Four-dimensional Kondo Problem

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In 1934, de Haas, de Boer, and van den Berg observed that a small concentration of magnetic impurities in a metal leads to a minimum in the temperature dependence of the resistance. Thirty years later, this effect was explained by Jun Kondo, who used perturbation theory to study the exchange interaction of a localized magnetic impurity with the conduction electrons of the host metal.[1] Further developments of the theory of magnetic impurities in metals led to many powerful methods that can be applied in a variety of related problems, including the Hartree-Fock solution to the Anderson model, many-body calculations, scaling, Bethe ansatz solution of the spin-half *s*-*d* model, variational methods, and $1/N$ expansion.

In this talk, I am going to describe certain results of my recent work with E. Witten [8], where we studied an analog of this problem in four-dimensional gauge theory. Thus, an analog of impurity in four-dimensional gauge theory is a "surface operator," which is supported on a 2-dimensional surface $D \subset M$ in the space-time manifold M. Surface operators have many interesting applications to various problems in physics and mathematics, including applications to the geometric Langlands program and knot homologies. They naturally belong to the list of non-local operators and, therefore, can be useful for understanding the physics of four-dimensional gauge theory (phases, non-perturbative phenomena, *etc.*). In order to put this in a slightly broader perspective, it is convenient to classify non-local operators in four-dimensional gauge theory by their codimension:

- codimension 4: the operators of codimension 4 are the usual local operators $\mathcal{O}(p)$ supported at a point $p \in M$. These are most familiar operators in this list, which

[1] In a way, Kondo's perturbative calculation of the resistance is the first known example of asymptotic freedom, in which the coupling becomes non-perturbatively strong at low temperatures/low energies.

have been extensively studied *e.g.* in the context of the AdS/CFT correspondence. Typical examples of local operators can be obtained by considering gauge-invariant combinations of the fields in the theory, *e.g.* $\mathcal{O}(p) = \mathrm{Tr}(\phi^n \ldots)$.

- codimension 3: line operators. Important examples of such operators are Wilson and 't Hooft operators, which are labeled, respectively, by a representation, R, of the gauge group, G, and by a representation ${}^L R$ of the dual gauge group ${}^L G$.
- codimension 2: surface operators. These are perhaps least studied among the operators and defects listed here, and will be precisely our main subject.
- codimension 1: domain walls and boundaries.

While particular examples of surface operators have appeared (explicitly or implicitly) in various contexts in physics [1, 6, 7, 14, 15, 17] and mathematics [4, 5, 11, 12] literature, they have not been studied systematically in the context of four-dimensional gauge theory. In particular, in order to make surface operators a standard tool in gauge theory and put them on the same footing as the other operators listed above, one would like to address systematically a number of basic questions:

- How can one define surface operators?
- What are they classified by?
- Are there supersymmetric surface operators?
- What are the correlation functions of surface operators?
- What is the OPE algebra of line operators in the presence of a surface operator?
- How do surface operators transform under dualities?

$\vdots$

Following [8], I will describe some progress towards addressing these questions for a certain class of surface operators in a topological gauge theory. Already with simple arguments one can see interesting properties of surface operators and preliminary answers to these questions. For example, according to the general formal rules of topological quantum field theory (TQFT), correlation function of a surface operator supported on a compact surface D embedded in a 4-manifold M is a number (namely, the partition function of the theory). This number depends on parameters of the theory and on the surface operator, and in various examples provides interesting invariants of embedded surfaces in 4-manifolds, cf. [11, 12].

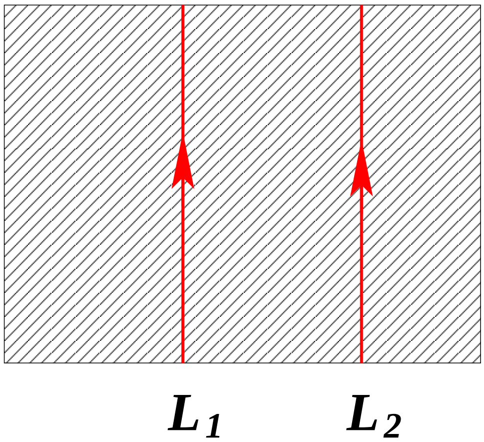

Figure 1: Line operators confined to a surface operator do not commute.

