The Hitchin functionals and the topological B-model at one loop

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Based on a paper written with E. Witten.
Please refer to [1] for the complete list of references and the details.

Introduction

In [2, 3, 4] it was suggested that the Hitchin functional [5] is related to the topological B model [6, 7, 8, 9]. The topological B-model studies maps of a Riemann surface Σ into a target space X, and it couples to deformations of the complex structure of X. The Hitchin functional S_H can be viewed as an (effective) target space theory action on X. In some sense it is a real analogue of the Kodaira-Spencer gravity [9]. Roughly speaking, the Hitchin functional S_H is defined on the space of all almost complex structures on the real six dimensional manifold X, while its critical points are integrable complex structures. Using special properties of three-forms in six dimensions, variations of almost complex structures can be mapped to variations of real three-forms, and S_H is defined on the functional space of real three-forms. It is interesting to compare the free energy of the target space theory with action S_H with the free energy of topological B-model.

Construction of S_H

Our goal is to construct, following Hitchin, a functional defined on the space of almost complex structures, such that its critical points are integrable complex structures. We use special properties of three-forms in six dimensions.

Namely, any non degenerate three-form ρ can be represented as a sum of two three-forms

$$\rho = \alpha + \beta,$$

where α and β are decomposable into a product of one-forms

$$\alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$$

and

$$\beta = \beta_1 \wedge \beta_2 \wedge \beta_3.$$

The three-forms α and β are uniquely defined up to permutation, while the one forms α_i and β_i are defined only up to an $SL(3)$ action. This statement is easily checked by dimensional counting. The space of three-forms in six dimensions has dimension $6!/3!3! = 20$.

The dimension of the $GL(6)$ group is 36. The dimension of the stabilizer $SL(3)\times SL(3)$ is $8+8 = 16$. Hence the dimension of the orbit of some canonical three-form $e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$ under the action of $GL(6)$ is $36-16 = 20$. We see that it is equal to the dimension of the space of three-forms.

The statement above was formulated for an algebraically complete field such as $\mathbb{C}$. If we want to make analogue statement for *real* three-form ρ, then using the property above, we represent $\rho = \alpha + \beta$, where α and β are (generally complex) decomposable three-forms. Then, from the reality condition $\rho = \bar{\rho}$ it follows, that (i) either $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$, or (ii) $\alpha = \bar{\beta}$. The cases (i) and (ii) can be viewed as the domains of 'positive' and 'negative' real three forms in six dimensions. The case (ii) is actually interesting for us, since it allows us to build an almost complex structure as following. Take as the holomorphic subspace 'z' the space spanned by $\alpha_1, \alpha_2, \alpha_3$, and as the corresponding antiholomorphic subspace '$\bar{z}$' the space spanned by $\beta_1, \beta_2, \beta_3$. Algebraically, the domains (i) and (ii) in the space of three-forms are separated by hypersufarce of codimension one, where the three-form degenerates. Therefore, if we start with a three-form of type (ii), and consider its small deformations, they will be again of type (ii), and hence will again define an almost complex structure.

So we have learned, that locally we can map almost complex structures on X to three-forms on X. Now we want to construct the functional $S_H(\rho)$, such that its critical points will be integrable complex structures. The natural functional $S_H[\rho]$ is just the volume form (up to a normalization)

$$S = i \int \alpha \wedge \beta,$$

where are α and β are (locally) uniquely constructed by the three-form ρ as above.

Let us introduce $\hat{\rho} = i(\bar{\alpha} - \alpha)$, and $\Omega \equiv \alpha = \rho + i\hat{\rho}$, so that Ω is the $(3,0)$ form. So we have $\rho = \mathrm{Re}\Omega$, and $\hat{\rho} = \mathrm{I}m\Omega$. The functional $S_H(\rho)$ is equivalently

$$S_H(\rho) = i \int \alpha \wedge \bar{\alpha} = \frac{1}{2} \int \rho \wedge \hat{\rho} = -\frac{i}{4} \int \Omega \wedge \bar{\Omega}$$

The formula for S_H

Let X be a real six dimensional manifold, let $\rho \in \bigwedge^3(TX)$ is a real three-form on X (of type (ii)). Given ρ let us define map $K(\rho)$ from TX to $\bigwedge^5(TX^*)$ as

$$v \mapsto (v \cdot \rho) \wedge \rho \tag{1}$$

where $(v \cdot \rho)$ means contraction of vector v and the three-form ρ. Since $\bigwedge^5(TX^*) \simeq \bigwedge^6(TX^*) \otimes TX$ the map K can be considered to be in $\mathrm{End}(TX)$ with values in $\bigwedge^6(TX^*)$. Therefore $\sqrt{-\frac{1}{6}\mathrm{tr}K^2} \in \bigwedge^6(TX^*)$ can be integrated over X and defines S_H

$$S_H(\rho) = \int_X \sqrt{-\frac{1}{6}\mathrm{tr}K(\rho)^2} \tag{2}$$

The property of the form to be of type (ii) means that $\mathrm{tr}K^2 < 0$.

