REMODELING THE B-MODEL

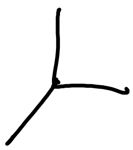
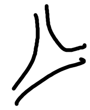
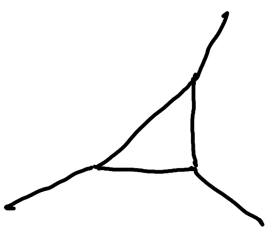
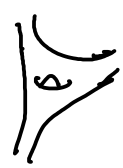
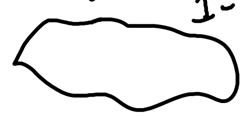
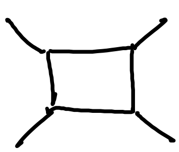
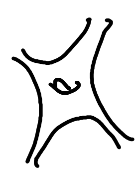
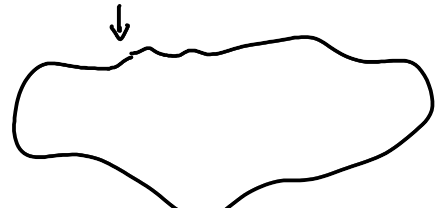
Goal: ① Find a B-model formalism to compute unambiguously all open/closed amplitudes for toric CY_3
["mirror formalism" to top. vertex]

② Expand the open/closed amplitudes at <u>other</u> points in the moduli space (such as orbifold points)
(B-model well suited for that)

Ref^s: Mariño '06, VB-Klemm-Mariño-Pasquetti (in progress), . . .

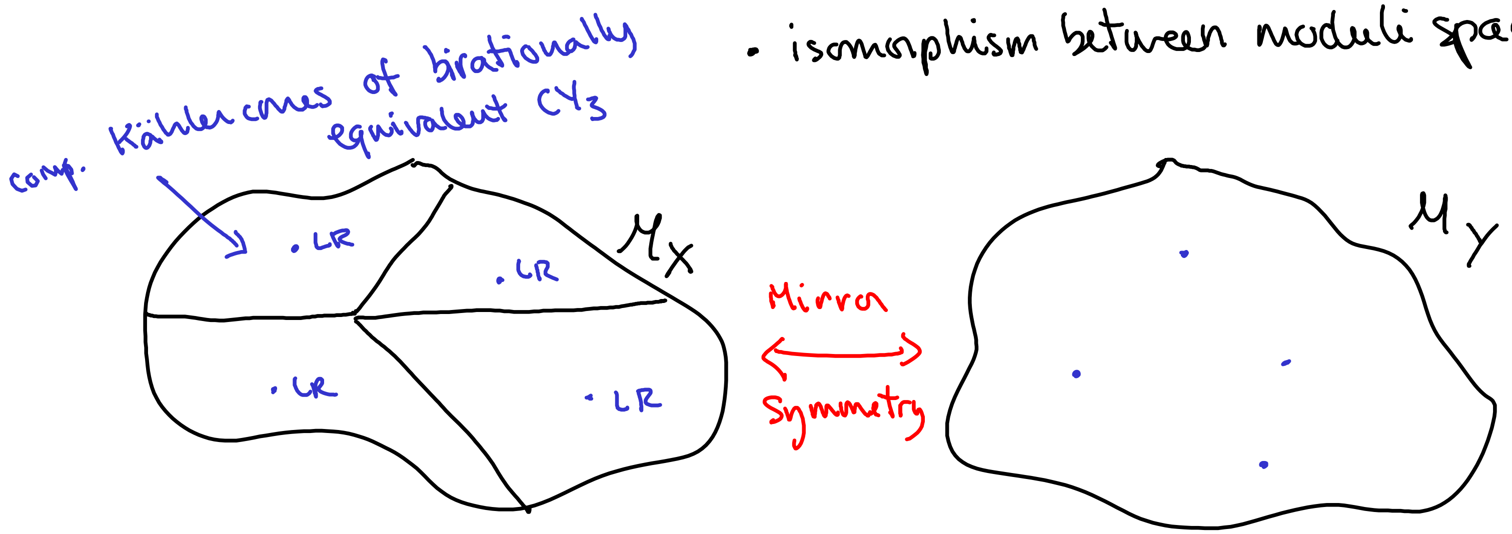
Plan:
① Mirror Symmetry (closed)
② Top. Strings and orbifold GW invariants
③ Open geometry
④ Formalism
⑤ Examples: • framed vertex
• CS theory on lens space
• $\mathbb{C}^3/\mathbb{Z}_3$?

