Localization for Wilson Loops in Chern-Simons Theory

Chris Beasley
Harvard University

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Introduction

The topic of my talk today is Chern-Simons gauge theory, and my goal is simply to explain to you some very beautiful mathematical facts about the theory. This talk is a sequel to a talk I gave at the Simons Workshop last year, which was based upon a previous paper [1] with E. Witten. The talk today is based upon a follow-up paper [2] to appear.

More or less as a means to establish notation, let me begin by reminding you of some elementary features of Chern-Simons theory.

Chern-Simons theory is an intrinsically three-dimensional gauge theory, defined on a compact, oriented three-manifold M. Throughout the talk, the Chern-Simons gauge group will be G, a compact, connected, simply-connected, and simple Lie group. For example, G could be $SU(N)$ for some N.

As a classical functional of the gauge field A, which is a connection on the trivial G-bundle over M, the Chern-Simons action $\mathbf{CS}(A)$ is given by

$$\mathbf{CS}(A) \,=\, \int_M \mathrm{Tr}\big(A{\wedge}dA + \frac{2}{3}A{\wedge}A{\wedge}A\big)\,. \tag{1}$$

If $G = SU(N)$, the symbol 'Tr' denotes the trace in the fundamental representation. More generally, with our assumptions on the group G, 'Tr' is fixed up to normalization as an invariant, negative-definite quadratic form on the Lie algebra $\mathfrak{g}$ of G.

The Chern-Simons action $\mathbf{CS}(A)$ has two important properties. First, $\mathbf{CS}(A)$ is a manifestly topological action, as only an orientation – not a metric – on M appears in (1). Second, though the

Chern-Simons action is not manifestly gauge-invariant (and indeed it is *not* gauge-invariant), the Chern-Simons action is almost gauge-invariant in the sense that it is invariant those gauge transformations which can be continuously connected to the identity. Otherwise, under homotopically non-trivial, "large" gauge transformations, the Chern-Simons action merely shifts by an integral multiple of $8\pi^2$, assuming that the form 'Tr' is suitably normalized.

The most basic observable in Chern-Simons theory on M is the partition function $Z(k)$,

$$Z(k) \,=\, \frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{\mathcal{A}} \mathcal{D}A \, \exp\left[i \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right]. \tag{2}$$

So long as the level k is an integer, the exponential of the Chern-Simons action appearing in (2) is completely gauge-invariant, as required to discuss a physically-sensible path integral. Formally, $Z(k)$ is described by an integral over the infinite-dimensional affine space $\mathcal{A}$ of all connections on M, and as is standard in gauge theory, $Z(k)$ is normalized by dividing by the volume of the infinite-dimensional group $\mathcal{G}$ of gauge transformations acting on $\mathcal{A}$.

In my previous talk, I focused attention exclusively on the Chern-Simons partition function. However, no discussion of Chern-Simons theory can be complete without also including the Wilson loop operators, whose expectation values are related [3] to knot invariants such as the celebrated Jones polynomial.

Quite generally, a Wilson loop operator $W_R(C)$ in any gauge theory on a manifold M is described by the data of an oriented, closed curve C which is smoothly[1] embedded in M and which is decorated by an irreducible representation R of the gauge group. As a classical functional of the connection A, the Wilson loop operator is given simply by the trace in R of the holonomy of A around C,

$$W_R(C) \,=\, \mathrm{Tr}_R \, P \exp\left(-\oint_C A\right). \tag{3}$$

To describe the expectation value of $W_R(C)$ in the Lagrangian formulation of Chern-Simons theory, let me first introduce the absolutely-normalized Wilson loop path integral,

$$Z(k; C, R) \,=\, \frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{\mathcal{A}} \mathcal{D}A \;\; W_R(C) \;\, \exp\left[i \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right]. \tag{4}$$

In terms of $Z(k)$ and $Z(k; C, R)$, the Wilson loop expectation value is then given by the ratio

$$\big\langle W_R(C) \big\rangle \,=\, \frac{Z(k; C, R)}{Z(k)}. \tag{5}$$

[1] The condition that C be smoothly embedded in M is merely for convenience and is not strictly required to define $W_R(C)$ as a sensible operator in gauge theory. Indeed, the Wilson loop expectation value in Chern-Simons theory can be computed exactly even for the case that C is an arbitrary closed graph [4] in M.

Among other results, one which I plan to sketch today is how the Wilson loop path integral in (4) can be computed exactly, at least in special cases.

Of course, you do not need me to explain to you how to compute $Z(k; C, R)$. In 1989, Witten [3] explained quite precisely how to compute the arbitrary Wilson loop path integral in Chern-Simons theory, based upon the canonical quantization of the theory and its relation to rational conformal field theory. What I want to discuss here is a completely different way to compute $Z(k; C, R)$, a way which is based much more closely upon the path integral formulation of Chern-Simons theory.

To the naked eye, Chern-Simons theory appears to be a purely bosonic gauge theory. Nonetheless, I will show that for certain observables, Chern-Simons theory is secretly supersymmetric, in the sense that the theory can be reformulated to admit a conserved, nilpotent scalar supercharge Q of the sort that arises naturally in topological twists of supersymmetric Yang-Mills theory (as for instance lead to Donaldson/Seiberg-Witten theory in four dimensions). Just as in topological Yang-Mills theory, the presence of Q in Chern-Simons theory implies that for Q-closed observables, the path integral in (4) localizes onto a finite-dimensional moduli space of supersymmetric solutions, and the integrand of (4) reduces to a cohomology class on that moduli space.

The cohomological reformulation of Chern-Simons theory turns out to be interesting for at least two reasons. First, this reformulation of Chern-Simons theory provides a new and essentially classical interpretation for (some of) the *a priori* quantum topological invariants that are defined in the Hamiltonian approach to Chern-Simons theory. Second, from a more pragmatic perspective, the supersymmetric Wilson loop operators in Chern-Simons theory give nice examples in gauge theory for which the Wilson loop path integral can be computed exactly, in a more-or-less direct fashion. Unfortunately, I will not have time in this talk to perform explicit computations of the Wilson loop path integral in Chern-Simons theory, but see [2] for concrete examples.

Some Experimental Evidence

Before proceeding further, I want to present some experimental evidence that the stationary-phase approximation to the Wilson loop path integral is exact, at least for a particularly simple class of Wilson loop operators in a particularly simple class of three-manifolds. This evidence is based upon remarkable work [5] by R. Lawrence and L. Rozansky, who performed a detailed analysis of the exact formulae for $Z(k; C, R)$ that arise from conformal field theory. In general, the exact expressions for $Z(k; C, R)$ have a rather complicated, arithmetic dependence on the level k. However, as demonstrated in [5], the dependence of $Z(k; C, R)$ on k dramatically simplifies for special choices of M and C.

Specifically, M must be a Seifert manifold, which is the total space of a nontrivial circle bundle over a Riemann surface Σ,

$$\begin{array}{ccc} S^1 & \longrightarrow & M \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \Sigma \end{array} \,. \tag{6}$$

Here Σ is allowed to have orbifold points, and the circle bundle is allowed to be a corresponding orbifold bundle, so long as M itself is smooth.

