Sigma Models, NS-fluxes and Generalized Kähler Geometry

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Introduction

This talk will be a review of old results and a presentation of new ones, and will be based on work I did along many years with several collaborators, most prominently Ulf Lindström and Rikard von Unge, and more recently Maxim Zabzine and my student Itai Ryb. Recent relevant papers include [1, 2, 3].

The motivation from string theory of these ideas comes from the fact that in the NS-NS sector the fundamental string carries B-field charge. This is reflected in a direct coupling in the world-sheet string Lagrangian of the form

$$\mathcal{L} \sim (G + B)\partial\phi\bar{\partial}\phi\,, \tag{1}$$

and leads naturally to the study of two-dimensional σ-models of such a form. It turns out that when we combine this Lagrangian with $N = (2, 2)$ supersymmetry, we discover a lot of very interesting geometry; indeed, susy σ-models seem to be intent on teaching us very interesting mathematics.

The plan of the talk is as follows:

• I will start reviewing some old results of my paper of 1984 with Gates and Hull on bihermitian geometry in $N = (1,1)$ superspace [4].

• I will then explain a mixture of old and recent results on the $N = (2,2)$ superspace description of such geometries. This includes work with various collaborators as well as work of Sevrin and his collaborators.

• Next I will discuss recent work that we have done on isometries in such spaces, including current work in progress and lots of open questions.

• Finally, I will switch gears and describe how this fits into the framework of the Generalized (Kähler) Geometry of Hitchin, Gualtieri and others. We will see that the sigma model approach leads naturally to these geometric concepts, which look rather abstract otherwise. For this reason it is very convenient to approach the subject by starting from a Lagrangian and letting the mathematical structures emerge naturally.

Review of Gates-Hull-Roček

We start with an $N = (1,1)$ supersymmetric action in two dimensions, formulated in $N = (1,1)$ superspace; this is automatically which is invariant under the usual $N = (1,1)$ supersymmetry generators $q_\pm$

$$\delta\phi^i \sim \epsilon^+ q_+ \phi^i + \epsilon^- q_- \phi^i \,. \tag{2}$$

We are interested in finding additional supersymmetries, and we expect them to have the form

$$\delta\phi^i \sim \epsilon^+ J^i_{+j} D_+ \phi^j + \epsilon^- J^i_{-j} D_- \phi^j \,, \tag{3}$$

where $D_\pm$ are the left and right spinor derivatives of $N = (1,1)$ superspace that anticommute with $q_\pm$, and we have introduced two tensors $J^i_{\pm j} = J^i_{\pm j}(\phi^k)$. In my conventions, the signs $\pm$ denote the Lorentz charge of left- and right-moving spinors rather than the U(1) R-symmetry charge.

Requiring the supersymmetry algebra (3) to be closed, and to leave invariant the supersymmetric version of (1), implies that:

- The tensors $J_\pm$ are almost complex structures, *i.e.* $J_-^2 = J_+^2 = 1$, and they are integrable, *i.e.*, a coordinate system exists in which they are constant (not simultaneously, unless $[J_+, J_-] = 0$).
- The metric g is bihermitian, *i.e.* is hermitian with respect to both complex structures, so we have $g(J_\pm, J_\pm) = g$.

- The two-forms $w_\pm = gJ_\pm$ (the would be Kähler-forms)are not closed, $dw_\pm \neq 0$, so the geometry is not Kähler, and we have $H = dB \sim +J_+ dw_+ = -J_- dw_-$. In the case $J_- = J_+$, it follows that $H = dw_\pm = 0$ and the manifold becomes Kähler. So we obtained a particular deformation of the notion of a Kähler manifold.
- Using the three-form H and the metric g, one constructs two connections that preserve the tensors $J_\pm$, *i.e.,* $\nabla^\pm J_\pm = 0$.

Notice that we can read off the complex structures $J_\pm$ from the supersymmetry transformations. We use this repeatedly below.

$N = (2,2)$ superspace formulation of GHR

This was a brief summary of $N = (2,2)$ supersymmetry in $N = (1,1)$ superspace; for more tensors $J_\pm$'s, one can also get $N = (4,4)$ in $N = (1,1)$ language.