① MIRROR SYMMETRY

X = toric CY_3	$Y = \left\{ \begin{array}{c} \text{conic} \\ \downarrow \\ \mathbb{C}^* \times \mathbb{C}^* \end{array} \right.$, where fiber deg. to 2 lines over R.S. $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$
① $\mathbb{C}^3$ 	 $ww' = 1 + x + y$, $w, w' \in \mathbb{C}$, $x, y \in \mathbb{C}^*$ $\Rightarrow \Sigma = \{1 + x + y = 0\} \subset \mathbb{C}^* \times \mathbb{C}^*$ genus 0 $(\mathbb{P}^1 \setminus \{0, 1, \infty\})$
② local $\mathbb{P}^2$, $Q = (-3\ 1\ 1\ 1)$ 	$ww' = \underbrace{y^2 + y + xy + e^{-t} x^3}_{\Sigma,\ \text{genus } 1,\ 3 \text{ punctures}}$ fibration  Σ $\downarrow$  1-param. c.s. moduli space $[z = e^{-t}]$
③ local $\mathbb{P}^1 \times \mathbb{P}^1$ $Q = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix}$ 	$ww' = \underbrace{x^2 y + x y^2 + xy + e^{-t_1} y + e^{-t_2} x}_{\Sigma,\ \text{genus } 1,\ 4 \text{ punctures}}$  Σ $\downarrow$  2-param. c.s. mod. space $[z_1 = e^{-t_1},\ z_2 = e^{-t_2}]$

Some features of mirror symmetry

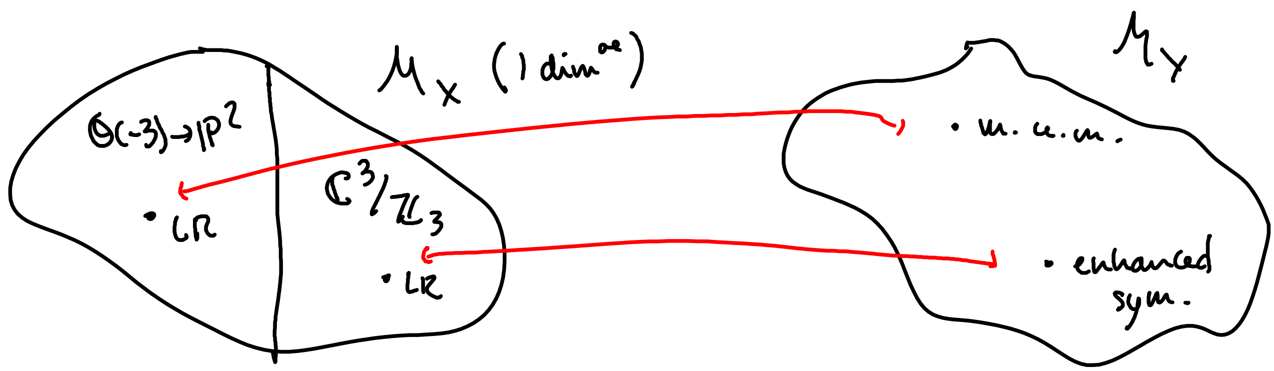
• isomorphism between moduli spaces



stringy Kähler m.s. $\qquad\qquad$ comp. struct. m.s.

↑ described by secondary fan

EXAMPLE: local $\mathbb{P}^2$



2 Kähler cones

KEY: $\mathcal{M}_X$ contains Kähler cones of topologically distinct manifolds, including sing. ones like orbifolds

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MIRROR MAP(S): - local iso. in each patch, mapping Kähler par. to a basis of periods

$*$ involves choice of periods, different for each Kähler cone

$*$ basis of periods are related by symp transf^{ns}.