Amongst all three-manifolds, the Seifert manifolds can be usefully characterized as those which admit a locally-free $U(1)$-action, which rotates the fibers of (6). By definition, a locally-free $U(1)$-action is one for which the generating vector field is nowhere vanishing. Equivalently, all stabilizers under a locally-free $U(1)$ action are proper (necessarily discrete) subgroups of $U(1)$.

If M is a Seifert manifold, then M carries a distinguished class of Wilson loop operators which respect the Seifert $U(1)$-action. For these operators, the curve C is an orbit of the Seifert $U(1)$, and hence C appears in (6) as the S^1 fiber over a point $\sigma \in \Sigma$. Assuming that σ is a smooth (non-orbifold) point of Σ, the topology of C in M does not depend upon the choice of σ, which can be continuously varied. Let me refer to the special Wilson loop operators which wrap the generic Seifert fibers of M as "Seifert loop operators" to distinguish them from more general Wilson loop operators in M, about which I will also have some things to say.

According to [5], if M is a Seifert manifold and C is a generic Seifert fiber in M, then the exact result for $Z(k; C, R)$ derived from conformal field theory can be rewritten very compactly as a finite sum of analytic expressions, each expression being either a contour integral or the residue of a meromorphic function, such that the summands in $Z(k; C, R)$ correspond in a one-to-one fashion with the connected components in the moduli space of flat connections on M. Since the flat connections on M are the critical points of the Chern-Simons action, this form for $Z(k; C, R)$ strongly suggests that the stationary-phase approximation to the Seifert loop path integral is exact.

Let me give a concrete example for the sort of formulae that appear in [5]. In this example, I will take M to be S^3, which is a Seifert manifold by virtue of the Hopf fibration over $\mathbb{CP}^1$. The Hopf fiber is a great circle in S^3, so the associated Seifert loop operator wraps the unknot.

However, the unknot in S^3 is not the only knot associated to a Seifert loop operator. Much more generally, S^3 admits many locally-free $U(1)$-actions which are not necessarily free, and by a basic theorem of Moser [6], the knots which can be embedded as Seifert fibers in S^3 are precisely the torus knots.

Briefly, a torus knot $\mathcal{K}_{p,q}$ is described by a trigonometric embedding of S^1 into a two-torus $S^1 \times S^1$ itself embedded in S^3, under which

$$\mathrm{e}^{i\theta} \longmapsto \left(\mathrm{e}^{ip\theta},\, \mathrm{e}^{iq\theta}\right) \,, \qquad\qquad \gcd(p,q) \,=\, 1\,. \tag{7}$$

Here θ is an angular coordinate on S^1, and (p, q) are a pair of non-zero, relatively-prime[2] integers which determine the embedding. If either p or q is equal to 1, then $\mathcal{K}_{p,q}$ is equivalent to the unknot. Otherwise, for distinct pairs $p > q > 1$, the associated torus knots are non-trivial and topologically distinct.[3]

In Figure 1, I present three torus knots which can be drawn with few crossings, including the trefoil, which is $\mathcal{K}_{3,2}$.

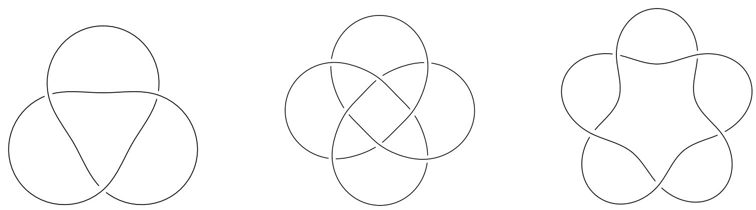

Figure 1: *The Torus Knots* $\mathcal{K}_{3,2}$, $\mathcal{K}_{4,3}$, *and* $\mathcal{K}_{5,2}$

If S^3 is embedded as the unit sphere in $\mathbb{C}^2$, then $\mathcal{K}_{p,q}$ appears as the generic orbit of the $U(1)$ which acts on the complex coordinates (w_1, w_2) of $\mathbb{C}^2$ with respective charges p and q,

$$\big(w_1, w_2\big) \overset{e^{i\theta}}{\longmapsto} \big(e^{ip\theta} w_1,\, e^{iq\theta} w_2\big)\,. \tag{8}$$

So long as both w_1 and w_2 are non-zero, the orbit representing $\mathcal{K}_{p,q}$ is embedded in the two-torus parametrized by the phases of w_1 and w_2. Dividing $S^3 \subset \mathbb{C}^2$ by the $U(1)$-action in (8), we then see that $\mathcal{K}_{p,q}$ sits as the general fiber in a Seifert fibration of S^3 over a genus-zero Riemann surface with two orbifold points. Upstairs on S^3, the orbifold points correspond to the two degenerate orbits where either w_1 or w_2 vanishes and upon which a cyclic subgroup $\mathbb{Z}_q$ or $\mathbb{Z}_p$ in $U(1)$ acts trivially.

The formulae presented by Lawrence and Rozansky in [5] are all for the simplest case that the Chern-Simons gauge group G is $SU(2)$. Irreducible representations of $SU(2)$ are uniquely labelled by their dimension, and I let $\mathbf{j}$ denote the irreducible representation of $SU(2)$ with dimension j.

According to Lawrence and Rozansky, the exact result for the Wilson loop path integral associated to a torus knot $\mathcal{K}_{p,q} \subset S^3$ decorated with the representation $\mathbf{j}$ can be written quite suggestively as

$$Z\big(k; \mathcal{K}_{p,q}, \mathbf{j}\big) \;=\; \frac{1}{2\pi i}\,\frac{1}{\sqrt{pq}}\,\exp\left[-\frac{i\pi}{2(k+2)}\left(\frac{p}{q} + \frac{q}{p} + pq\,\big(j^2 - 1\big)\right)\right] \times$$

[2] If p and q share a common factor $d > 1$, the map in (7) is a degree d covering of the embedding determined by the reduced pair $(p/d, q/d)$.

[3] For fixed (p, q), the knots $\mathcal{K}_{p,q}$ and $\mathcal{K}_{q,p}$ are equivalent under ambient isotopy. Also, $\mathcal{K}_{-p,-q}$ trivially agrees with $\mathcal{K}_{p,q}$ up to orientation. Finally, $K_{p,-q}$ is the mirror image of $\mathcal{K}_{p,q}$. All torus knots (other than the unknot) are chiral, meaning that $\mathcal{K}_{p,q}$ is not isotopic to its mirror.

$$\times \int_{-\infty}^{+\infty} dx \;\; \chi_{\mathbf{j}}\Big(\mathrm{e}^{\frac{i\pi}{4}}\,\frac{x}{2}\Big)\,\sinh\Big(\mathrm{e}^{\frac{i\pi}{4}}\,\frac{x}{2p}\Big)\,\sinh\Big(\mathrm{e}^{\frac{i\pi}{4}}\,\frac{x}{2q}\Big)\,\exp\Big(-\frac{(k+2)}{8\pi}\frac{x^2}{pq}\Big)\,,$$
$$j = 1, \ldots, k+1\,. \tag{9}$$

Here $\chi_{\mathbf{j}}$ is the character of $SU(2)$ in the representation $\mathbf{j}$,

$$\chi_{\mathbf{j}}(y) \,=\, \frac{\sinh(j\,y)}{\sinh(y)} \,=\, \mathrm{e}^{\,(j-1)y} \,+\, \mathrm{e}^{\,(j-3)y} \,+\, \cdots \,+\, \mathrm{e}^{\,-(j-3)y} \,+\, \mathrm{e}^{\,-(j-1)y}\,, \tag{10}$$

and $\mathbf{j}$ in (9) runs without loss over the finite set of irreducible representations which are integrable in the $SU(2)$ current algebra at level k.