The supersymmetry algebra (3) closes only on-shell *except* when $[J_+, J_-] = 0$ (which includes the Kähler case). This suggests that for $[J_+, J_-] = 0$, a formulation using $N = (2,2)$ superspace exists; this was found already in [4]. The resulting space is called a 'Bihermitian Local Product' (BiLP) geometry. The name is chosen because when $[J_+, J_-] = 0$, we have $(J_- J_+)^2 = 1$, so the tangent space is the product of two subspaces with eigenvalues $+1$ and -1 under the local product structure $J_- J_+$. These subspaces do not necessarily have the same dimension.[1]

The way it works is worth reviewing: In $N = (2,2)$ superspace, we have left and right chiral and antichiral derivatives; since the conventions vary, here are mine:

	Left	*Right*
chiral	$\mathbb{D}_+$	$\mathbb{D}_-$
antichiral	$\bar{\mathbb{D}}_+$	$\bar{\mathbb{D}}_-$

($\pm$ indicates Lorentz charge, and the bar indicates $U(1)_R$ charge). Chiral superfields obey $\bar{\mathbb{D}}_\pm \Phi = 0$ (that is, they are (cc) or chiral-chiral; their conjugates are (aa)) whereas twisted chiral superfields obey $\bar{\mathbb{D}}_+ \chi = \mathbb{D}_- \chi = 0$ ((ca) or chiral-antichiral with (ac) conjugates). When we reduce to $N = (1,1)$ superspace, we keep the real part of $\mathbb{D}$ as the $N = (1,1)$ spinor derivative D, and explicitly expand in the imaginary part. The chirality and twisted chirality constraints, e.g., $0 = \bar{\mathbb{D}}\Phi = \frac{1}{2}(D + iQ)\Phi$, imply

$$Q_\pm \Phi = J D_\pm \Phi \ , \quad Q_\pm \chi = \pm J D_\pm \chi \quad , \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4}$$

[1] By the way, back then I asked my student Tom Buscher to study certain aspects of the beta functions for this case, and this led to his discovery of the Buscher rules and T-duality [5]. This is an example of what supersymmetric σ-models can lead to in the hands of a smart student.

Thus $J_+ = J_- = J$ for chiral superfields and $J_+ = -J_- = J$ for twisted chiral superfields; in both cases (and in the more interesting mixed case) $[J_+, J_-] = 0$. Note that if all the the fields are twisted chiral, then we by a change of convention it is clear that the geometry is Kähler. The interesting case arises when we have both chiral and twisted-chiral fields.

Semichiral superfields

A few years later, in 1987, Ulf Lindström, Tom Buscher, and I discovered semichiral superfields [6]. These are superfields that come as a quartet $(c,\cdot)$,$(a,\cdot)$,$(\cdot,c)$,$(\cdot,a)$:

$$\bar{\mathbb{D}}_+ X_L = \bar{\mathbb{D}}_- X_R = \mathbb{D}_+ \bar{X}_L = \mathbb{D}_- \bar{X}_R = 0$$

We realized immediately that reducing to $N = (1,1)$ superspace now gives:

$$Q_+ X_L = J D_+ X_L \ , \quad Q_+ X_R = \Psi_{R+} \ , \quad (+, L, R \leftrightarrow -, R, L)$$

where Ψ is an auxiliary $N = (1,1)$ spinor superfield and by an abuse of notation $\{X_{L,R}\}$ now stands for the pair $\{X_{L,R}, \bar{X}_{L,R}\}$. To find the complex structure, one needs to solve for the auxiliary spinor Ψ. Unlike the case we considered above, now the form of the complex structures $J_\pm$ *do* depend on the form of the action, and one finds that the resulting $J_\pm$ do *not* commute.

Not much happened with these models for almost a decade, until in 1996, when Sevrin and Troost studied them in [7]. They found the explicit coordinate system that diagonalizes the complicated complex structures (such coordinates must exist, since the complex structures are integrable). They speculated that chiral, twisted chiral, and semichiral multiplets were sufficient to describe all $N = (2,2)$ nonlinear σ-models, but were unable to prove this was the case. Their conjecture was finally proved in [1], and we now discuss the ideas that enter in the proof.