(2) CLOSED TOPOLOGICAL STRINGS and ORBIFOLD GW INVARIANTS

A-MODEL (X) $\longleftrightarrow$ B-MODEL (Y)

(Kähler) (c. struct.)

part. fct^{n}: $Z_A[X] = Z_B[Y]$

• define $Z = \exp\Big(\sum_{g=0}^{\infty} g_s^{2g-2} F_g \Big)$ genus g amplitude; string coupling

* the phys. top. string part. fct^{n} Z is defined <u>all over</u> the moduli space
  - → it is not holomorphic [Walcher's talk]
  - → non-hol. encoded by HA eq^{ns} [BCOV]

<u>INSTEAD</u> • take the phys. Z_A, and take its limit at a LR pt deep inside a Kähler cone, i.e.

$$\tilde{Z}_A = \lim_{LR} Z_A$$

then $\tilde{F}_g \rightsquigarrow$ GW theory of CY_3 in this Kähler cone.
holomorphic

<u>EXAMPLE</u>: • local $\mathbb{P}^2 \rightsquigarrow$ 2 Kähler cones

$$Z \rightsquigarrow \text{GW theory of } \begin{cases} \mathcal{O}(-3) \to \mathbb{P}^2 & \tilde{Z}^{\infty} \\ \mathbb{C}^3/\mathbb{Z}_3 & \tilde{Z}^{orb} \end{cases}$$

* in this talk I will always work w/ the limiting hol. objects $\tilde{F}_g$ [not the physical ones]

MIRROR SIDE:

FACTS:

① $\tilde{F}_0$: prepotential of Spec. Geometry in the good basis of periods, (for the mirror map)

$$t_D = \frac{\partial \tilde{F}_0}{\partial t}.$$

② $\tilde{Z}$: wave-fct^{n} in geometric quantization of $H^3(Y,\mathbb{C})$ in real polarization corresponding to this basis of periods.

$\Rightarrow$ $\tilde{Z}$ transforms in metaplectic rep^{n} (Bogoliubov transf^{n}) under change of basis of periods.

$\rightsquigarrow$ we can extract $\tilde{Z}^{orb}$ from knowledge of $\tilde{Z}^{\infty}$.

[Aganagic, VB, Klemm]

EX: local $\mathbb{P}^2 \longrightarrow$ GW $[\mathbb{C}^3/\mathbb{Z}_3]$

(proved at genus 0 by Coates et al, some higher genus, VB-Cavalieri, etc.)

Many questions: • integrality a la Gopakumar-Vafa? (counting BPS states?)

• rel^{n} between GW and DT? (different than for smooth CY_3)

• "OSV" and rel^{n} w/ black holes?

etc.

Other point of view:

- from wave-fct^{n} behavior under monodromy [or hol. anomaly]
  extract recursive rel^{ns} directly at one pt in moduli space
  $\tilde{F}_g$ = lower genus data + h_g
  ← modular fct^{n} which is not fixed
- equivalent to HA eq^{ns}, but holomorphic
- same eq^{ns} at any LR pt, and h_g is same everywhere
- need $\tilde{F}_0, \tilde{F}_1 \rightsquigarrow$ all $\tilde{F}_g$ up to amb. h_g {for any LR point}

NEXT STEP:

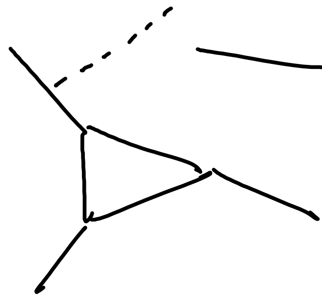
- find new recursive rel^{ns} for $\hat{F}_g$ which are completely fixed [no modular ambiguity h_g]
- extend to open amplitudes

$\underbrace{\hspace{20em}}$

need new formalism

③ OPEN GEOMETRY

- we need to fix a special Lagrangian (SLAG) submanifold of X on which open strings end

ex:  $\longrightarrow$ topology $\mathbb{C}\times S^1$, ends on a toric leg

[Agamagic-Vafa]

Open amplitudes: • maps w/ boundaries ending on this SLAG.