On the other hand, quantization of a four-dimensional topological theory on a 4-manifold $M = \mathbb{R} \times Y$ with a surface operator on $D = \mathbb{R} \times K$ gives rise to a functor that associates to this data (namely, a 3-manifold Y, a knot/link $K \subset Y$, and parameters of the surface operator) a vector space, the space of quantum ground states,

$$\mathcal{H}_{Y;K,\text{parameters}} \tag{1}$$

Therefore, in such "static" case the correlation function of a surface operator is a vector space, rather than a number. This simple yet important property of surface operators has direct applications to knot homologies.

Another interesting property of surface operators which is easy to understand has to do with the question about the OPE algebra of line operators in the presence of a surface operator. Without a surface operator, the OPE algebra of line operators in a four-dimensional TQFT is commutative simply because one can continuously exchange positions of line operators by moving them around each other in four dimensional space. (This process can be regarded as varying the metric on M, which has no effect in topological theory.) In the presence of a surface operator, however, there can be additional line operators which are supported on the surface operator and can not move into the rest of the 4-manifold M. Since such line operators are confined to the surface $D \subset M$, they can not be passed through each other without encountering a singularity. As a result, the OPE algebra of such lines operators in general is non-commutative.

Definition of Surface Operators

First, let us recall that a Wilson line operator labeled by a representation R of the gauge group G is defined as

$$W_R(K) = \mathrm{Tr}_R \, \mathrm{Hol}_K(A) = \mathrm{Tr}_R \left(P\mathrm{exp} \oint A \right) \tag{2}$$

In general, there is no analogous "electric" definition of surface operators in four-dimensional gauge theory. However, one can define operators supported on a surface D by requiring the gauge field A (and, possibly, other fields) to have a prescribed singularity along D:

$$\mathrm{Hol}_\ell(A) \in \mathfrak{C} \tag{3}$$

where ℓ is a small loop that links surface $D \subset M$ in the space-time 4-manifold M, and $\mathfrak{C}$ is a fixed conjugacy class in the gauge group G.

A careful definition of surface operators — which we present below for a particular example of gauge theory — essentially gives an answer to the question about classification of surface operators. In general, parameters of surface operators can be divided into discrete data and continuous parameters. In a way, the former is analogous to the choice[2] of a representation that labels line operators, *cf.* (2), while the latter are a novel feature of surface operators. Moreover, it turns out that understanding these continuous parameters is the key to addressing other important questions about the properties of surface operators. For example, the non-commutative structure of line operators supported on a surface operator that was mentioned above is described by the fundamental group of the suitable (sub)space of continuous parameters:

$$\pi_1(\{\mathrm{parameters}\}) \tag{4}$$

It turns out that many interesting four-dimensional gauge theories admit (supersymmetric) surface operators, which have a number of nice properties. Let us focus on a particular topological gauge theory obtained by a topological twist of maximally supersymmetric (that is, $\mathcal{N} = 4$) Yang-Mills theory in four dimensions. This theory has been extensively studied in the context of string dualities, in particular in the AdS/CFT correspondence [13]. It has many remarkable properties and, in fact, might even be integrable in the large N limit. Bosonic fields in this theory include the gauge connection A and six Higgs fields, ϕ_i, $i = 1, \ldots, 6$ which transform as $\mathbf{6}$ of the global R-symmetry group $SO(6)_{\mathcal{R}}$.