The general $N = (2,2)$ Superspace Lagrangian

The general Lagrangian is the most naive thing that you would guess: a generalized Kähler potential that is simply a function (locally) of all the types of superfields we have seen:

$$K(X_L, \bar{X}_L, X_R, \bar{X}_R, \Phi, \bar{\Phi}, \chi, \bar{\chi}) \tag{5}$$

This function is not globally defined–it can be shifted by generalized Kähler transformations. These follow immediately from the superspace measure, which can be expressed as $\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-$. We can therefore change the potential (5) by any term which is annihilated by this measure, and the most general transformation of this type is

$$\Delta K = f_L(X_L, \Phi, \chi) + f_R(X_R, \Phi, \bar{\chi}) + c.c. \ ,$$

where $\bar{\mathbb{D}}_+ f_L = \bar{\mathbb{D}}_- f_R = 0$. These transformations of the potential turn out to be very different from the standard Kähler case, because the transformation functions f_L and f_R themselves are ambiguous. To see this, note that if we change

$$\Delta f_L = g(\Phi) + h(\chi) \ , \quad \Delta f_R = -g(\Phi) - \bar{h}(\bar{\chi}) \, , \tag{6}$$

then ΔK remains invariant. As we go from one open set to another, we expect that we need ΔK shifts to make K nonsingular (by analogy with Kähler transformations). However, in triple overlaps, we may find $\Delta f_{L,R}$ terms, so K is not just a section as in the Kähler case, but we really have to use gerbes. I am not an expert on these, but we are currently exploring this aspect, which I believe has not been noted before.

Another important point is that, much like in the Kähler case, when we have multiple complex structures for the left and the right movers, this leads to generalized hyper-Kähler models.

Question: Are there explicit examples of these geometries for compact spaces?
Answer: If we are interested in concrete examples of these Lagrangians, the best known case with a compact target space is the Hopf fibration of $SU(2) \times U(1)$, which can be described with a single chiral and a single twisted chiral multiplet. There are examples involving other group manifolds. The group $SU(3)$ involves one chiral, one twisted chiral and one quartet of semichirals. There are other compact generalized Kähler manifolds which have been constructed in the mathematics literature, but whose explicit realization in terms of superfields is not known yet.

Question: Are these non-linear sigma models conformal invariant?
Answer: First of all if you are in the $N = (4, 4)$ case you are guaranteed to have conformal invariance at the quantum level. This happens for example in the $SU(2) \times U(1)$ case. Otherwise you have to look at the beta functions, and generalized Monge-Ampere equations show up. This was first studied by Grisaru and collaborators [8], and recently Halmagyi and Tomasiello found a nice way of rewriting and understanding these equations [9].

Diagonalizing $J_\pm$

We will now see how to find the right coordinates in these geoemtries. This is not just an academic exercise, since, as we will see, we will gain a deep insight into the meaning of the potential K in (5).

Returning to the issue of Ψ, we reduce to $N = (1, 1)$ superspace by rewriting the measure as $D_+ D_- Q_+ Q_-$, and pushing $Q_\pm$ into the Lagrangian; the $\Psi_{R+} = Q_+ X_R$ field equation comes from integrating out its conjugate variable, $\Psi_{L-} = Q_- X_L$; this is easy to calculate:

$$Q_+ \left[(Q_- X_L) \frac{\partial K}{\partial X_L} \right] = -(Q_- X_L) Q_+ \left(\frac{\partial K}{\partial X_L} \right) + (J D_+ Q_- X_L) \frac{\partial K}{\partial X_L} \ , \tag{7}$$

which, after integration by parts, leads to the equation of motion

$$Q_+ \left(\frac{\partial K}{\partial X_L} \right) = J D_+ \left(\frac{\partial K}{\partial X_L} \right) \ . \tag{8}$$

Comparing with (4), this clearly means that the coordinates $X_L, Y_L \equiv \frac{\partial K}{\partial X_L}$ diagonalize J_+; similarly, the coordinates $X_R, Y_R \equiv \frac{\partial K}{\partial X_R}$ diagonalize J_-.

This is a beautiful example of how σ-models in superspace "know" profound things about the geometry. We immediately recognize this situation as arising in Poisson geometry: $K(X_L, X_R)$ is the generating function of a canonical transformation from the coordinates X_L, Y_L to the coordinates X_R, Y_R. However, we need to identify the appropriate Poisson structure. To do this, we consider the geometric meaning of the different superfields that we have introduced.

Geometry of the superfields

We would like now to uncover the geometrical significance of the different superfields that we have introduced. From the definition of the different superfields, we have that

$$\{d\Phi\} = \ker(J_+ - J_-) \tag{9}$$

$$\{d\chi\} = \ker(J_+ + J_-) \tag{10}$$

$$\{dX_{L,R}\} = \mathrm{coker}\Big((J_+ - J_-)(J_+ + J_-) \Big) = \mathrm{coker}([J_+, J_-]) \tag{11}$$

Since the cotangent space is a direct sum of the rhs of the above three equations, we see that our three types of superfields cover all the directions for a given pair of complex structures $J_\pm$, and the integrability conditions show that the superfields make sense as coordinates.