- from CS, each brane comes w/ a framing integer $f \in \mathbb{Z}$ which must be fixed [different $f \leftrightarrow$ different amplitudes]

$\Rightarrow$ choice of "location" on toric diagram and $f \in \mathbb{Z}$

Mirror: A-brane $\longrightarrow$ $H(x,y) = 0 = ww'$

- moduli space parameterized by R.S.: Σ: $\{H(x,y)=0\} \subset \mathbb{C}^* \times \mathbb{C}^*$.

$\Rightarrow$ location + framing A-brane

$\updownarrow$

choice of paramaterization of Σ {i.e. (x,y)} [choice of open string parameter]

BUT (CRUCIAL): $\Sigma \subset \boxed{\mathbb{C}^* \times \mathbb{C}^*}$

$\Rightarrow$ $\underbrace{SL(2,\mathbb{Z})}_{\text{group of reparameterizations of a curve} \subset \mathbb{C}^*\times\mathbb{C}^*}$ acts as $(x,y) \mapsto (x^a y^b, x^c y^d)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

[think of e^u, $u \mapsto au + bv$]

	A-MODEL	B-MODEL
RESULT:	• location and framing	choice of $SL(2,\mathbb{Z})$ parameterization
GEOMETRY:	• toric CY_3 • SLAG ending on a toric leg w/ a choice of $f \in \mathbb{Z}$	• R.S. (mirror curve) $\Sigma = \{H(x,y)=0\} \subset \mathbb{C}^* \times \mathbb{C}^*$ • choice of $SL(2,\mathbb{Z})$ parameterization of Σ

WARNING: - this $SL(2,\mathbb{Z})$ is very different than

symp. transf^n of periods

reparameterizations of Σ (only affects open string amplitudes)

moving in moduli space (changes everything)

④ FORMALISM

[Mariño '06, Eynard-Orantin '07, VB-Klemm-Mariño-Pasquetti]

IDEA: Eynard-Orantin:

- matrix model → spectral curve $\mathcal{E} \subset \mathbb{C}^2$
- free energies F_g & correlation fct^ns $W_k^{(g)}$ defined recursively (sol^n of loop eq^ns, see Klemm's talk)

$*$ everything is geometric on $\mathcal{E}$, and can be defined whether $\mathcal{E}$ is the spectral curve of a MM or not

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RECALL: • B-model: we have a mirror curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$

STRATEGY: • use EO to generate open/closed top. string amplitudes (F_g and $W_k^{(g)}$), suitably modified for curves in $\mathbb{C}^* \times \mathbb{C}^*$ rather than $\mathbb{C}^2$.

WARNING: this is experimental mathematics:

• we have no matrix model or large N transition to justify our claim (yet :-)), but IT WORKS! :-)

FORMALISM

We start w/ $\Sigma = \{H(x,y)=0\} \subset \mathbb{C}^* \times \mathbb{C}^*$.

INGREDIENTS:

$$\begin{array}{ccc} \Sigma & \ni & q \\ \downarrow & & \downarrow \\ \mathbb{C}^* & \ni & x(q) \end{array}$$

- ram. pts $q_i \in \Sigma_i$ of projection $\pi: \Sigma_i \to \mathbb{C}^*$ on x-axis $\left(\frac{\partial H}{\partial x}(q_i) = 0 \right)$
- near q_i, there are 2 pts $q, \bar{q} \in \Sigma_i$ s.t. $x(q) = x(\bar{q})$
- mero. diff$^{\text{ae}}$ $\Phi = \log y \frac{dx}{x}$ [coming from symp-form $\frac{dx}{x} \wedge \frac{dy}{y}$ on $\mathbb{C}^* \times \mathbb{C}^*$]

  $\downarrow$ governs comp. struct. deformations

- Bergmann kernel $B(p,q)$:
  - unique mero. diff$^{\text{ae}}$ w/ double pole at $p=q$ and no other pole, and normalized by

$$\oint_{A_I} B(p,q) = 0$$

$\nwarrow$ canonical basis of cycles. $\uparrow$ (important choice)

$$dE_{q,\bar{q}}(p) = \frac{1}{2} \int_{\bar{q}}^{q} d\xi \, B(p,\xi)$$

EX: Σ_i genus 0, $B(p,q) = \dfrac{dz(p)\,dz(q)}{(z(p)-z(q))^2}$

RECURSION:

Step 1: - generate mero diff$^{\underline{\text{als}}}$ $W_k^{(g)}(p_1,\dots,p_k)$

fix $W_1^{(0)}(p_1)=0$, $\quad W_2^{(0)}(p_1,p_2) = B(p_1,p_2)$

$$\boxed{\begin{aligned} W_k^{(g)}(p_0,\dots,p_{k-1}) = &-\sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_{q,\bar q}(p_0)}{\Phi(q)-\Phi(\bar q)} \Bigg(W_{k+1}^{(g-1)}(q,\bar q, p_1,\dots,p_{k-1}) \\ &+ \sum_{\ell=0}^{g} \sum_{m=0}^{k-1} \sum_{\sigma\in S(k-1)} \frac{1}{m!(k-1-m)!} W_{m+1}^{(g-\ell)}(q, p_{\sigma(1)},\dots,p_{\sigma(m)}) \\ &\qquad \cdot W_{k-m}^{(\ell)}(\bar q, p_{\sigma(m+1)},\dots,p_{\sigma(k-1)})\Bigg) \end{aligned}}$$

w/ $\sigma\in S(k-1)$ permutations of $k-1$ elements.

Step 2: - generate fct$^{\underline{\text{ns}}}$ F_g

$$\boxed{F_g = \frac{1}{2-2g} \sum_{q_i} \operatorname*{Res}_{q=q_i} \phi(q)\, W_1^{(g)}(q)}$$

w/ $\quad d\phi(p) = \Phi(p)$

any anti-derivative

CLAIM: • F_g + mina map $\rightarrow$ closed amplitudes

• $A_k^{(g)} = \int W_k^{(g)}$ + mina map $\rightarrow$ open amplitudes

COMMENTS

- gluing procedure, e.g.

$$F_1 \quad = \quad W_1^{(1)} \quad \bullet \quad \text{disk}$$

etc.

- fund. objects: disk & annulus [closed built from open]
- F_g invariant under $SL(2,\mathbb{Z})$, but $A_k^{(g)}$ depend on parameterizations of Σ: framing + location of the brane
- recalling previous discussion, this generates hol. $\tilde{F}_g$ and corr. fct$^{\underline{ns}}$ $\tilde{W}_k^{(g)}$, i.e. the limits of the physical amplitudes at a given point in moduli space

* <u>no</u> ambiguity in recursion rel$^{\underline{ns}}$

$\Rightarrow$ if we know disk + annulus at a point, we can generate unambiguously all open/closed amplitudes at this point.

$\rightsquigarrow$ this works for <u>any</u> point in moduli space

CHECKS:

LR point of:

- framed vertex
- framed inner/outer branes in resolved conifold
- " " " " local $\mathbb{P}^2$
- " " " " local $\mathbb{P}^1 \times \mathbb{P}^1$

  local F_1

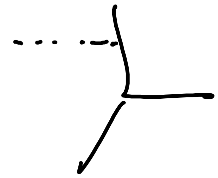
  local F_2

Other points

- blow-down $\mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow$ CS theory on $S^3/\mathbb{Z}_2$
- blow-down $\mathbb{P}^2 \rightsquigarrow$ $\mathbb{C}^3/\mathbb{Z}_3$

⑤ EXAMPLES:

① FRAMED VERTEX:



$$H(x,y) = x + y + 1 = 0$$

Framing: $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$

$$(\tilde{x},\tilde{y}) = (x\, y^f,\ y)$$

$$\rightarrow \quad H_f(x,y) = x + y^{f+1} + y^f = 0$$

Disk: $A_1^{(0)} = \int \log y \,\frac{dx}{x}$

solve $y = y(x) = -1 - (-1)^f x + f x^2 + \cdots$

mirror map: $X = -x$,

$$\rightarrow A_1^{(0)} = -(-1)^f X - \frac{1}{4}(1+2f)X^2 - \cdots$$

Annulus: Σ genus $0 \Rightarrow B(x_1,x_2) = \dfrac{dy_1(x_1)\, dy_2(x_2)}{(y_1(x_1) - y_2(x_2))^2}$

$$A_2^{(0)} = \int \left(B(p,q) - \frac{dx_1 dx_2}{(x_1 - x_2)^2} \right)$$

$$= \log\left(-y_1(x_1) + y_2(x_2)\right) - \log(-x_1 + x_2)$$

$$= \frac{1}{2} f(f+1) X_1 X_2 + \frac{(-1)^f}{3} f(1+3f+2f^2)(X_1^2 X_2 + X_1 X_2^2) + \cdots$$