The formula for $Z(k; \mathcal{K}_{p,q}, \mathbf{j})$ in (9) deserves a number of comments. First, $Z(k; \mathcal{K}_{p,q}, \mathbf{j})$ is written as a contour integral over the real axis. This contour integral is to be interpreted as the stationary-phase contribution from the trivial connection on S^3 to the full Wilson loop path integral in (4). Since the trivial connection is indeed the only flat connection on S^3, the stationary-phase approximation to the Wilson loop path integral is exact for torus knots.

In the special case $\mathbf{j} = \mathbf{1}$, for which $Z(k; \mathcal{K}_{p,q}, \mathbf{j})$ reduces to the Chern-Simons partition function $Z(k)$, we have already provided in [1] a very precise derivation of (4) starting from the Chern-Simons path integral. In that derivation, the contour integral over x arises geometrically as an integral over the Cartan subalgebra of $SU(2)$, regarded as the group of constant gauge transformations on S^3. The constant gauge transformations are the stabilizer of the trivial connection in the group of all gauge transformations, and the presence of this stabilizer group plays an important role in the semi-classical analysis of the Chern-Simons path integral. In the computation in [1], the Gaussian term involving x^2 in the integrand of (9) is essentially classical[4], arising from the tree-level Chern-Simons action, and the two sinh factors are derived from one-loop determinants over the modes of A.

One interesting feature of the formula for $Z(k; \mathcal{K}_{p,q}, \mathbf{j})$ is that, if $\mathbf{j} = \mathbf{1}$, then all dependence on p and q cancels between the (p, q)-dependent prefactor and the contour integral over x. However, if $\mathbf{j}$ is non-trivial, then $Z(k; \mathcal{K}_{p,q}, \mathbf{j})$ *does* depend on the pair (p, q), as required for instance if (9) is to reproduce the (absolutely-normalized) Jones polynomial of the torus knot when $\mathbf{j} = \mathbf{2}$.

Since all dependence on the representation $\mathbf{j}$ enters the integrand of (4) through the $SU(2)$ character $\chi_{\mathbf{j}}$, one is naturally tempted to identify $\chi_{\mathbf{j}}$ as the avatar of the Seifert loop operator itself when the path integral in (4) is reduced to the contour integral in (9). The identification of the Seifert loop operator with a character is quite elegant, and a basic goal in [2] is to obtain this identification directly from the path integral via non-abelian localization.

[4]The Gaussian term in (9) is not entirely classical, as we see the famous renormalization $k \mapsto k+2$ appearing there.

Non-Abelian Localization

Briefly, as introduced by Witten in [7], non-abelian localization provides a cohomological interpretation for a special class of symplectic integrals which are intimately related to symmetries. These integrals take the canonical form

$$Z(\epsilon) \,=\, \int_X \exp\left[\Omega - \frac{1}{2\epsilon}\,(\mu,\mu)\right]. \tag{11}$$

Here X is an arbitrary symplectic manifold with symplectic form Ω. We assume that a Lie group H acts on X in a Hamiltonian fashion with moment map $\mu : X \to \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the dual of the Lie algebra $\mathfrak{h}$ of H. We also introduce an invariant quadratic form $(\,\cdot\,,\,\cdot\,)$ on $\mathfrak{h}$ and dually on $\mathfrak{h}^*$ to define the invariant function $S = \frac{1}{2}(\mu,\mu)$ appearing in the integrand of $Z(\epsilon)$. Finally, ϵ is a coupling parameter.

The symplectic integral in (11) possesses two important properties [7], which for sake of time I will merely state. First, this integral can be rewritten so that it is effectively supersymmetric, with a conserved, nilpotent scalar supercharge Q. The existence of Q then leads to a cohomological interpretation for $Z(\epsilon)$ of the sort well-known in the context of the Duistermaat-Heckman formula. Second, by adding a suitable Q-trivial term to the integrand of (11), one can show that the canonical symplectic integral localizes onto the critical points in X of the invariant function $S = \frac{1}{2}(\mu,\mu)$.

Example: Two-Dimensional Yang-Mills Theory

Given the special form of $Z(\epsilon)$, one should not be surprised that this integral has special properties. But why consider such an integral in the first place? One answer, following Witten in [7], is that the path integral of two-dimensional Yang-Mills theory assumes precisely the canonical symplectic form in (11).

To explain the latter observation, I will simply exhibit the counterparts of X, Ω, H, and μ relevant to describe Yang-Mills theory on a Riemann surface Σ.

The Yang-Mills path integral is formally an integral over the affine space $\mathcal{A}$ of connections on a fixed principal G-bundle P over Σ, so clearly we must set

$$X \,=\, \mathcal{A}\,. \tag{12}$$

The affine space $\mathcal{A}$ also possesses a natural symplectic form Ω given by the intersection pairing on Σ,

$$\Omega \,=\, -\int_\Sigma \mathrm{Tr}\big(\delta A \wedge \delta A\big)\,. \tag{13}$$

Here δ denotes the exterior derivative acting on $\mathcal{A}$. Since A serves as a coordinate on $\mathcal{A}$, δA is a one-form on $\mathcal{A}$, and Ω is a two-form on $\mathcal{A}$ which is manifestly non-degenerate and closed. Of course on Σ, δA transforms as a section of the bundle $\Omega^1_\Sigma \otimes \mathrm{ad}(P)$ of adjoint-valued one-forms.

The obvious group which acts on $\mathcal{A}$ is the group $\mathcal{G}$ of gauge transformations. As shown long ago by Atiyah and Bott [8], the action of $\mathcal{G}$ on $\mathcal{A}$ is Hamiltonian with moment map given by the curvature $F_A = dA + A\wedge A$. That is, since elements in the Lie algebra of $\mathcal{G}$ appear on Σ as sections of the adjoint bundle $\mathrm{ad}(P)$, the curvature F_A, as a section of $\Omega^2_\Sigma \otimes \mathrm{ad}(P)$, can naturally be considered as a function on $\mathcal{A}$ taking values in the dual of the Lie algebra of $\mathcal{G}$.