In coordinates one can write the matrix K_b^a as following

$$K_a^b(\rho) = \frac{1}{2!3!}\epsilon^{bi_2i_3i_4i_5i_6}\rho_{ai_2i_3}\rho_{i_4i_5i_6}. \tag{3}$$

We can check that S_H is a nontrival nonlinear function of ρ, namely it is written as the square root of the certain quartic polynomial of ρ_{ijk}.

Critical points of $S_H(\rho)$ and integrability condition

Let us consider the space of closed three-forms, and moreover, let us fix the cohomology class as $\rho = \rho_0 + db$, where ρ_0 is some representative of $H^3(X, \mathbb{R})$.

Hitching proved that at the critical points of $S_H(\rho_0 + db)$ with respect to b, the almost complex structure constructed by $\rho = \rho_0 + db$ is integrable. To prove this statement consider a variation of $S_H(\rho_0 + db)$

$$\delta S_H = \frac{1}{2}\int \rho \wedge \hat{\rho} = \frac{1}{2}\int \delta\rho \wedge \hat{\rho} + \frac{1}{2}\int \rho \wedge \delta\hat{\rho} = \int \delta\rho \wedge \hat{\rho}$$

where the last equation holds because $\rho\hat{\rho}$ is an homogeneous function of ρ of the second degree. Therefore

$$\delta S_H = \int \delta b \wedge d\hat{\rho}$$

and at the critical point we have $d\hat{\rho} = 0$. Since $d\rho = 0$ by construction, we have $d\Omega = d(\rho + i\hat{\rho}) = 0$. From the fact that the (3,0) form Ω is closed follows [5] the integrability condition for the complex structure.

Moreover, Hitchin proved that modulo $Diff(X)$ group the Hessian of the functional is nondegenerate in the critical point. Therefore, locally an element of $H^3(X, \mathbb{R})$ defines a complex structure on

X together with (3,0) holomorphic form Ω, which will be called CY structure for a short. Locally, $H^3(X,\mathbb{R})$ is the moduli space of CY structures on X. (Compare it with the traditional parametrization of moduli space of topological B-model, where the tangent space is $H^{(3,0)}(X,\mathbb{C})\oplus H^{(2,1)}(X,\mathbb{C})$. The dimension is the same, but the slice in $H^3(X,\mathbb{C})$ is chosen differently).

Correspondence between topological B-model and S_H at classical level

So we consider the field theory of closed real three forms with fixed cohomology class in six dimensions, defined by the action S_H. Symbolically, the quantum field theory is defined by the following functional integral

$$e^{-F_H} = Z_H[\rho_0] = \int Dbe^{-S_H[\rho_0+db]}$$

So the partition function Z_H is a function of the cohomology class $\rho_0 \in H^3(X,\mathbb{R})$. Classicaly, the free energy $F_H = S_H$ evaluated in the critical point. In the critical point of S_H the manifold X has CY structure and

$$F_H[\rho] = -\frac{i}{4}\int \Omega\wedge\bar{\Omega}$$

In a canonical basis of A_i, B_i cycles in $H_3(X,\mathbb{Z})$ using bilinear Riemann identities, S_H can be written as

$$F_H[\rho] = -\frac{i}{4}(X^i\bar{F}_i - F^i\bar{X}_i),$$

where $X^i = \int_{A_i}\Omega$ and $F_i = \int_{B^i}\Omega$.

