Generalized Kähler Potentials For Supergravity

Nick Halmagyi
Enrico Fermi Institute,
University of Chicago,
Chicago, IL 60637, USA
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Introduction

Generalized complex geometry was introduced by Hitchin and students [1, 2] and naturally includes the (not necessarily closed) NS B-field into its structure. Subsequently it was shown by the Paris group [3] that even with non-vanishing Ramond fluxes, four dimensional reductions of type II supergravity are examples of generalized complex manifolds, in fact they are generalized Calabi-Yau manifolds .

There is reason to hope that a deeper understanding of these geometrical structures will result in fruitful interplay between worldsheet and spacetime methods. In the case of Ramond backgrounds of course, any insight at all would be beneficial as this is a notoriously hard problem. In fact there was some progress by Linch and Vallilo [4]. Within the framework of Berkovitz' hybrid formalism, they considered a worldsheet theory which is first order flux deformation of a Calabi-Yau background. They found that when accounting correctly for the warp factor which of course couples the spacetime and internal CFT's, the full resulting theory retains (2,2) superconformal symmetry. In addition they found that the physical state conditions are of precisely the same form as the supersymmetry conditions found in [3].

However before embarking on the difficult problem of type II backgrounds with Ramond flux, one is lead first to revisit backgrounds with only the NS threeform turned on. There has recently been much success in relating generalized complex geometry and the target space geometry of the tree level nonlinear sigma model with NS flux [5]. There it was shown that in order to realize the most general (2,2) sigma model one must include left and right semi-chiral superfields. As is familiar from (2,2) sigma models, the entire worldsheet Lagrangian can be derived from a single function known as the *Kähler Potential*.

It was shown by Gualtieri [2] that the most general on-shell (2,2) sigma model, known to realize so called *bi-Hermitian* geometry, is equivalent to generalized Kähler geometry. One success of the off-shell worldsheet formalism is to demonstrate that these generalized Kähler geometries should be entirely derivable from a potential much like ordinary Kähler geometry. The difference in the generalized case is that the passage from the potential to the metric (and B-field) is non-linear, this might lead one to regard generalized Kähler geometry as *nonlinear* Kähler geometry. This also provides hope that searching for supergravity solutions may be easier in terms of the pure spinors or even Kähler potential since this represents a certain linearization of the problem.

This is a short summary of the work [6], where we provided a supergravity interpretation for the generalized Kähler potential and derived a generalized Monge Ampére equation for it. We also demonstrated that the simplest example of a type II, NS background, the so called Lunin-Maldacena background belongs in the class of backgrounds which are described by worldsheet semi-chiral superfields and we derive the Kähker potential in this case.

1 The Generalized Kähler Potential from Spacetime

We first decompose the two ten dimensional Majorana-Weyl spinors in a 4-6 split

$$\epsilon_i = \zeta_+ \otimes \eta_{i+} + c.c. \tag{1}$$

and form the pure spinors which are really just convenient sums of spinor bilinears

$$\Phi_\pm = e^B \left(\eta_{1+} \otimes \eta^\dagger_{2\pm} \right). \tag{2}$$

The conditions for supersymmetry are then simply

$$d\Phi_\pm = 0. \tag{3}$$

So although the analysis of [5] included arbitrary dimensional target spaces and thus arbitrary numbers of chiral, twisted chiral superfields[1] since we are concerned with reductions to four dimensions we will be concerned only with a single pair of semi-chiral superfields and a single chiral superfield. Of course it is merely convention to take a chiral superfield, we could just as well take a twisted chiral superfield. With this in mind, the generalized Kähler potential is given by

$$K = K(q, \overline{q}, P, \overline{P}, w, \overline{w}). \tag{4}$$

The co-ordinates given by lowercase letter q, p are leftmoving semi-chiral co-ordinates, those with uppercase letters are right moverign semi-chiral co-ordinates and w is a chiral superfield. The

[1] the number of semi-chiral superfields is restricted to appear in multiples of four

generalized Kähler potential only depends on half the left and half the right moving co-ordinates, the remaining co-ordinates are obtained from

$$p = \partial_q K, \qquad Q = \partial_P K.$$

The main result of [6] is that through algebraic manipulations, the pure spinors can be written purely in terms of K:

$$\begin{aligned} \Phi_+ &= \exp\left(-8\partial\bar{\partial}K\right) && (5)\\ \Phi_- &= dw \wedge \exp\left(-8(dq \wedge d(\partial_q K) + d(\partial_P K) \wedge dP)\right) \\ &= dw \wedge \exp\left(-8(dq \wedge dp + dQ \wedge dP)\right) && (6) \end{aligned}$$

and the differential constraint on the pure spinors is a Monge Ampére equation for K, namely

$$\frac{\det(K_H)}{K_{qP}K_{\overline{q}\overline{P}} - K_{q\overline{q}}K_{P\overline{P}}} = \text{constant}. \tag{7}$$

Here we have denoted second derivatives of K by K_{qP} etc. and K_H is the 3×3 matrix of second derivatives w.r.t. q, P, w and their conjugates. This equation has in fact been discovered previously by Grisaru et. al. from a one-loop sigma model analysis [7].