Thus,

$$H \,=\, \mathcal{G}\,, \qquad\qquad \mu \,=\, F_A\,. \tag{14}$$

Finally, the Lie algebra of $\mathcal{G}$ admits an invariant form given by

$$(\phi,\phi) \,=\, -\int_\Sigma \mathrm{Tr}\big(\phi\wedge\star\phi\big)\,. \tag{15}$$

Here ϕ is an element of the Lie algebra of $\mathcal{G}$, transforming on Σ as a section of $\mathrm{ad}(P)$. With respect to (15), the invariant function $S = \frac{1}{2}(\mu,\mu)$ appearing in the canonical symplectic integral over $\mathcal{A}$ is immediately the Yang-Mills action,

$$S \,=\, \frac{1}{2}(\mu,\mu) \,=\, -\frac{1}{2}\int_\Sigma \mathrm{Tr}\big(F_A\wedge\star F_A\big)\,. \tag{16}$$

The metric on $\mathrm{Lie}(\mathcal{G})$ in (15) is defined using a duality operator $\star$ on Σ. For two-dimensional Yang-Mills theory, $\star$ relates zero-forms to two-forms, and to obtain such an operator, we only require a symplectic structure, as opposed to a metric, on Σ. Given a symplectic form ω on Σ, we define $\star$ by the condition $\star 1 = \omega$. The symplectic form ω is invariant under all area-preserving diffeomorphisms of Σ, and this large group acts as a symmetry of two-dimensional Yang-Mills theory. As a result, two-dimensional Yang-Mills theory is essentially a topological gauge theory.

In the remainder of the talk, I want to explain how to cast the Seifert loop path integral (4) as a symplectic integral of the canonical form (11). Once this step is accomplished, the general arguments in [7] imply that the Seifert loop path integral localizes onto critical points of the classical action $S = \frac{1}{2}(\mu,\mu)$, as we observed experimentally for $Z(k;\mathcal{K}_{p,q},\mathbf{j})$. Furthermore, using a non-abelian localization formula derived in [1], one can perform exact computations of the Seifert loop path integral and thus obtain a cohomological description for the Seifert loop operator itself.

Two key ideas are required to obtain a symplectic description of the Seifert loop path integral. The first idea, which appears in §3 of [1], pertains to the basic Chern-Simons path integral in (2) and really has nothing to do with the Wilson loop operator. In contrast, the second idea concerns the Wilson loop operator itself and really has nothing to do with Chern-Simons theory. Nonetheless, both of these ideas fit together in a very elegant way.

The Symplectic Geometry of Chern-Simons Theory

The path integral which describes the partition function of two-dimensional Yang-Mills theory automatically assumes the canonical symplectic form in (11). As a special case of our general study of $Z(k; C, R)$, I now want to review how the path integral (2) which describes the partition function $Z(k)$ of Chern-Simons theory on a Seifert manifold M can also be cast as such a symplectic integral.

In order to obtain a symplectic interpretation of the two-dimensional Yang-Mills path integral, we found it necessary to introduce a symplectic structure on Σ. To obtain a corresponding symplectic interpretation for the Chern-Simons path integral, we must introduce the analogous geometric structure on the three-manifold M – namely, a contact structure.

Globally, a contact structure on M is described by an ordinary one-form κ, a section of Ω^1_M, which at each point of M satisfies the contact condition

$$\kappa \wedge d\kappa \neq 0\,. \tag{17}$$

By a classic theorem of Martinet [9], any compact, orientable[5] three-manifold admits a contact structure, so we do not necessarily assume M to be Seifert at this stage. However, if M is a Seifert manifold, then we certainly want the contact form κ to respect the $U(1)$-action on M. Such a contact form can be exhibited as follows.

We recall that the Seifert manifold M is the total space of an S^1-bundle of degree n over Σ,

$$\begin{array}{ccc} S^1 & \xrightarrow{n} & M \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \Sigma \end{array}, \tag{18}$$

or an orbifold version thereof. For simplicity, I will phrase the following construction of κ in the language of smooth spaces, but the orbifold generalization is immediate.

Regarding M as the total space of a principal $U(1)$-bundle, we take κ to be a $U(1)$-connection on this bundle which satisfies

$$d\kappa = n\,\pi^*(\omega)\,. \tag{19}$$

Here ω is any unit-area symplectic form on Σ, and we recall that a $U(1)$-connection on Σ appears upstairs on M as an ordinary one-form. Because $d\kappa$ represents the Euler class of the S^1-bundle over Σ, the degree n necessarily appears in (19). As an abelian connection, κ is automatically invariant under the $U(1)$-action on M. Also, since the pullback of κ to each S^1 fiber is non-vanishing, the contact condition in (17) is satisfied so long as $n \neq 0$ and the bundle is non-trivial, as we assume.

I began this talk with the observation that Chern-Simons theory is an intrinsically three-dimensional gauge theory. However, one of the more interesting results in [1, 2] is to show that Chern-Simons theory is *not quite* a three-dimensional gauge theory, since one of the three components of A can be completely decoupled from all topological observables.

[5]Any three-manifold admitting a contact structure must be orientable, since the nowhere vanishing three-form $\kappa \wedge d\kappa$ defines an orientation.

In order to decouple one component of A from the Chern-Simons path integral, we introduce a new, infinite-dimensional "shift" symmetry $\mathcal{S}$ which acts on A as

$$\delta A \,=\, \sigma\,\kappa\,. \tag{20}$$

Here σ is an arbitrary adjoint-valued scalar, a section of $\Omega^1_M \otimes \mathfrak{g}$, that parametrizes the action of $\mathcal{S}$ on $\mathcal{A}$.

Of course, the Chern-Simons action $\mathbf{CS}(\,\cdot\,)$ does not respect the shift of A in (20), so we must play a little path integral trick, of the sort familiar from path integral derivations of T-duality or abelian S-duality. See §8 in [10] for a nice review of the path integral derivations of these dualities.

We first introduce a new field Φ which transforms like σ as an adjoint-valued scalar, a section of $\Omega^1_M \otimes \mathfrak{g}$, and which is completely gauge-trivial under $\mathcal{S}$. Thus $\mathcal{S}$ acts on Φ as

$$\delta\Phi \,=\, \sigma\,. \tag{21}$$

Next, we consider a new, shift-invariant action $S(A,\Phi)$ incorporating both A and Φ such that, if Φ is set identically to zero via (21), then $S(A,\Phi)$ reduces to the Chern-Simons action for A. This condition fixes $S(A,\Phi)$ to be

$$\begin{aligned} S(A,\Phi) \,&=\, \mathbf{CS}(A-\kappa\,\Phi)\,,\\ &=\, \mathbf{CS}(A) - \int_M \Big[2\kappa\wedge\mathrm{Tr}(\Phi F_A) - \kappa\wedge d\kappa\,\mathrm{Tr}(\Phi^2)\Big]\,. \end{aligned} \tag{22}$$

Finally, using (22) we introduce an *a priori* new path integral[6] over both A and Φ,

$$\widetilde{Z}(k) \,=\, \int \mathcal{D}A\,\mathcal{D}\Phi\,\exp\!\left[i\frac{k}{4\pi}\,S(A,\Phi)\right]. \tag{23}$$

On one hand, if we set $\Phi \equiv 0$ using the shift-symmetry in (21), $\widetilde{Z}(k)$ immediately reduces[7] to the usual Chern-Simons partition function $Z(k)$. Hence,

$$\widetilde{Z}(k) \,=\, Z(k)\,. \tag{24}$$

On the other hand, due to the trivial fact that $\kappa\wedge\kappa = 0$, the field Φ appears only quadratically in $S(A,\Phi)$. So we can simply perform the Gaussian integral over Φ in (23) to obtain a new path integral description of the Chern-Simons partition function,

$$Z(k) \,=\, \int \mathcal{D}A\,\exp\!\left[i\frac{k}{4\pi}\,S(A)\right], \tag{25}$$

[6] I will not be very careful about the overall normalizations for the path integrals that appear here, but see §3 of [1] for a detailed accounting of formal normalization factors.

