Instanton corrected hypermultiplet moduli spaces

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August 7th, 2007

Talk at the Fifth Simons Workshop in Mathematics and Physics
Stony Brook University, July 30 - August 31, 2007

We review recent progress in understanding non-perturbative instanton corrections to the hypermultiplet moduli space in type II string compactifications on Calabi-Yau threefolds.

Instanton corrections recently played a prominent role both in the context of string cosmology (KKLT models) and particle physics model building. A setup where one might hope to understand these corrections using string dualities and supersymmetry constraints on the low energy effective action (LEEA) is type II string theory compactified on a Calabi-Yau threefold (CY_3). In this case the LEEA is a $\mathcal{N} = 2$, $d = 4$ supergravity action which generically receives quantum corrections from the world sheet conformal field theory (α'-corrections), from higher genus world sheets (perturbative g_s-corrections), and from instanton corrections due to Euclidean branes wrapping supersymmetric cycles of the CY_3 (non-perturbative g_s-corrections). While the α'-corrections to the classical LEEA are well understood, the non-perturbative g_s corrections [1] still pose many open questions.

Besides the supergravity gravity multiplet these LEEA contain n_V vector- and n_H hypermultiplets. Supersymmetry implies that the total moduli space $\mathcal{M}$ of these theories factorizes into a local product $\mathcal{M} = \mathcal{M}_{\mathrm{VM}} \otimes \mathcal{M}_{\mathrm{HM}}$ where $\mathcal{M}_{\mathrm{VM}}$ and $\mathcal{M}_{\mathrm{HM}}$ are parameterized by the scalars of the VM and HM, respectively. This factorization has the profound consequence that only those subsectors which contain the dilaton (volume modulus) receive g_s (α') corrections. Furthermore, supersymmetry dictates that $\mathcal{M}_{\mathrm{VM}}$ be a special Kähler manifold, and $\mathcal{M}_{\mathrm{HM}}$ a quaternion-Kähler manifold.

For both IIA and IIB compactifications the dilaton sits in a hypermultiplet. Thus determining the g_s corrections to the LEEA requires understanding $\mathcal{M}_{\mathrm{HM}}$. In this context it turns out to be useful that in the presence of $n_H + 1$ commuting shift symmetries[1], $\mathcal{M}_{\rm HM}$ can equivalently be described by rigidly superconformal tensor multiplet (TM) actions [2]

$$\mathcal{L} = \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \, H(\eta^I) \, . \tag{1}$$

Here $\eta^I = v^I/\zeta + x^I - \zeta \bar{v}^I$ denotes a $\mathcal{N} = 2$ tensor superfield, $H(\eta^I)$ is a weakly homogeneous function of degree 1 (i.e., logarithmic terms are allowed) with no explicit ζ-dependence, and $\mathcal{C}$ is an arbitrary contour in the complex ζ-plane. Taking $\mathcal{L}$ as a function of the scalar fields $v^I, \bar{v}^I, x^I$, one can define a "tensor potential" [3]

$$\chi = -\mathcal{L} + x^I \, \frac{\partial \mathcal{L}}{\partial x^I} \, . \tag{2}$$

Coupling the rigidly superconformal TM Lagrangian to conformal supergravity and carrying out the superconformal quotient both $\mathcal{L}$ and χ can be used to determine $\mathcal{M}_{\rm HM}$ [3]. Working with χ thereby has the virtue that (discrete) symmetries of the supergravity theory, as e.g., the SL(2, $\mathbb{Z}$) invariance of the type IIB string, are reflected by the invariance of χ.

Building on the off-shell description [4] of the c-map [5] the perturbatively corrected hypermultiplet moduli space has been determined in [6]

$$\mathcal{L}(v, \bar{v}, x) = \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \, \left[\frac{F(\eta^\Lambda)}{\eta^0} \pm \frac{i \, \chi_E}{(2\pi)^3} \, 2\zeta(2) \, \eta^0 \, \ln(\eta^0) \right] \, . \tag{3}$$

Here $F(\eta^\Lambda)$ is the holomorphic prepotential underlying the dual vector multiplet geometry, η^0 is an additional TM acting as a conformal compensator and the contour encloses the branch cut between $\zeta = 0$ and one of the zeros of $\zeta\eta^0$. The second term encodes the universal one-loop correction with the upper (lower) sign correspond to type IIB (type IIA) strings, respectively, with χ_E being the Euler number of the CY_3.