3-HOLE: • ram. points: $\frac{\partial H}{\partial y} = 0$

$\rightarrow$ one pt at $y = \frac{-f}{f+1} := q_1$

$$A_3^{(0)} = \int \operatorname*{Res}_{y=q_1} \frac{x(y)\, y\; dy_1(x_1)\, dy_2(x_2)\, dy_3(x_3)}{(y-y_1(x_1))^2 (y-y_2(x_2))^2 (y-y_3(x_3))^2} \left(\frac{dx}{dy}\right)^{-1}$$

$$= \frac{f^2}{f+1} \prod_{i=1}^{3} \frac{1}{f+(f+1)y_i(x_i)}$$

$$= -(-1)^f f^2 (1+f)^2 X_1 X_2 X_3 + \cdots$$

etc. always works!

- genus 1 examples much more complicated computationally, but always works
- the procedure is algorithmic and could be implemented in a computer code

② LOCAL $\mathbb{P}^1 \times \mathbb{P}^1$:

$$H(x,y) = x^2 y + x y^2 + xy + e^{-t_1} y + e^{-t_2} x = 0$$

$$z_1 = e^{-t_1}, \quad z_2 = e^{-t_2}$$

LR : works

other point: [Aganagic-Klemm-Mariño-Vafa]

$$X_1 = 1 - \frac{z_1}{z_2}, \qquad X_2 = \frac{1}{\sqrt{z_2}\left(1 - \frac{z_1}{z_2}\right)}$$

$X_1, X_2 \to 0$ ["orbifold" point, $\mathbb{P}^1 \times \mathbb{P}^1 \to$ zero size]

THEN : Closed top. strings $\longleftrightarrow$ $\underbrace{\text{CS theory on lens space } S^3/\mathbb{Z}_2}$

because geometric transition :

$$\text{local } \mathbb{P}^1 \xleftrightarrow[\text{down } \mathbb{P}^1]{\text{blow}} \text{sing. cm.} \xleftrightarrow{\text{deform}} T^*S^3$$

now

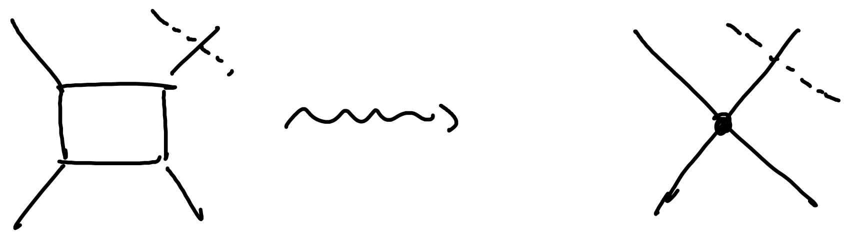
$/\mathbb{Z}_2 \downarrow$ $\qquad\qquad\qquad\qquad$ $/\mathbb{Z}_2 \downarrow$

blow up extra $\mathbb{P}^1$ (fixed pt under $\mathbb{Z}_2$)

$\downarrow$

$$\rightsquigarrow \text{local } \mathbb{P}^1 \times \mathbb{P}^1 \longleftrightarrow \text{sing.} \longleftrightarrow T^*(S^3/\mathbb{Z}_2)$$

What we now have



(schematically)

open amplitudes

$$\boxed{\text{framed unknot in } CS(S^3/\mathbb{Z}_2)}$$

<u>IT WORKS!</u>

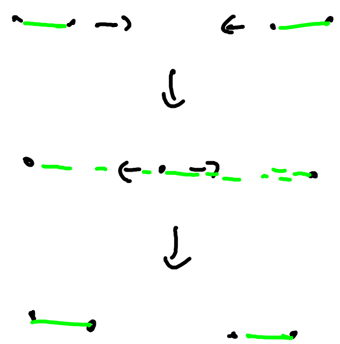
(rather involved calculation,
<u>strong</u> check of validity of
the formalism)