[7] We note that the Jacobian associated to the gauge-fixing condition $\Phi \equiv 0$ for $\mathcal{S}$ is trivial.

where

$$S(A) \,=\, \mathbf{CS}(A) \,-\, \int_M \frac{1}{\kappa\wedge d\kappa}\mathrm{Tr}\Big[(\kappa\wedge F_A)^2\Big]\,. \tag{26}$$

In performing the Gaussing integral over Φ, we use the contact condition on κ in (17), since this condition ensures that quadratic term for Φ in (22) is everywhere non-degenerate on M. In particular, the inverse "$1/\kappa\wedge d\kappa$" appearing in (26) is defined as follows. Because $\kappa\wedge d\kappa$ is everywhere non-vanishing, we can always write $\kappa\wedge F_A = \varphi\,\kappa\wedge d\kappa$ for some section φ of $\Omega^0_M \otimes \mathfrak{g}$. Thus, we set $\kappa\wedge F_A/\kappa\wedge d\kappa = \varphi$, and the second term in $S(A)$ becomes $\int_M \kappa\wedge\mathrm{Tr}(F_A\varphi)$.

By construction, $S(A)$ is invariant under the shift of A in (20), as can be checked directly. Thus, we have obtained a new description of the Chern-Simons partition function for which one component of A completely decouples from the path integral. Further, we have yet to use the condition that M be a Seifert manifold, so the reformulation of $Z(k)$ via the shift-invariant action in (26) holds for any three-manifold M endowed with a contact structure.

Symplectic Data

If M is a Seifert manifold, an additional miracle occurs, and the path integral in (25) becomes an integral of the canonical symplectic form to which non-abelian localization applies. For sake of time, let me merely summarize the symplectic data associated to (25).

First, the space over which we integrate in (25) and which must play the role of X is the quotient of the affine space $\mathcal{A}$ by the group $\mathcal{S}$,

$$X \,=\, \mathcal{A}/\mathcal{S}\,. \tag{27}$$

In three dimensions, the affine space $\mathcal{A}$ is not symplectic. However, once we take the quotient by $\mathcal{S}$ in (27), the space $\mathcal{A}/\mathcal{S}$ carries a natural symplectic form Ω given by

$$\Omega \,=\, -\int_M \kappa\wedge\mathrm{Tr}(\delta A\wedge\delta A)\,. \tag{28}$$

Clearly Ω is both gauge-invariant and shift-invariant, and Ω descends to a non-degenerate symplectic form on $\mathcal{A}/\mathcal{S}$.

We must now find a Hamiltonian group acting on $\mathcal{A}/\mathcal{S}$ such that the shift-invariant action $S(A)$ is the square of the corresponding moment map. As an initial guess, motivated by the example of two-dimensional Yang-Mills theory, one might consider the group $\mathcal{G}$ of gauge transformations. However, this guess cannot be correct. By construction, the square of the moment map for the action of $\mathcal{G}$ on $\mathcal{A}/\mathcal{S}$ would be invariant under $\mathcal{G}$. However, $S(A)$ is plainly not invariant under $\mathcal{G}$, since the Chern-Simons term appearing in (26) is not invariant under the large gauge transformations in $\mathcal{G}$ (while the remaining term in (26) manifestly is invariant under $\mathcal{G}$).

As a second guess, one might replace $\mathcal{G}$ with its identity component $\mathcal{G}_0$, which does preserve the shift-invariant action $S(A)$. However, one can show that $\mathcal{G}_0$ is obstructed from acting in a Hamiltonian fashion on $\mathcal{A}/\mathcal{S}$ by a non-trivial Lie algebra cocycle c,

$$c(\phi,\psi) \,=\, -\int_M d\kappa\wedge\mathrm{Tr}(\phi\, d\psi)\ . \tag{29}$$

Here ϕ and ψ are elements of the Lie algebra of $\mathcal{G}_0$, transforming as sections of $\Omega^0_M\otimes\mathfrak{g}$ on M. Parenthetically, this cocycle is closely related to a cocycle that appears in the theory of loop groups [11], which also provides useful background for the identification of H below.

To remedy the situation, we consider the central extension[8] $\widetilde{\mathcal{G}}_0$ of $\mathcal{G}_0$ determined by $c(\phi,\psi)$,

$$U(1)_{\mathbf{Z}} \xrightarrow{c} \widetilde{\mathcal{G}}_0 \longrightarrow \mathcal{G}_0\,. \tag{30}$$

Here we use the subscript '**Z**' to emphasize that $U(1)_{\mathbf{Z}}$ is central in $\widetilde{\mathcal{G}}_0$. The natural action of $\mathcal{G}_0$ on $\mathcal{A}/\mathcal{S}$ extends to an action by $\widetilde{\mathcal{G}}_0$, for which $U(1)_{\mathbf{Z}}$ acts trivially. By construction, the action of $\widetilde{\mathcal{G}}_0$ on $\mathcal{A}/\mathcal{S}$ is then Hamiltonian.

However, $\widetilde{\mathcal{G}}_0$ is still not the group which is to play the role of the Hamiltonian group H! In order to define the canonical symplectic integral in (11), the Lie algebra of H must carry a non-degenerate, invariant quadratic form $(\,\cdot\,,\,\cdot\,)$. But the Lie algebra of $\widetilde{\mathcal{G}}_0$ does not admit such a form, essentially because we have no generator to pair with the generator of the central $U(1)_{\mathbf{Z}}$.

However, we have also yet to apply the Seifert condition on M. We do so now. To avoid confusion, let me denote the Seifert $U(1)$ acting on M by $U(1)_{\mathbf{R}}$, to distinguish it from the central $U(1)_{\mathbf{Z}}$. The action by $U(1)_{\mathbf{R}}$ on M induces a corresponding action on both $\widetilde{\mathcal{G}}_0$ and $\mathcal{A}/\mathcal{S}$, and we finally take H to be the semi-direct product

$$H \,=\, U(1)_{\mathbf{R}} \ltimes \widetilde{\mathcal{G}}_0\,. \tag{31}$$

The Lie algebra of H does admit a non-degenerate, invariant quadratic form $(\,\cdot\,,\,\cdot\,)$, under which the generators of $U(1)_{\mathbf{R}}$ and $U(1)_{\mathbf{Z}}$ are paired. Furthermore, the action of H on $\mathcal{A}/\mathcal{S}$ is Hamiltonian with moment map μ (for which a completely explicit though perhaps not so illuminating formula exists), and the corresponding invariant function $S=\frac{1}{2}(\mu,\mu)$ on $\mathcal{A}/\mathcal{S}$ is precisely $S(A)$. Given the amount of symmetry respected by both $\frac{1}{2}(\mu,\mu)$ and $S(A)$, the latter result could hardly have been otherwise.