There are three topological twists of $\mathcal{N} = 4$ super-Yang-Mills theory which correspond to different homomorphisms from the $SO(4)$ symmetry group of the four-dimensional Euclidean

[2]For example, as we explain below, in a theory with gauge group $G = SU(N)$ this choice includes the choice of a partition of N.

space to the global R-symmetry group [16]:

$$\kappa : \; SO(4) \to SO(6)_{\mathcal{R}} \tag{5}$$

One of these twists — the so-called GL twist [10] — leads to a physical framework for realizing the geometric Langlands program in gauge theory.

We will be interested in surface operators in this theory (both twisted and untwisted version) which are half-BPS, *i.e.* preserve 8 real supersymmetries. Following [8], we describe a large class of half-BPS surface operators which break the gauge group down to a subgroup $\mathbb{L} \subset G$ (the so-called "Levi subgroup") and which also break the global R-symmetry group,

$$SO(6)_{\mathcal{R}} \to SO(4) \times SO(2) \tag{6}$$

by introducing a singularity for two components of the Higgs field, say $\varphi = \phi_1 + i\phi_2$,

$$\varphi = \frac{1}{2}(\beta + i\gamma)\frac{dz}{z} + \dots \tag{7}$$

Here, $z = x^2 + ix^3 = re^{i\theta}$ is a local complex coordinate, normal to the surface $D \subset M$, and the dots stand for less singular terms. In general, the definition of a surface operator also includes a singularity for the gauge field,

$$A = \alpha d\theta + \dots , \tag{8}$$

which corresponds to the holonomy (3). In order to obey the supersymmetry equations, the parameters α, β, and γ must take values in (the $\mathbb{L}$-invariant part of) $\mathfrak{t}$, the Lie algebra of the maximal torus $\mathbb{T}$ of G. Moreover, gauge transformations shift values of α by elements of the cocharacter lattice, Λ_{cochar}. Hence, α takes values in $\mathbb{T} = \mathfrak{t}/\Lambda_{\text{cochar}}$.

In addition to the classical (or "geometric") parameters (α, β, γ), the surface operators of this type are also labeled by quantum parameters η, the "theta angles" of the two-dimensional theory on $D \subset M$. It is easy to see that parameters η take values in (the ${}^L\mathbb{L}$-invariant part of) the maximal torus ${}^L\mathbb{T}$ of the Langlands/GNO dual group ${}^LG$.

We can summarize all this by saying that maximally supersymmetric ($\mathcal{N} = 4$) super-Yang-Mills theory admits a large class of surface operators labeled by a choice[3] of the Levi subgroup $\mathbb{L} \subset G$ and continuous parameters[4]

$$(\alpha, \beta, \gamma, \eta) \in (\mathbb{T} \times \mathfrak{t} \times \mathfrak{t} \times {}^L\mathbb{T})/\mathcal{W} \tag{9}$$

invariant under the Weyl group $\mathcal{W}_{\mathbb{L}}$ of $\mathbb{L}$. These surface operators naturally correspond to the so-called Richardson conjugacy classes in the complexified gauge group $G_{\mathbb{C}}$, *cf.* (3) and, in a theory with gauge group $G = SU(N)$, cover all half-BPS surface operators which correspond to singularities with simple poles.

[3] In a theory with gauge group $G = SU(N)$ this choice is equivalent to a choice of a partition of N.

[4] Similar surface operators exist in $\mathcal{N} = 2$ supersymmetric gauge theories; the only difference is that they don't have parameters β and γ.

Duality for Surface Operators

Many supersymmetric gauge theories are believed to enjoy interesting duality symmetries. Thus, the maximally supersymmetric Yang-Mills theory discussed above is conjectured to enjoy electric-magnetic duality symmetry which, among other things, exchanges electric and magnetic charges and transforms the gauge group and the coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ as:

$$\begin{aligned} G &\to {}^L G \\ \tau &\to -\frac{1}{n_{\mathfrak{g}} \tau} \end{aligned} \tag{10}$$

Here, ${}^L G$ is the Langlands or GNO dual group, and $n_{\mathfrak{g}} = 1$, 2, or 3 depending on G. (More specifically, $n_{\mathfrak{g}}$ is the ratio of the length squared of the long and short roots of G.) How does electric-magnetic duality act on a surface operator?