Let us recall that the genus zero free energy of the topological B model is given by the formula

$$F_B[X] = \frac{1}{2}X^iF_i$$

What is the relation between F_H and F_B? First of all we have to identify what variables they depend on. The free energy F_H is a function of ρ which is a real part of the (3,0) form Ω, in other words it is a function on $H^3(X,\mathbb{R})$. While F_B is traditionally considered to be a function of $H^{(3,0)}(X,\mathbb{C})\oplus H^{(2,1)}(X,\mathbb{C})$, or, equivalently, a function of periods X^i. Going to real variables $\operatorname{Re}X^i, \operatorname{Im}X^i, \operatorname{Re}F^i, \operatorname{Im}F^i$ we see that

$$F_B \equiv F_B[\operatorname{Re}X^i, \operatorname{Im}X^i]$$

while

$$F_H \equiv F_H[\operatorname{Re}X^i, \operatorname{Re}F^i]$$

Thus F_B and F_H depend on different variables. However the relation between $\operatorname{Im}F_B$ and F_H is very simple. Namely F_H is a Legendre transform of $\operatorname{Im}F_B$ with respect to $\operatorname{Im}X^i$ and vice verse.

Quantization of $S_H(\rho)$ in quadratic order

Now we want to compare F_B and F_H beyond the classical limit. The leading quantum correction is given by the determinants coming from the Gaussian functional integral for the action expanded up to the second order near the critical point ρ_c.

The quadratic term of $S_H(\rho_c + db)$ in terms of Hodge decomposition $b = b_{20} + b_{11} + b_{02}$ is simply

$$S_H = \int \partial b_{11} \wedge \bar{\partial} b_{11} \tag{4}$$

First we note the trivial degeneracy of the action with respect to the components b_{20} and b_{02}, they are simply absent in the quadratic expansion. It was mentioned before that the quadratic part of the Hitchin functional is nondegenerate up to the diffeomorphism group. Indeed, the modes b_{20} and b_{02} are precisely identified with diffeomorphisms generated by some vector field ξ as following

$$\delta\rho = L_\xi \rho = (i_\xi d + d i_\xi)\rho = d(i_\xi \rho) \tag{5}$$

Since at the critical point $\rho = \mathrm{Re}\,\Omega$ and Ω is (3,0) form we see that $\delta\rho = d(i_\xi \rho)$, is represented as d of (2,0) plus (0,2) form.

So the modes b_{20} and b_{02} are gauged away by $\mathrm{Diff}(X)$ symmetry. There still remains the gauge symmetry of the form $b_{11} = \bar{\partial} b_{10} + \partial b_{01}$. Moreover these gauge transformations are degenerate, namely they vanish if $b_{10} = \partial b_{00}$ and $b_{01} = \bar{\partial} b_{00}$. Quantization of gauge theories with degenerate gauge symmetries is most conveniently done in the Batalin-Vilkovysky formalism [10, 11]. There the set of fields and ghosts fields is extended by set of antifields. The antifields are canonically conjugate to fields with respect to odd symplectic structure $\{,\}$ on the functional space, which is called antibracket. The Q transformations of fields are viewed as an odd vector field in the functional phase space associated with the master action S, so the field f transforms as $Qf = S, f$. There is a requirement that Q^2, or equivalently, $\{S, S\} = 0$. After extending the physical action to the master action S one integrates over some Lagrangian submanifold in the BV phase space. Equivalently that can be done by introducing gauge fixing fermion

The parity of the ghosts fields is alternating with its level, so ghosts for ghosts are bosons, and so on. In the quadratic case, like we have, it is easy to make Gaussian integration. The result will be an alternating product of determinants arising from Gaussian integrals for the fields and ghosts of all levels.

The complete technical details can be looked in [1], and the result turns out to be expressed in terms of Ray-Singer torsions.

$$Z_{H,1-loop} = \frac{I_1}{I_0}. \tag{6}$$

Comparison with genus one free energy of the topological B-model

Now we need to compare this result with the genus one free energy of the topological B-model. The genus one free energy of the topological B-model was computed in [9] and turned out to be equal to some product of $\bar{\partial}$ Ray-Singer torsions I_p for the bundles of holomorphic p-forms $\Omega^{p,0}(X)$ on X:

$$F_1^B = \log \prod_{p=0}^{n} (I_p)^{(-1)^{p+1}p} = -\log \frac{I_1}{I_0^3}. \tag{7}$$

Here I_p stands for

$$I_p = \left(\prod_{q=0}^{n} ({\det}' \Delta_{pq})^{(-1)^{q+1}q} \right)^{1/2}. \tag{8}$$

In terms of determinants

$$F_1^B = \frac{1}{2} \log \prod_{p=0}^{n} \prod_{q=0}^{n} ({\det}' \Delta_{pq})^{(-1)^{p+q}pq}. \tag{9}$$

We can see now that $Z_{H,1-loop} = I_1/I_0$ and $Z_{B,1-loop} = I_1/I_0^3$ differ by a factor of I_0^2. Does this points to failure of the correspondence between target space theory with Hitchin functional and the topological B-model?