The presentation here is a regrettably quick sketch of a fairly miraculous result, and I refer the reader to §3 of [1] for a complete discussion.

[8] A Lie algebra two-cocycle always determines a central extension of algebras. Provided that the cocycle is properly quantized, as is the cocycle in (29), the central extension of algebras lifts to a corresponding central extension of groups.

Generalization to Wilson Loops

I now want to explain how the prior statements concerning the Chern-Simons partition function can be generalized to allow for insertions of Wilson loop operators. As it happens, only one new idea is required.

We clearly need a new idea, because a naive attempt to reapply the previous path integral manipulations to the Wilson loop path integral in (4) runs immediately aground. To illustrate the difficulty with the direct approach, let us consider the obvious way to rewrite the Wilson loop path integral in a shift-invariant form,

$$Z(k; C, R) \,=\, \int \mathcal{D}\!A\, \mathcal{D}\Phi\; \mathcal{W}_R(C)\, \exp\left[i\frac{k}{4\pi}\mathbf{CS}(A - \kappa\,\Phi)\right]. \tag{32}$$

Here $\mathcal{W}_R(C)$ denotes the generalized Wilson loop operator defined not using A but using the shift-invariant combination $A - \kappa\,\Phi$, so that

$$\mathcal{W}_R(C) \,=\, \mathrm{Tr}_R\, P\exp\left[-\oint_C \big(A - \kappa\,\Phi\big)\right]. \tag{33}$$

Exactly as for our discussion of (23), we can use the shift symmetry to fix $\Phi \equiv 0$, after which the path integral in (32) reduces trivially to the standard Wilson loop path integral in (4).

However, to learn something useful from (32) we must perform the path integral over Φ, and as it stands, this integral is not easy to do. Because the generalized Wilson loop operator $\mathcal{W}_R(C)$ is expressed in (33) as a complicated, non-local functional of Φ, the path integral over Φ in (32) is not a Gaussian integral that we can trivially evaluate as we did for (23).

A more fundamental perspective on our problem is the following. Let us return to the description of the ordinary Wilson loop operator $W_R(C)$ as the trace in the representation R of the holonomy of A around C,

$$W_R(C) \,=\, \mathrm{Tr}_R\, P\exp\left(-\oint_C A\right). \tag{34}$$

As observed by Witten in one of the small gems of [3], this description of $W_R(C)$ should be regarded as intrinsically quantum mechanical, for the simple reason that $W_R(C)$ can be naturally interpreted in (34) as the partition function of an auxiliary quantum system attached to the curve C. Briefly, the representation R is to be identified with the Hilbert space of this system, the holonomy of A is to be identified with the time-evolution operator around C, and the trace over R is the usual trace over the Hilbert space that defines the partition function in the Hamiltonian formalism.

Because the notion of tracing over a Hilbert space is inherently quantum mechanical, any attempts to perform essentially classical path integral manipulations involving the expressions in (33) or (34) are misguided at best. Rather, if we hope to generalize the simple, semi-classical path integral manipulations which we used to study the Chern-Simons partition function to apply to the Wilson loop path integral, we need to use an alternative description for the Wilson loop operator that is itself semi-classical.

More precisely, we want to replace the quantum mechanical trace over R in (34) by a path integral over an auxiliary bosonic field U which is attached to the curve C and coupled to the connection A as a background field, so that schematically

$$W_R(C) \,=\, \int \mathcal{D}U \, \exp\Big[i\,\mathbf{cs}_R\big(U; A|_C\big)\Big]\,. \tag{35}$$

Here $\mathbf{cs}_R\big(U; A|_C\big)$ is an action, depending upon the representation R, which is a local, gauge-invariant, and indeed topological functional of the defect field U and the restriction of A to C. Not surprisingly, this semi-classical description (35) of $W_R(C)$ turns out to be the key ingredient required to reformulate the Wilson loop path integral in a shift-invariant fashion.

The idea of representing the Wilson loop operator by a path integral as in (35) is a very old and very general piece of gauge theory lore. In the context of four-dimensional Yang-Mills theory, this idea goes back (at least) to work of Balachandran, Borchardt, and Stern [12] in the 1970's. See also [13] and §7.7 in [10] for other appearances of the path integral in (35).

As I mentioned, the basic idea behind the path integral description (35) of the Wilson loop operator is very simple. We interpret the closed curve C as a periodic "time" for the field U, and we apply the Hamiltonian formalism to rewrite the path integral over U axiomatically as the quantum mechanical trace of the corresponding time-evolution operator around C,

$$W_R(C) \,=\, \mathrm{Tr}_{\mathbb{H}}\, P \exp\left(-i \oint_C \widehat{\mathbf{h}}\right). \tag{36}$$

Here $\mathbb{H}$ is the Hilbert space obtained by quantizing U, and $\widehat{\mathbf{h}}$ is the Hamiltonian which acts on $\mathbb{H}$ to generate infinitesimal translations along C.

Comparing the conventional description of the Wilson loop operator in (34) to the axiomatic expression in (36), we see that the two agree if we identify

$$\begin{aligned} R \,&\longleftrightarrow\, \mathbb{H}\,,\\ P\exp\left(-\oint_C A\right) \,&\longleftrightarrow\, P\exp\left(-i\oint_C \widehat{\mathbf{h}}\right). \end{aligned} \tag{37}$$

Hence to make the Wilson loop path integral in (35) precise, we need only exhibit a classical theory on C, for which the gauge group G acts as a symmetry, such that upon quantization we obtain a Hilbert space $\mathbb{H}$ isomorphic to R and for which the time-evolution operator around C is given by the holonomy of A, acting as an element of G on R.

A Semi-Classical Description of the Wilson Loop Operator

Let me now tell you what classical theory to place on C to realize the quantum identifications in (37).

Of the two identifications in (37), the more fundamental by far is the identification of the irreducible representation R with a Hilbert space, obtained by quantizing some classical phase space upon which G acts as a symmetry. So before we even consider what classical theory must live on C to describe the Wilson loop operator, we can ask the simpler and more basic question — what classical phase space must we quantize to obtain R as a Hilbert space?

As is well-known, the question above is beautifully answered by the Borel-Weil-Bott theorem [14]. In order to recall this theorem, let me first fix a maximal torus $T \subset G$, for which $\mathfrak{t} \subset \mathfrak{g}$ is the associated Cartan subalgebra. Given the irreducible representation R and some choice of positive roots for G, I also introduce the associated highest weight α. Canonically, the weight α lies in the dual $\mathfrak{t}^*$ of $\mathfrak{t}$, but given the invariant form 'Tr' on $\mathfrak{g}$, we are free to identify $\mathfrak{t}^* \cong \mathfrak{t}$ and hence to regard α as an element of $\mathfrak{t}$,

$$\alpha \in \mathfrak{t}^* \cong \mathfrak{t}\,. \tag{38}$$

Though mathematically unnatural, the convention in (38) leads to more transparent physical formulae below.