2 Generalized Complex Structure Moduli

Probably the simplest example of a background with nonvanishing NS threeform flux is the Lunin-Maldacena bakcground[2]. This backgound is obtained by a solution generating element of the U-duality group of type II strings on flat space. Whilst the metric and B-field have a somewhat complicated form we have shown the the Kähler potential is simply

$$K = qP + \overline{q}\overline{P} + \gamma \sum_i |z_i|^2 + \gamma^2(|z_2|^2 - |z_3|^2)(|z_3|^2 - |z_1|^2) \tag{8}$$

and the leftmoving and rightmoving co-ordinates are given by

$$\begin{aligned} q &= \log(z_1) - \frac{\gamma}{2}(|z_2|^2 - |z_3|^2), \qquad p = \log(z_2) - \frac{\gamma}{2}(|z_3|^2 - |z_1|^2) \\ Q &= \log(z_1) + \frac{\gamma}{2}(|z_2|^2 - |z_3|^2), \qquad P = \log(z_2) + \frac{\gamma}{2}(|z_3|^2 - |z_1|^2). \end{aligned}$$

The parameter γ measures the deformation away from flat space. In the full RR background this is the strength of the superpotential deformation

$$W = z_1[z_2, z_3] + \gamma z_1\{z_2, z_3\}. \tag{9}$$

[2] it should be emphasized that Lunin and Maldacena derive both a Ramond background and an NS solution, we are concerned here with the NS solution

This background is an example of a general procedure devised by Gualtieri to produce a generalized Kähler geometry from a Kähler geometry. Now Gualtieri has shown that the moduli of generalized complex structures are given by the cohomology groups

$$\oplus_{p+q=2} H^p(M, \Lambda^q T_{(1,0)}), \tag{10}$$

and Wijnholt observed that this may account for the gravity duals to Leigh-Strassler marginal deformations of SCFT's. We take $\mathbb{C}^3$ as an example of a Calabi-Yau cone but the arguments hold for any cone over a regular Sasaki-Einstein manifold. The level set of $\mathbb{C}^3$ is S^5 which is a $U(1)_R$ fibration over $\mathbb{C}P^2$. Since by the standard rules of AdS/CFT, superpotential deformations are non-normalizable modes in the bulk we expect exactly marginal deformations of $\mathcal{N} = 4$ SYM to come from holomorphically extending a modulus of $\mathbb{C}P^2$ over $\mathbb{C}^3$. Now dim $\left(H^0(\mathbb{C}P^2, \Lambda^2 T_{(1,0)})\right) = 2$ and this is indeed the expected number of exactly marginal deformations of $\mathcal{N} = 4$ SYM.

The mechanism for generating generalized Kähler structures proposed by Gualitieri is to act on the pure spinors as follows:

$$\Phi_\pm = \exp(\beta_{ij}\Gamma^{ij})\Phi_\pm, \tag{11}$$

where β_{ij} is a section of $\Lambda^2 T_{(1,0)}$ and importantly, this has been chosen such that Φ_+ is invariant. Gualtieri's procedure however does not guarantee that the resulting six dimensional manifold will be generalized Calabi-Yau, in other words, it may not satisfy the Monge Ampére equation. We have shown that this is indeed the mechanism which drives the Lunin-Maldacena geometry and the choice of β_{ij} is given by:

$$\Phi_\pm = \exp[\gamma(4\iota_{\partial_{\varphi^1}} \wedge \iota_{\partial_{\varphi^2}} + \frac{1}{4} dr_1^2 \wedge dr_2^2) + \text{cycl. perm.}] \wedge \Phi_{\pm 0}. \tag{12}$$

where $\Phi_{\pm 0}$ are the pure spinors of the undeformed Calabi-Yau background and $z_i = r_i e^{i\varphi_i}$.

The main direction for future work is to incorporate Ramond fluxes, we hope this discoveries will enable us to make progress in that direction.

References

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