The resolution of this discrepancy turns out to be nice, and actually natural from the view point of the topological B-model.

Generalized complex structures and the resolution of the discrepancy

Namely, we have to consider not variations of complex structure on X, but variations of the *generalized complex structure* on X. The notion of *generalized complex structure* was introduced by Hitchin [12] and then developed in the thesis [13].

An almost complex structure can be viewed as a section $J \in End(TX)$ with condition $J^2 = -1$. A generalized complex structure is defined similarly, by extending TX to $TX \oplus TX^*$. So, a generalized complex structure is a section $\mathcal{J} \in End(TX \oplus TX^*)$, which satisfies $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$.

Generalized complex structures naturally interpolate between complex structures and symplectic structures. For example, a complex structure J defines the generalized complex structure

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & J^{-1} \end{pmatrix}. \tag{10}$$

And a symplectic structure ω defines the generalized complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{11}$$

In the standard case we had a correspondence between complex structures and real three-forms. In the generalized case Hitchin's construction gives a correspondence between generalized complex structures and an odd real form of mixed degree $\rho_1 + \rho_3 + \rho_5$. The generalized Hitchin functional is defined in a similar fashion on a space of odd-forms with the fixed cohomology class $H^{odd}(X, \mathbb{R})$. The critical points of that functional give integrability conditions for the generalized complex structure defined by $\rho = \rho_1 + \rho_3 + \rho_5$. The Hessian of the functional is again non-degenerate modulo diffeomorphisms of X and B-field transforms [12]. To compute the partition function in quadratic order we again take the quadratic term of the action and follow BV quantization for careful treatment of (degenerate) gauge transformation. The details can be looked again in [1]. The quadratic term of the extended Hitchin functional looks like

$$S_{H,ext} = \int b_{11} \wedge \partial\bar{\partial} b_{11} + b_{00} \wedge \partial\bar{\partial} b_{22} \tag{12}$$

It is the second term of this expression which differs from the one that we had in the case of Hitchin functional for complex structures. And it turns out, that after BV completion, careful gauge fixing and the Gaussian integration the second term produces precisely the determinants I_0^{-2} that were missing in comparison between $Z_{H,1-loop}$ and $Z_{B,1-loop}$. The conclusion is that

$$Z_{H_{ext},1-loop} = \frac{I_1}{I_0^3} = Z_{B,1-loop} \tag{13}$$

Interpretation of the results on the B-model side

It is interesting that the emergence of the generalized complex structure concept can be observed on the topological B-model side.

Let us recall that the physical states in B-model are identified with sections of $H^q_{\bar{\partial}}(\bigwedge^p(TX))$, that is closed $(-p, q)$ forms modulo $\bar{\partial}$ exact ones. The case $(-1, 1)$ corresponds to Beltrami differentials $\mu^i_{\bar{j}}$, which represent infinitesimal deformation of complex structure. The operators with $p + q = 2$ can be used to construct deformation of Lagrangian of ghost number zero by the descend relations

$$\{Q, \mathcal{O}^{(1)}\} = d\mathcal{O}^{(0)} \tag{14}$$
$$\{Q, \mathcal{O}^{(2)}\} = d\mathcal{O}^{(1)} \tag{15}$$

If $(-1, 1)$ deformations stand for deformations of the complex structure, what is the geometrical meaning of the $(-2, 0)$ and $(0, 2)$ deformations?

In the picture of generalized complex geometry these deformations are identified with the non diagonal blocks β^{ij} and B_{ij} in the matrix of generalized complex structure

$$\mathcal{J} = \begin{pmatrix} J^i_j & \beta^{ij} \\ B_{ij} & -J^j_i \end{pmatrix}. \tag{16}$$

In the algebraic approach to the open B-model, where B-branes are roughly speaking holomorphic vector bundles on X, (or, more precisely, derived categories of coherent sheaves on X), deformations have the following meaning. The deformations of type (-1,1) correspond to the usual deformations of complex structure, parameterized by Beltrami differential $\mu^i_{\bar{j}}$. The deformations of type (-2,0), parameterized by holomorphic bivector β^{ij} correspond to the noncommutative Kontsevich deformation of the multiplication. And deformations of type $(2,0)$, parameterized by two-form $B_{\bar{i}\bar{j}}$ correspond to deformation of holomorphic vector bundles into gerbes, see [14, 15, 16] and references therein.

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