,. \tag{43}$$

Comparing this to the discussion in Section 3 we find the speed of sound and the ratio of bulk viscosity to entropy density at $T = T_H$:

$$v_s = 0\,, \qquad\qquad \frac{\zeta}{s} = 1/10\pi\,. \tag{44}$$

Note that these results are exact to all orders in $1/k$.

Discussion

We have computed transport coefficients in Little String Theory at Hagedorn temperature. Our result for the correlation function in the sound channel agrees with predictions of hydrodynamics up to the terms quartic in spatial momentum. To account for those terms, one needs to improve the hydrodynamic description, possibly by including higher-derivative terms in the constitutive relation (4). It would certainly be interesting to extend the analysis to temperatures other than the Hagedorn temperature. This would correspond to including higher loop corrections in the string amplitudes.

In addition to the seemingly well-defined hydrodynamic poles, for sufficiently large values of spatial momenta all the correlators have an identical set of singularities, including the poles on the real axis in the complex ω-plane, and the poles on the negative and positive imaginary axis. Also, one of the poles in the correlators formally corresponds to a mode propagating with the speed of light. Normally, retarded correlators cannot have poles both in the lower and upper half-planes in a stable system, and, moreover, one does not expect a purely propagating mode to exist in a thermal medium. Since the characteristic wavelength of the poles with finite gap is $\sqrt{k}l_s$, it is conceivable that their existence is related to the non-locality of LST.

Poles of similar nature arise in the correlators of LST in a double scaling limit. Authors of [6] observed massless poles that do not correspond to physical states in the $U(1)^{k-1}$ super Yang-Mills theory which naively is supposed to be a good description of LST at low energies. From the world-sheet point of view, the relevant correlators on the cigar are saturated in the bulk, far from the tip. It has been argued in [6] that these poles appear due to the UV/IR mixing, i.e. highly massive states do not decouple in the infrared of LST. Massive poles, which are analogous to the non-hydrodynamic poles described at the end of section 6, were found to be of similar origin [6]. These poles, coming from non-locality and the UV/IR mixing should be distinguished from the other, more conventional poles, which correspond to the normalizable states at the tip of the cigar. These poles correspond to physical states in LST.

The instability of the high-energy phase of LST appears to be similar to that of a Schwarzschild black hole whose specific heat also diverges to minus infinity as $E \to \infty$. We find it curious that the speed of sound in LST vanishes precisely at Hagedorn temperature. As we mentioned in the Introduction, in more conventional systems such a behavior might be associated with a phase transition.

We found that the hydrodynamic pole does not receive α' (or, equivalently, $1/k$) corrections, and the ratio η/s is equal to the universal value $1/4\pi$. This should be contrasted with the results of [4], where the curvature corrections to the near-extremal D3-branes were investigated. In [4] it was found that such corrections increase η/s. In the system with RR flux, turning on α' corrections is associated with departing from the infinite value of the t'Hooft coupling in the dual gauge theory. This fits well with the proposal that the viscosity bound should be saturated in strongly coupled systems. The case without RR flux studied in this paper appears to be fundamentally different. Indeed, there is no known way in which the theory on k NS5 branes becomes weakly coupled at large energy densities, even when k is small. It would be interesting to see what effect lowering energy density has on the value of η/s.

In [7] a large class of LST vacua dual to string theory compactified on a singular n-dimensional Calabi-Yau manifold was constructed. The backgrounds considered in [7] have a general form

$$\mathbf{R}^{d-1,1} \times \mathbf{R}_\phi \times \mathcal{N} \tag{45}$$

where $\mathbf{R}_\phi$ describes the linear dilaton direction and $\mathcal{N}$ defines a superconformal theory whose detailed properties are discussed in [7]. In (45) $d = 10 - 2n$. Maximally supersymmetric LST in 5+1 dimensions discussed in our paper correspods to the Calabi-Yau being a two-dimensional ALE space. (In this case $\mathcal{N} = SU(2)_k$). Choosing the Calabi-Yau to be a singular three-fold can give rise to NS5 branes wrapping various Riemann surfaces.

String theory calculations in our paper generalize straightforwardly to these cases. Indeed, introducing finite temperature to the system described by (45) and performing Wick rotation, we end up with the background

$$\mathbf{R}^{d-1} \times \frac{SL(2)_{k'}}{U(1)} \times \mathcal{N} \tag{46}$$

where k' is determined by requiring (46) to be a consistent background for superstring propagation (total central charge of the worldsheet matter should be equal to 15). The computations of the stress-energy Green's function in Section 5 do not involve the $\mathcal{N}$ theory, and therefore are unaltered. Hence, our computation of η/s and ζ/s is valid for a large class of LSTs. In general, k' does not need to be an integer. When $d \geq 4$, k' is bounded from below by $k' = 1$, which in $d = 4$ corresponds to the Calabi-Yau being the singular conifold. The holographic computation requires j to define non-normalizable state in the cigar. In the hydrodynamic limit $j \to 0$, which indeed corresponds to the non-normalizable state, as long as $k' \geq 1$. The bound on k' can be violated when $d = 2$.

References

[1] A. Parnachev and A. Starinets, "The silence of the little strings," arXiv:hep-th/0506144.

[2] G. Policastro, D. T. Son and A. O. Starinets, "From AdS/CFT correspondence to hydrodynamics," JHEP **0209**, 043 (2002) [arXiv:hep-th/0205052].

[3] P. Kovtun, D. T. Son and A. O. Starinets, "Holography and hydrodynamics: Diffusion on stretched horizons," JHEP **0310**, 064 (2003) [arXiv:hep-th/0309213].

[4] A. Buchel, J. T. Liu and A. O. Starinets, "Coupling constant dependence of the shear viscosity in N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **707**, 56 (2005) [arXiv:hep-th/0406264].

[5] D. Kutasov and D. A. Sahakyan, "Comments on the thermodynamics of little string theory," JHEP **0102**, 021 (2001) [arXiv:hep-th/0012258].

[6] O. Aharony, A. Giveon and D. Kutasov, "LSZ in LST," Nucl. Phys. B **691**, 3 (2004) [arXiv:hep-th/0404016].

[7] A. Giveon, D. Kutasov and O. Pelc, "Holography for non-critical superstrings," JHEP **9910**, 035 (1999) [arXiv:hep-th/9907178].