The Borel-Weil-Bott theorem concerns the geometry of the orbit $\mathcal{O}_\alpha \subset \mathfrak{g}$ which passes through α under the adjoint action of G. Equivalently, as a homogeneous space $\mathcal{O}_\alpha$ can be realized as a quotient G/G_0, where G_0 is the stabilizer of α under the adjoint action of G. Explicitly, the identification between G/G_0 and $\mathcal{O}_\alpha$ is given by the map

$$g\,G_0 \longmapsto g\,\alpha\,g^{-1}\,, \qquad g \in G\,. \tag{39}$$

As will be essential in a moment, $\mathcal{O}_\alpha$ is a compact complex manifold which admits a natural Kähler structure invariant under G. As an easy example, if $G = SU(2)$ and α is any non-zero[9] weight, then $\mathcal{O}_\alpha = SU(2)/U(1)$ can be identified as $\mathbb{CP}^1$ endowed with the Fubini-Study metric.

In a nutshell, the Borel-Weil-Bott theorem states that the irreducible representation R can be realized geometrically as the space of holomorphic sections of a certain unitary line bundle $\mathcal{L}(\alpha)$ over $\mathcal{O}_\alpha$. That is,

$$R \,\cong\, H^0_{\bar\partial}\big(\mathcal{O}_\alpha,\, \mathcal{L}(\alpha)\big)\,, \tag{40}$$

where the action of G on the sections of $\mathcal{L}(\alpha)$ is induced from its action on $\mathcal{O}_\alpha$.

As a unitary line bundle over a Kähler manifold, $\mathcal{L}(\alpha)$ carries a canonical unitary connection Θ_α, which is also invariant under G. The connection Θ_α enters the path integral description of the

[9] Of course, if $\alpha = 0$, then $\mathcal{O}_\alpha$ is merely a point.

Wilson loop operator, so let me exhibit it explicitly. When pulled back to G, the line bundle $\mathcal{L}(\alpha)$ trivializes, and the connection Θ_α appears as the following left-invariant one-form on G,

$$\Theta_\alpha \,=\, \mathrm{Tr}\big(\alpha\cdot g^{-1}dg\big)\,. \tag{41}$$

I have introduced the connection Θ_α because its curvature $\nu_\alpha = d\Theta_\alpha$ is precisely the Kähler form on $\mathcal{O}_\alpha$. As a result, $\mathcal{L}(\alpha)$ is a prequantum line bundle over $\mathcal{O}_\alpha$, and the Borel-Weil-Bott isomorphism in (40) identifies R as the Hilbert space obtained by Kähler quantization of $\mathcal{O}_\alpha$.

Perhaps more physically, the Borel-Weil-Bott theorem can be interpreted as identifying the space of groundstates for a charged particle moving on $\mathcal{O}_\alpha$ in the presence of a background magnetic field $B = \nu_\alpha$. Briefly, because of the non-zero magnetic field, the wavefunctions which describe this particle transform on $\mathcal{O}_\alpha$ not as functions but as sections of the line bundle $\mathcal{L}(\alpha)$. As standard, the Hamiltonian which describes free propagation on $\mathcal{O}_\alpha$ is proportional to the Laplacian $\triangle$ acting on sections of $\mathcal{L}(\alpha)$, and by Hodge theory, the kernel of $\triangle$ can be identified as the space of holomorphic sections of $\mathcal{L}(\alpha)$. Hence the role of (40) is to realize the representation R in terms of Landau levels on $\mathcal{O}_\alpha$.

Given the quantum mechanical interpretation of R above, the corresponding path integral description (35) for the Wilson loop operator follows immediately. Ignoring the coupling to A for a moment, if we simply wish to describe the low-energy effective dynamics of an electron moving on $\mathcal{O}_\alpha$ in the background magnetic field $B = \nu_\alpha$, we consider a one-dimensional sigma model of maps

$$U : C \longrightarrow \mathcal{O}_\alpha\,, \tag{42}$$

with sigma model action

$$\mathbf{cs}_R(U) \,=\, \oint_C U^*(\Theta_\alpha) \,=\, \oint_C \mathrm{Tr}(\alpha\cdot g^{-1}dg)\,. \tag{43}$$

Here $U^*(\Theta_\alpha)$ denotes the pullback of Θ_α to a connection over C. If U is lifted as a map to $\mathcal{O}_\alpha = G/G_0$ by a corresponding map

$$g : C \longrightarrow G\,, \tag{44}$$

then the pullback of Θ_α appears explicitly as in (43). As a word of warning, I will freely switch between writing formulae in terms of U or g as convenient.

From a physical perspective, the first-order action $\mathbf{cs}_R$ simply describes the minimal coupling of the charged particle on $\mathcal{O}_\alpha$ to the background magnetic field specified by Θ_α, and we have omitted the second-order kinetic terms for U as being irrelevant at low energies. From a more geometric perspective, $\mathbf{cs}_R$ is a one-dimensional Chern-Simons action for the abelian connection $U^*(\Theta_\alpha)$ over C. As such, the quantization of the parameter $\alpha \in \mathfrak{t}$ as a weight of G follows just as for the quantization of the Chern-Simons level k. Physically, the quantization of α follows from the quantization of flux on a compact space.

This sigma model on C clearly respects the action of G on $\mathcal{O}_\alpha$ as a global symmetry. To couple the sigma model to the restriction of the bulk gauge field A, we simply promote the global action of G on $\mathcal{O}_\alpha$ to a gauge symmetry. That is, we consider the covariant version of (43),

$$\begin{aligned}\mathbf{cs}_R(U; A|_C) \,&=\, \oint_C U^*(\Theta_\alpha(A)) \,=\, \oint_C \mathrm{Tr}(\alpha \cdot g^{-1} d_A g)\,,\\ d_A g \,&=\, dg + A|_C \cdot g\,.\end{aligned} \tag{45}$$

In the second line, we indicate the action of the covariant derivative d_A on g. The action by d_A on g descends to a corresponding covariant action by d_A on U as well.

I now claim that the quantization of the gauged sigma model on C with action $\mathbf{cs}_R(U; A|_C)$ leads to the identifications in (37) required to describe the Wilson loop operator. First, the classical equation of motion for U simply asserts that U is covariantly constant,

$$d_A U \,=\, 0\,. \tag{46}$$

As a result, the classical phase space for U can be identified with the orbit $\mathcal{O}_\alpha$, and by the Borel-Weil-Bott isomorphism in (40), the corresponding Hilbert space $\mathbb{H}$ for U is identified as the representation R. Similarly, since the classical time-evolution for U is given by parallel transport, the quantum time-evolution operator around C is immediately given by the holonomy of A, acting as an element of G on R.

The Shift-Invariant Wilson Loop In Chern-Simons Theory

Because A only enters as a background field in (45), the path integral description (35) of $W_R(C)$ is completely general and applies to any gauge theory in any dimension. Nonetheless, this description of the Wilson loop operator is precisely what we need to obtain a shift-invariant formulation of the Wilson loop path integral in Chern-Simons theory.