TOP. STRING SIDE:

LR ⟶ orbifold: S-duality transf$^{\underline{n}}$

(see that by looking at ram. points, i.e. cuts)

"exchange" of the cuts)



S-duality

$\Rightarrow$ easy to compute $B(x_1, x_2)$ at orbifold using

Akemann's formula for hyperelliptic curves

[which is in terms of the cuts]

just anal. continuation, no modularity

$\Downarrow$

SO we have disk, annulus $\leftarrow$ S-duality + Akemann

$\Rightarrow$ Generate all other amplitudes recursively at the orbifold point

CS SIDE:

CS ($S^3/\mathbb{Z}_2$): 2-Matrix model.

$$Z(N_1,N_2,g_s) = \int dM_1 dM_2 \exp\left\{-\frac{1}{2g_s}\mathrm{Tr}M_1^2 - \frac{1}{2g_s}\mathrm{Tr}M_2^2 + V(M_1) + V(M_2) + W(M_1,M_2)\right\}$$

$$V(M) = \frac{1}{2}\sum_{k=1}^{\infty} a_k \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \mathrm{Tr}M^s \,\mathrm{Tr}M^{2k-s}$$

$$W(M_1,M_2) = \sum_{k=1}^{\infty} b_k \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \mathrm{Tr}M_1^s \,\mathrm{Tr}M_2^{2k-s}$$

$$a_k = \frac{B_{2k}}{k(2k)!} \qquad b_k = \frac{2^{2k}-1}{k(2k)!} B_{2k}.$$

UNKNOT: $\quad W_{\vec{k}}(N_1,N_2,g_s) = \frac{1}{Z(N_1,N_2,g_s)} \mathrm{Tr}_{\vec{k}} \langle e^M \rangle$

w/, in terms of eigenvalues m^1_i, m^2_j of M_1, M_2,

$e^M = \mathrm{diag}(e^{m^1_1}, \ldots, e^{m^1_{N_1}}, -e^{m^2_1}, \ldots, -e^{m^2_{N_2}}$

$\rightsquigarrow$ $W_k^{(g)}$ match top. string calculation!

CRUCIAL POINT: MIRROR MAP at orbifold pt

- closed mirror map found in [AKMV]
- open mirror map?

→ by matching disk amplitude, very simple open mirror map:

$$X = \frac{x}{x_1 x_2}$$

[would be nice to have a better justification]

⇒ good results for $W_2^{(0)}$, $W_3^{(0)}$, $W_1^{(1)}$, ...

③ LOCAL $\mathbb{P}^2$: – "open" amplitudes for $\mathbb{C}^3/\mathbb{Z}_3$?

"PROBLEMS": • open mirror map?

• transf^n is not S-duality, but complicated $SL(2,\mathbb{C})$ transf^n [VB-Aganagic-Klemm]

Disk: • find open mirror map by requiring monodromy-invariance,

$$X = x\frac{\sigma}{\psi}$$

← orbifold parameter

← comp. struct. parameter, orbifold at $\psi = 0$

[nothing to compare with]

Annulus: • not clear how to implement $SL(2,\mathbb{C})$ transf$^{\underline{n}}$ of $B(x_1, x_2)$, but should be possible

$\Rightarrow$ all the other amplitudes recursively

Q: • what would these amplitudes compute? [• open orbifold GW invariants? • large N transition?]

SUMMARY

• new B-model formalism for open/closed amplitudes on toric CY_3, NO ambiguity

• can be used at other pts in moduli space
$\rightarrow$ open/closed amplitudes for sing. CY_3 [orbifolds, ...]

OPEN QUESTIONS

- is there a matrix model (and/or large N transition) behind this story???

$\rightsquigarrow$ non-perturbative insight on top. strings

- Compact CY_3?
  - need to find something to replace Σ_i, or generalize the formalism to higher-dim$^{\text{al}}$ manifolds

- rel$^{\text{n}}$ w/ Walcher's open hol. anomaly eq$^{\text{ns}}$?

- etc...

THANK YOU! :)