It was argued in [8], a surface operator labeled by a Levi subgroup $\mathbb{L}$ and continuous parameters $(\alpha, \beta, \gamma, \eta)$ under electric-magnetic duality transforms into a surface operator of the same type, but with different parameters. Namely, the duality essentially acts trivially on the parameters β and γ, and acts on the other parameters as:

$$\begin{aligned} \mathbb{L} &\to {}^L \mathbb{L} \\ (\alpha, \eta) &\to (\eta, -\alpha) \end{aligned} \tag{11}$$

Surface Operators and the Geometric Langlands Program

In the gauge theory approach to the geometric Langlands program [10], surface operators are the key ingredients that allow to incorporate ramification [8]. Understanding the ramified case is important for a number of reasons. First, ramification is inescapable in the "classical" Langlands correspondence which relates irreducible representations of the Galois group $\sigma : \ \mathrm{Gal}(\overline{F}/F) \to {}^L G$, on the one hand, and irreducible automorphic representations of $G(\mathbb{A}_F)$, on the other hand.[5] Moreover, in the presence of ramification certain new phenomena appear: the algebra of Hecke operators becomes non-commutative, while (for certain types of ramification) the relation to D-modules becomes tricky, the formulation of the geometric Langlands correspondence in terms of D-branes still remains tractable, *etc.* All this motivates the study of ramification in the framework of the four-dimensional gauge theory which, as we illustrate below, leads to important lessons for the geometric Langlands program.

[5]Here, F can be a number field or a function field, $\overline{F}$ is its algebraic closure, and $\mathbb{A}_F$ is the ring of adeles of F.

The physical approach to the geometric Langlands program is based on a twisted version of the $\mathcal{N} = 4$ super-Yang-Mills theory compactified on a Riemann surface C [10]. The topological reduction of this theory leads to a $\mathcal{N} = 4$ sigma-model [2, 10], whose target space is a hyper-Kahler manifold $\mathcal{M}_H$, the moduli space of solutions to the Hitchin equations on C [9]:

$$\begin{aligned} & F_A - \phi \wedge \phi = 0 \\ & d_A \phi = 0 \quad , \quad d_A \star \phi = 0 \end{aligned} \tag{12}$$

Here, A is the gauge field, a connection on G-bundle $E \to C$, and ϕ is the Higgs field, an ad(E)-valued one-form on C. After the topological reduction, the electric-magnetic duality (10) of the $\mathcal{N} = 4$ super-Yang-Mills theory becomes mirror symmetry which relates topological sigma-models on dual Hitchin fibrations,

A-model on $\mathcal{M}_H(G, C, \text{ramification})_{\omega_K}$	$\Longleftrightarrow$	B-model on $\mathcal{M}_H({}^L G, C, {}^L \text{ramification})_J$

where we incorporated ramification, which corresponds to introducing surface operators in the four-dimensional gauge theory. Interpreting D-branes as objects in certain categories identified by this duality, one recovers the basic claims of the geometric Langlands correspondence with ramification. In this approach, different kinds of surface operators correspond to different types of ramification. In particular, surface operators (6) - (8) considered above describe the so-called *tame* ramification [8].

We note that, in the mathematical literature, ramification is usually described in terms of filtered local systems (on the Galois side) or in terms of parabolic bundles (on the automorphic side) which involve a choice of the parabolic structure at a ramification point p, that is a reduction of the structure group of a G-bundle $E \to C$ at a point p to a parabolic subgroup $\mathcal{P}$. In particular, the two sides appear very asymmetrically. The symmetry is restored in the gauge theory description, where the data on both sides is encoded in the parameters of the surface operator, namely in the choice of the Levi subgroup $\mathbb{L}$ and in the continuous parameters $(\alpha, \beta, \gamma, \eta)$. Moreover, the duality transformation of these parameters (11) automatically leads to the expected map between singularity of the filtered local system and the dual choice of the parabolic structure.