There are also two remarkable facts which I will mention without proof:

• The first is that both $\pi_\pm = (J_+ \pm J_-) g^{-1}$ and $\sigma = [J_+, J_-] g^{-1}$ are Poisson structures! That is, they define a good Poisson bracket because they obey the Jacobi identities. When any of them is invertible (e.g., π_+ in the Kähler case), their inverse is a symplectic (closed and nondegenerate). This is a nontrivial calculation using the fact that the covariant derivatives with torsion preserve $J_\pm$.

• The second fact is that using the hermiticity of g, one discovers that the two-form σ is the real part of a holomorphic $(2,0)$ Poisson structure with respect to *both* $J_\pm$. In other words, it has only $(2,0)$ and $(0,2)$ components, but no $(1,1)$ piece.

This last point is what we were looking for, because it tells us what is it that the canonical transformation generated by K is preserving: σ! The X and Y coordinates are nothing but Darboux coordinates for σ:

$$\sigma = \mathrm{Re}\left(\frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Y}\right) \tag{12}$$

both for $X = X_L$, $Y = Y_L$ and $X = X_R$, $Y = Y_R$

This gives a beautiful interpretation of the $N = (2, 2)$ superspace Lagrangian: it is the generating function of the canonical transformation that preserves σ and takes us between coordinates that diagonalize J_+ and coordinates that diagonalize J_-. A caveat is that this does not determine the Φ, χ dependence; but that is more data that is needed for the geometry, just as in the case without $X_{L,R}$.

An interesting point is that whenever we perform a canonical transformation we use half of the old coordinates and half of the new coordinates. The breaking up of the left coordinates into X_L and Y_L is arbitrary, and similarly for the right coordinates. The arbitrariness when selecting the $\{X_{L,R}\}$ from the set $\{X_{L,R}, Y_{L,R}\}$ is a choice of *polarization*. One can ask: is this choice intrinsic to the geometry? The answer is no. It is very easy to change polarization by changing the generalized Kähler potential to

$$\tilde{K} = K - XY \tag{13}$$

Note that the change of polarization is very natural in the sigma-model language–it looks like a kind of duality for the semichiral fields. This follows because the semichiral chiral condition is solved by, e.g., $\bar{\mathbb{D}}_+ X_L = 0 \Rightarrow X_L = \bar{\mathbb{D}}_+ \Psi_-$, where Ψ_- is some unconstrained spinor; hence, when used as a Lagrange multiplier, it imposes the semichiral condition again. (A similar analysis shows that a chiral superfield imposes a *linear* constraint, not a chiral one).

The changes of polarization become very interestingly when we think of the global structure of the manifold, because we might have to change polarizations as we go from patch to patch, and nothing is known about this point.

This interpretation of K as the generator of canonical transformations is the crucial element used to show that all generalized Kähler geometries can be described by these superfields [1].

Isometries and Topological Models

This is an area on which we have just written the paper [3], and is being actively pursued. My prejudice is that to understand the isometries one look at them through the sigma model.

The isometries can be gauged, and notions of generalized T-duality and generalized mirror symmetry arise.

The idea is that there are four kinds kinds of isometries, classified according to the properties of the vector k generating the isometry. The latter can be chiral, twisted chiral, mixed or pure semichiral, as we see in the table below.

Chiral	$k = \pi_+\xi$
Twisted chiral	$k = \pi_-\xi$
Mixed	$k = general$
Pure semi	$k = \sigma\xi$

Here ξ is some one-form, and $\pi_\pm, \sigma$ are the Poisson structures described above. We are studying the moment maps associated to this isometries and we have some results on this already. One interesting question that arises is the following. In the case of abelian isometries, we know that a polarization can be chosen which is preserved by the isometries, but it is not known if this is also possible for non-abelian isometries. This is an important open problem.

Another subject under study is the topological twist of these models. In general when one does, *e.g.*, an A-twist for chiral fields, it is a B-twist for twisted chiral fields, but it should be more complicated for semichiral fields.

Question: What happens when you gauge an isometry?
Answer: We have studied some examples with $U(1)$ isometries. The resulting metric can be singular, but that does not mean that the gauged theory is singular, as we know from the example of the gauged $SU(2)/U(1)$ WZW models.

