Quaternionic geometry, BPS black holes and topological strings

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The following are the notes I used to give this talk — they are written in a rather abbreviated fashion, but all the content is there.

1 Preface

We'll describe some applications of quaternionic geometry to supergravity with 8 supercharges: to supersymmetric BH, their quantization, topological strings (and an extension thereof). Some of these things are usually studied using special geometry, we suggest quaternionic geometry — and the geometry of twistor spaces over quaternionic manifolds — will eventually be useful.

The reason is the reduction from $d = 4$ to $d = 3$, which has applications to:

- classical BH solutions (their description and also solution generating),
- their quantization,
- construction of partition functions (compactify Euclidean time),
- extended duality symmetries.

Work with M. Gunaydin, B. Pioline, S. Vandoren, A. Waldron. In progress!

2 c-map

Begin with $\mathcal{N} = 2$ supergravity coupled to $n-1$ abelian vector multiplets, e.g. Type IIA on CY with $h^{1,1} = n-1$ (ignoring hypers). $2n-2$ scalar moduli of CY, n gauge fields. Chiral part of action is $\int d^4\theta \mathcal{F}(X^0, X^a)$, $\mathcal{F}$ homogeneous of degree 2. Moduli $t^a = X^a/X^0$.

Theory reduced on S^1 can first be described in components: has $4n$ scalars. $2n-2$ of these are moduli t^a from $d = 4$, $2n$ are Wilson lines ζ^Λ, $\tilde{\zeta}_\Lambda$, 2 are gravity sector e^U, σ. Low energy theory has quaternionic Kähler moduli space $\mathcal{M}$. Classically, $2n+2$ triholomorphic isometries: shifts of $2n$ Wilson lines, σ, U. They form an extended Heisenberg group.

Constant curvature $R = 4n(n+2)\nu$. Metric purely in terms of special geometry data.

3 Quaternionic Kähler geometry

QK manifolds $\mathcal{M}$ have $Sp(1) \times Sp(n)$ holonomy, i.e. $T_{\mathbb{C}}\mathcal{M} = H \otimes E$ preserved by Levi-Civita connection. So replace $\mu \to AA'$. Natural target spaces for SUGRA coupled to hypers with 8 supercharges, e.g. in $d = 4$ or $d = 3$. Not Kähler: S^2 of compatible almost complex structures at each point. These define $\mathcal{Z} = \mathbb{P}(H) \to \mathcal{M}$, complex. Also have $\mathcal{S} = H \to \mathcal{M}$, hyperkähler, with natural $SU(2)$ action rotating the complex structures, metric

$$\mathrm{d}s^2_{\mathcal{S}} = |\mathrm{D}\pi^{A'}|^2 + \frac{\nu}{4} r^2 \mathrm{d}s^2_{\mathcal{M}} \,. \tag{1}$$

($R = 4n(n+2)\nu$.) Superconformal formalism naturally involves $\mathcal{S}$, in much the same way as the rigid special Kahler manifold appears in vector multiplet sector.

For σ-model into $\mathcal{M}$, fermions are sections of E, SUSY parameters are sections of H.

$$\delta_\eta \psi^A_\alpha = \partial^\mu \phi^{AA'} (\gamma_\mu)^\beta_\alpha \eta_{A'\beta} \tag{2}$$

4 SUSY configurations

SUSY black holes in $d = 4$ have a time translation isometry, so can be viewed as configs in $d = 3$.

Consider SUSY-preserving radially symmetric configs: geodesics on $\mathcal{M}$, with $p = h \otimes e$. (Gloss over analytic continuation.) ie, $p^{[AA'} p^{B]}_{A'} = 0$, all minors of $2 \times 2n$ matrix $p^{AA'}$ vanish. A complicated quadratic constraint, solution space not linear!

But SUSY configs have natural lift to $\mathcal{S}$, provided by h. This lift gives economical description of the SUSY configs. Pick coordinates $(u^{\aleph}, u^{\bar{\aleph}})$ for "natural" $\mathbb{C}$ str on $\mathcal{S}$. SUSY condition means all $u^{\bar{\aleph}}$ constant, ie $p_{\aleph} = 0$. If $\mathcal{M}$ has $2n+2$ triholomorphic isometries (as in c-map case) then also $p_{\bar{\aleph}} = g_{\aleph\bar{\aleph}} \frac{du^{\aleph}}{dt} = c_{\bar{\aleph}}$. Using $g_{\aleph\bar{\aleph}} = \partial_{u^{\aleph}} \partial_{u^{\bar{\aleph}}} \chi$, this means $\partial_{u^{\bar{\aleph}}} \chi = c_{\bar{\aleph}} t + d_{\bar{\aleph}}$. This is the attractor black hole (including NUT charge). Conversely, any such config with zero $SU(2)$ momentum descends to supersymmetric solution.

To write the solutions in terms of 4-d field content is harder: need to work out relation between $(u^{\aleph}, u^{\bar{\aleph}})$ and coordinates on $\mathcal{M}$, and determine χ.

5 Quantum version

This lift also allows to solve a *quantum* version of SUSY constraint: look at functions φ on $\mathcal{M}$ (naive quantization of radial flow), then impose

$$\left(\nabla^{AA'} \nabla^{B}_{A'} + \nu \epsilon^{AB} \right) \varphi = 0. \tag{3}$$

(Curvature term determined by quaternionic invariance.)