The description of ramification in terms of surface operators leads to valuable lessons and interesting new results. For example, as we explained earlier, in the presence of a surface operator the OPE algebra of line operators becomes non-commutative. In the sigma-model on $\mathcal{M}_H$, line operators correspond to functors acting on branes. According to (4), these functors form a group which often can be identified with the fundamental group of the (sub)space of parameters of a surface operator. To continue with our present example, let us consider surface operators which correspond to tame ramification and, for concreteness, focus on the B-model in complex structure J (which describes the Galois side of the geometric Langlands correspondence). Then, depending on whether one is interested in D-branes (as objects the derived category of coherent sheaves on $\mathcal{M}_H$) or in D-brane charges (classified by K-theory) one finds the following groups acting the on the K-theory/derived category of the moduli space of ramified Higgs bundles[6]:

Claim: *affine Weyl group* $\mathcal{W}_{\text{aff}}$ *acts on* $K(\mathcal{M}_H)$

affine Hecke algebra H_{aff} *acts on* $K^{\mathbb{C}^*}(\mathcal{M}_H)$

affine braid group B_{aff} *acts on* $D^b(\mathcal{M}_H)$

Hence, this result can be regarded as a categorification of the affine Hecke algebra, which in the local version of $\mathcal{M}_H$ was also obtained by Bezrukavnikov [3] using a "noncommutative counterpart" of the Springer resolution $\widetilde{\mathcal{N}} \to \mathcal{N}$. The action of $\mathcal{W}_{\text{aff}}$ and B_{aff} in the first and the last part of this claim can be understood as the monodromy action in the space of parameters of the surface operator [8].

For example, let us illustrate how this group action arises at the level of D-brane charges, which are classified by $K(\mathcal{M}_H)$. The space of D-brane charges $K(\mathcal{M}_H)$ varies as the fiber of a flat bundle over the space of parameters away from the points where $\mathcal{M}_H$ develops singularities. Since for the purposes of this question we are interested only in the geometry of $\mathcal{M}_H$, we can ignore the "quantum" parameter η. Hence, the relevant parameters are (α, β, γ), which take values in the space, *cf.* (9):

$$(\alpha, \beta, \gamma) \in (\mathfrak{t} \times \mathfrak{t} \times \mathfrak{t})/\mathcal{W}_{\text{aff}} \tag{13}$$

Moreover, $\mathcal{M}_H$ becomes singular precisely for those values of (α, β, γ) which are fixed by some element of $\mathcal{W}_{\text{aff}}$. The set of such points is at least of codimension three in $\mathfrak{t}^3$ (since it takes three separate conditions to be satisfied for (α, β, γ) to be fixed by some element of $\mathcal{W}_{\text{aff}}$). Therefore, the space of regular values of $(\alpha, \beta, \gamma) \in \mathfrak{t}^3$ where $\mathcal{M}_H$ is non-singular is connected and simply-connected, and since $\mathcal{W}_{\text{aff}}$ acts freely on this space, the fundamental group of the quotient is

$$\pi_1(\{(\alpha, \beta, \gamma)\}^{\text{reg}}) = \mathcal{W}_{\text{aff}} \tag{14}$$

This is the group that acts on D-brane charges, that is on $K(\mathcal{M}_H)$. In a similar way, one can deduce the action of the affine braid group B_{aff} on $D^b(\mathcal{M}_H)$ as the fundamental group of the Kähler moduli space. Indeed, for the B-model in complex structure J the complexified

[6]For simplicity, here we consider only one ramification point $p \in C$. For the case of ramification at several points, one finds several group actions, one for each ramification point.

Kähler parameters are $\beta + i\eta$, and from (9) one finds [8]:

$$\pi_1(\{(\beta,\eta)\}^{\mathrm{reg}}) = B_{\mathrm{aff}} \tag{15}$$

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