Let us first apply (35) to rewrite the Wilson loop path integral in (4) as a path integral over both A and U,

$$Z(k; C, R) \,=\, \frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{\mathcal{A}\times L\mathcal{O}_\alpha} \mathcal{D}A\, \mathcal{D}U\, \exp\left[i\frac{k}{4\pi}\mathbf{CS}(A) \,+\, \mathbf{cs}_R(U; A|_C)\right]. \tag{47}$$

Here we introduce the free loopspace $L\mathcal{O}_\alpha$ of $\mathcal{O}_\alpha$ to parametrize configurations of U.

Once we introduce the defect degree-of-freedom U coupling to A in (47), the classical equation of motion for A becomes

$$\mathcal{F}_A \stackrel{\text{def}}{=} F_A + \frac{2\pi}{k}\, U \cdot \delta_C \,=\, 0\,. \tag{48}$$

Here δ_C is a two-form with delta-function support on C which represents the Poincaré dual of the curve. Using δ_C, we rewrite $\mathbf{cs}_R(U; A|_C)$ as a bulk integral over M,

$$\mathbf{cs}_R\big(U; A|_C\big) \,=\, \oint_C \mathrm{Tr}\big(\alpha \cdot g^{-1} d_A g\big) \,=\, \int_M \delta_C \wedge \mathrm{Tr}\big(\alpha \cdot g^{-1} d_A g\big)\,, \tag{49}$$

from which (48) follows.

As we see from (48), in the presence of the operator $W_R(C)$, classical configurations for A are given by connections which are flat on the complement $M^o = M - C$ and otherwise have delta-function curvature along C. The singularity in A along C manifests itself on M^o as a non-trivial monodromy of the connection around a transverse circle linking C. Though I will not have time to say more, the moduli space of such flat connections with monodromies is the space onto which the Seifert loop path integral localizes. This space is directly related to the moduli space of representations of the knot group of C in G and, in suitable circumstances, is also related to the moduli space of (non-singular) flat connections on M.

At the cost of introducing defect degrees-of-freedom along C, we have managed to describe $W_R(C)$ in terms of a completely local – and indeed linear – functional of A. Consequently, the same path integral trick that we used to decouple one component of A from the Chern-Simons partition function applies immediately to (47). We simply replace A in (47) by the shift-invariant[10] combination $A - \kappa\,\Phi$, and we then perform the Gaussian integral over Φ. In the process, the only new ingredient is that we obtain an extra term linear in Φ from the coupling of A to U.

Without discussing any more details, let me present the resulting shift-invariant formulation for the Wilson loop path integral,

$$Z(k; C, R) \,=\, \frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{\mathcal{A}/\mathcal{S} \times L\mathcal{O}_\alpha} \mathcal{D}\!A\, \mathcal{D}\!U \;\exp\left[i \frac{k}{4\pi} S(A, U) \right], \tag{50}$$

where

$$S(A, U) \,=\, \mathbf{CS}(A) \,+\, \frac{4\pi}{k} \mathbf{cs}_R(U; A|_C) \,-\, \int_M \frac{1}{\kappa \wedge d\kappa} \mathrm{Tr}\big[(\kappa \wedge \mathcal{F}_A)^2 \big]\;. \tag{51}$$

The shift-invariant action $S(A, U)$ appears much as the corresponding shift-invariant action (26) for A alone, with the replacement therein of F_A by the generalized curvature $\mathcal{F}_A$. Thus for an arbitrary Wilson loop (or link) operator in Chern-Simons theory, the path integral can be rewritten such that one component of A completely decouples.

Moreover, if M is a Seifert manifold and C is a Seifert fiber of M, the shift-invariant Seifert loop path integral is again an integral of the canonical symplectic form in (11). The relevant symplectic space X is just the product

$$X \,=\, \mathcal{A}/\mathcal{S} \times L\mathcal{O}_\alpha\,, \tag{52}$$

where the loopspace $L\mathcal{O}_\alpha$ carries a natural symplectic (and indeed Kähler) form inherited from the Kähler form ν_α on $\mathcal{O}_\alpha$.

The Hamiltonian group H which acts on X is the same group that appears in (31). In fact, the loopspace $L\mathcal{O}_\alpha$ can be interpreted formally as an infinite-dimensional coadjoint orbit of H.

[10] The shift-symmetry $\mathcal{S}$ acts trivially on U.

Finally, the square of the moment map for the diagonal action of H on X is precisely the shift-invariant action $S(A,U)$ appearing in (51).

For a last bit of furious hand-waving, let me remark that the description of the Seifert loop operator as a character follows quite naturally from the appearance of the loopspace $L\mathcal{O}_\alpha$ in (50). In essence, non-abelian localization on $L\mathcal{O}_\alpha$ is related to index theory on $\mathcal{O}_\alpha$, and index theory on $\mathcal{O}_\alpha$ provides a classic derivation [15] of the famous Weyl character formula.

References

[1] C. Beasley and E. Witten, "Non-Abelian Localization For Chern-Simons Theory," J. Differential Geom. **70** (2005) 183–323, `hep-th/0503126`.

[2] C. Beasley, "Localization for Wilson Loops in Chern-Simons Theory," to appear.

[3] E. Witten, "Quantum Field Theory and the Jones Polynomial," Commun. Math. Phys. **121** (1989) 351–399.

[4] E. Witten, "Gauge Theories and Integrable Lattice Models," Nucl. Phys. B **322** (1989) 629–697.

[5] R. Lawrence and L. Rozansky, "Witten-Reshetikhin-Turaev Invariants of Seifert Manifolds," Commun. Math. Phys. **205** (1999) 287–314.

[6] L. Moser, "Elementary Surgery Along a Torus Knot," Pacific J. Math **38** (1971) 737–745.

[7] E. Witten, "Two-dimensional Gauge Theories Revisited," J. Geom. Phys. **9** (1992) 303–368, `hep-th/9204083`.

[8] M. Atiyah and R. Bott, "Yang-Mills Equations Over Riemann Surfaces," Phil. Trans. R. Soc. Lond. **A308** (1982) 523–615.

[9] J. Martinet, "Formes de contact sur les varietétés de dimension 3," Springer Lecture Notes in Math **209** (1971) 142–163.

[10] E.Witten, "Dynamics of Quantum Field Theory," in *Quantum Fields and Strings: A Course for Mathematicians, Vol. 2*, Ed. by P. Deligne et al., American Mathematical Society, Providence, Rhode Island, 1999.

[11] A. Pressley and G. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.

[12] A. P. Balachandran, S. Borchardt, and A. Stern, "Lagrangian And Hamiltonian Descriptions of Yang-Mills Particles," Phys. Rev. D **17** (1978) 3247–3256.

[13] D. Diakonov and V. Y. Petrov, "A Formula for the Wilson Loop," Phys. Lett. B **224** (1989) 131–135.

[14] R. Bott, "Homogeneous Vector Bundles," Ann. of Math. **66** (1957) 203–248.

[15] M. F. Atiyah and R. Bott, "A Lefschetz Fixed Point Formula for Elliptic Complexes, II," Ann. of Math. **88** (1968) 451–491.