Generalized Geometry

A few years ago, Hitchin proposed an idea to generalize the notion of complex geometry. We know that a complex geometry has a natural action and both the tangent and the cotangent space. Suppose we look at something that acts on the direct sum. This leads to define an object called Generalized Complex Structures on $T \oplus T^*$. This is a very useful object because it contains what would be a Ramond-Ramond flux (of course we do not know how to write a sigma model for that, but we can write supergravity models). There is a very nice review written by my collaborator Maxim Zabzine [2], where it is shown that these Generalized Complex Geometries can be naturally derived from a Hamiltonian superspace approach.

As it turns out, all the structures we have discussed fit nicely in the context of Generalized Complex Geometry. In the usual Kähler case, we can write several complex structures

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}$$

$$\mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

and one can show that there is a pair of $\mathcal{J}_1$ and $\mathcal{J}_2$ such that the metric satisfies

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = -\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_2\mathcal{J}_1$$

In the Generalized Kähler case which we have been discussing, we have

$$\mathcal{J}_1 = \frac{1}{2}\begin{pmatrix} J_+ + J_- & \omega_+^{-1} - \omega_-^{-1} \\ -\omega_+ + \omega_- & -J_+^t - J_-^t \end{pmatrix}$$

$$\mathcal{J}_2 = \frac{1}{2}\begin{pmatrix} J_+ - J_- & -\omega_+^{-1} - \omega_-^{-1} \\ \omega_+ + \omega_- & -J_+^t + J_-^t \end{pmatrix}$$

so when $J_- = J_+$, then $\mathcal{J}_1$ and $\mathcal{J}_2$ are reduced to the previous case. This pair of complex structures also satisfies

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = -\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_2\mathcal{J}_1$$

There is a general characterization of the Generalized Complex Geometries, precisely in terms of two complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$, which should satisfy the above properties, along with some integrability conditions. Now, Gualtieri in his PhD thesis [10] proved that having $\mathcal{J}_1$ and $\mathcal{J}_2$ which satisfy these properties is entirely equivalent to the bihermitian complex geometry that we had.

The integrability conditions are very similar to those of the ordinary complex structures, but the ordinary Lie bracket is replaced by new object called the "Twisted Courant Bracket". This new bracket does not obey in general the Jacobi identity, but it does obey it inside some interesting subspaces, as for example spaces which are holomorphic w.r.t. the generalized complex structures.

There is a supergravity formulation of these ideas, where Clifford algebras appear along with pure spinors, which arise in generalized Calabi-Yau geometries. This lies outside of the generalized Kähler framework that we have been discussing, and corresponds to turning on RR fluxes. Not surprisingly, no σ-model interpretation is known of them as yet.

Added note: RR fluxes can be treated on the world sheet in the formulations due to Berkovits. Understanding generalized geometry in this framework would be very interesting.

References

[1] U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, "Generalized Kaehler manifolds and off-shell supersymmetry," Commun. Math. Phys. **269**, 833 (2007) [arXiv:hep-th/0512164].

[2] M. Zabzine, "Lectures on generalized complex geometry and supersymmetry," arXiv:hep-th/0605148.

[3] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, "T-duality and Generalized Kahler Geometry," arXiv:0707.1696 [hep-th].

[4] S. J. Gates, C. M. Hull and M. Rocek, "Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **248**, 157 (1984).

[5] T. H. Buscher, "A Symmetry of the String Background Field Equations," Phys. Lett. B **194**, 59 (1987); "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models," Phys. Lett. B **201**, 466 (1988).

[6] T. Buscher, U. Lindstrom and M. Rocek, "NEW SUPERSYMMETRIC sigma MODELS WITH WESS-ZUMINO TERMS," Phys. Lett. B **202**, 94 (1988).

[7] A. Sevrin and J. Troost, "Off-shell formulation of N = 2 non-linear sigma-models," Nucl. Phys. B **492**, 623 (1997) [arXiv:hep-th/9610102].

[8] M. T. Grisaru, M. Massar, A. Sevrin and J. Troost, "The quantum geometry of N = (2,2) non-linear sigma-models," Phys. Lett. B **412**, 53 (1997) [arXiv:hep-th/9706218].

[9] N. Halmagyi and A. Tomasiello, "Generalized Kaehler Potentials from Supergravity," arXiv:0708.1032 [hep-th].

[10] M. Gualtieri, "Generalized complex geometry," arXiv:math/0401221.