If fix Noether (black hole) charges $k = 0$, q_Λ, p^Λ, then the lift leads to a natural solution,

$$\varphi(U, z^a, \bar{z}^{\bar{a}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) = \mathrm{e}^{2U + \mathrm{i} p^\Lambda \tilde{\zeta}_\Lambda - \mathrm{i} q_\Lambda \zeta^\Lambda} \; J_0(2\mathrm{e}^U |Z|) \tag{4}$$

where e^U is the scale of the time direction and

$$Z = \mathrm{e}^{\frac{1}{2}\mathcal{K}(X,\bar{X})} (p^\Lambda F_\Lambda(X) - q_\Lambda X^\Lambda). \tag{5}$$

This is derived by quaternionic Penrose transform: $\oint$ of a class in $H^1(\mathcal{Z}, \mathcal{O}(-2))$. (Analogue of the standard Penrose transform for $\mathbb{R}^4$.)

If we look near $U \to -\infty$, see that φ is peaked near attractor points. Appearance of Z seems appealing; but is this φ good for anything?

6 Geometry of holomorphic anomalies

Let $\mathcal{M}_s$ be moduli of complex structures on X: special Kähler manifold. $\mathcal{L}$ its canonical Hodge bundle. Introduce $\widetilde{\mathcal{M}}_s = \mathcal{L} \to \mathcal{M}_s$ (parameterize $\widetilde{\mathcal{M}}_s$ by periods of Ω). This is rigid special pseudo-Kähler, analog of $\mathcal{S}$. Introduce generating function of chiral correlators:

$$\Psi(X,\bar{X},x,\lambda) = \lambda^{\frac{\chi}{24}-1} \exp\left(\sum_{g=0}^{\infty}\sum_{k=0}^{\infty} \lambda^{2g-2+k} \langle O_{i_1}\cdots O_{i_k}\rangle_{g,(X,\bar{X})} x^{i_1}\cdots x^{i_k}\right) \tag{6}$$

Introduce periods of $\lambda^{-1}\Omega + x^i \delta_i \Omega$, call these x^I, parameterizing $H^{3,0}\oplus H^{2,1}$. Then Ψ is naturally a function on $\mathcal{T}\widetilde{\mathcal{M}}_s$, neither holomorphic nor antiholomorphic, obeying $3n$ differential equations.

Further introduce

$$y_I = (\tau - \bar{\tau})_{IJ}\bar{x}^J \tag{7}$$

and modify treatment of 0, 1 point functions in Ψ: then get $\bar{\Psi}$ purely holomorphic, defined on $\mathcal{T}^*\widetilde{\mathcal{M}}_s$, obeying

$$\left(\frac{\partial}{\partial X^I} - \frac{\mathrm{i}}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K}\right)\bar{\Psi}_{closed} = 0. \tag{8}$$

Just like a theta function (formally). Moreover, in Walcher's approach to holomorphic anomaly of the open string, similar manipulations give

$$\left[\frac{\partial}{\partial X^I} - \frac{\mathrm{i}}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K} + \mathrm{i}\nu_{IJ}\frac{\partial}{\partial y_J}\right]\bar{\Psi}_{open} = 0 \tag{9}$$

where ν is the Griffiths infinitesimal invariant ($\nu_{IJ} = \partial_I \partial_J \mathcal{T}$). So it can be absorbed by a shift $\Psi_{closed}(X^I, y_I) = \Psi_{open}(X^I, y_I + \nu_I)$.

Why does this occur? No obvious SUSY argument.

7 Hyperkähler structure, extended geometry of topological string

Note, $\mathcal{T}^*\widetilde{\mathcal{M}}_s$ is (one holomorphic-symplectic structure of) the rigid c-map space of $\widetilde{\mathcal{M}}_s$. So could try to use the SUSY in $d=3$ to "explain" this holomorphy. Plea for help: why does $\mathcal{T}^*\widetilde{\mathcal{M}}_s$ occur in these two "different" contexts?

Another clue that the full hyperkähler structure is good for something: one of the $\mathbb{C}$ str has complex coordinates $\mathrm{Re}X^I + i\zeta^I$, $\mathrm{Re}F_I + i\tilde{\zeta}_I$. This funny combination is just what appears in OSV.

Another clue that reduction to $d=3$ is relevant for holomorphic anomaly: Heisenberg symmetry of the projective c-map spaces $\mathcal{M}$ and $\mathcal{S}$ — related to the above by hyperkähler quotient. So this $d=3$ picture has a chance of explaining why this wavefunction property is there.

And finally, there's the fact that the SUSY configurations looked simple in this hyperkähler language.

Heat equation gives (formal) geometry of topological string. What's the analogous statement for the higher-derivative couplings in $d = 3$? Don't know, but a clue from representation theory: consider cases where $\mathcal{M} = G/K$. Then whatever the equations are they should be G-invariant. i.e. we look for a representation of G. Recall holomorphic discrete series related to sections of line bundles over symmetric spaces, holomorphic modular forms. Quaternionic discrete series similarly realized in $H^1(\mathcal{Z}, \mathcal{O}(-k))$. A rep in the limit of discrete series is very closely related to holomorphic anomaly; this rep has also been constructed by geometric quantization, using hyperkähler geometry of $\mathcal{S}$. H^1 related to indefinite metric in c-map space.

In some cases there's a modular form out there waiting to be used (Gan); in principle should contain all Gromov-Witten invariants plus more stuff.