Substituting (3) into (2) yields the tensor potential underlying the perturbatively corrected $\mathcal{M}_{\rm HM}$. For type IIB the corresponding expression naturally splits into a classical part, $\chi_{\rm cl}$, perturbative α' and g_s corrections, $\chi_{\rm pt}$, and non-perturbative world-sheet instanton contributions, $\chi_{\rm ws}$:

$$\begin{aligned} \chi_{\rm cl} &= 4 \, r^0 \, \tau_2^2 \, \frac{1}{3!} \, \kappa_{abc} \, t^a \, t^b \, t^c \, , \qquad \chi_{\rm pt} = \frac{1}{(2\pi)^3} \, r^0 \, \chi_E \, \left[\zeta(3) \tau_2^2 + 2\zeta(2) \right] \, , \\ \chi_{\rm ws} &= - \, \frac{r^0 \tau_2^2}{(2\pi)^3} \sum_{k_a} n_{k_a} \left[\mathrm{Li}_3(\mathrm{e}^{2\pi i k_a z^a}) + 2\pi k_a t^a \mathrm{Li}_2(\mathrm{e}^{2\pi i k_a z^a}) + c.c. \right] \, . \end{aligned} \tag{4}$$

These are determined by the triple intersection numbers κ_{abc}, the Euler number χ_E and the Gopakumar-Vafa invariants n_{k_a} of the CY_3. The one-loop correction gives rise to the second term in $\chi_{\rm pt}$ which is suppressed by $\tau_2^{-2} = g_s^2$ compared to the three-level terms.

[1] In the case of string compactifications these shift symmetries naturally arise from the reduction of the p-forms in the ten-dimensional type II supergravity actions and are preserved in the presence of D(-1) and D1 instanton corrections in type IIB or A-type membrane instanton corrections in type IIA.

In order to determine the D(-1) and D1-brane instanton corrections to the IIB LEEA we make use of the non-perturbative SL(2, $\mathbb{Z}$) invariance of the type IIB string. The modular transformation properties of the four-dimensional scalars can be found by tracing the transformation properties of the ten-dimensional IIB supergravity fields through the dimensional reduction [7]

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \quad t^a \mapsto t^a|c\tau + d|\,, \quad b^a \mapsto d\,b^a + c\,c^a\,, \quad c^a \mapsto b\,b^a + a\,c^a\,. \tag{5}$$

The modular transformation of the conformal compensator $r^0 \mapsto r^0|c\tau + d|$ is determined by lifting the action of SL(2, $\mathbb{Z}$) to superspace [8].

Applying the SL(2, $\mathbb{Z}$) transformations (5) to (4) one finds that $\chi_{\rm cl}$ is modular invariant while the α' and g_s corrections break the SL(2, $\mathbb{Z}$) invariance. Restoring SL(2, $\mathbb{Z}$) invariance by a modular completion of $\chi_{\rm pt}$ and $\chi_{\rm ws}$ gives

$$\begin{aligned} \chi^{\rm IIB}_{(-1)} &= \frac{r^0 \tau_2^{1/2}}{2(2\pi)^3}\,\chi_E \sum_{m,n}{}' \frac{\tau_2^{3/2}}{|m\tau + n|^3}\,, \\ \chi^{\rm IIB}_{(1)} &= -\frac{r^0 \tau_2^{1/2}}{(2\pi)^3} \sum_{k_a} n_{k_a} \sum_{m,n}{}' \frac{\tau_2^{3/2}}{|m\tau + n|^3} \left(1 + 2\pi|m\tau + n|\,k_a t^a\right) \mathrm{e}^{-S_{m,n}}\,, \end{aligned} \tag{6}$$

and determines the corrections of the D(-1) and D(1) instanton corrections, respectively. Here

$$S_{m,n} = 2\pi k_a \big(|m\tau + n|\,t^a - im\,c^a - in\,b^a\big)\,, \tag{7}$$

is the instanton action of a (m, n)-string with m units of D(1) and n units of fundamental string charge.

By the SYZ construction of mirror symmetry [9], the instanton contributions (6) are mirror to a certain class of membrane instantons, i.e., Euclidean D2-branes, wrapping so-called A-cycles of the mirror CY_3. The corresponding mirror map in the hypermultiplet sector is given by [10]

$$\phi_{\rm IIA} = \phi_{\rm IIB}\,, \quad A^1 = \tau_1\,, \quad A^a = -(c^a - \tau_1 b^a)\,, \quad z^a_{\rm IIA} = z^a_{\rm IIB}\,. \tag{8}$$

Furthermore, implementing mirror symmetry requires some rearrangements of the various contributions to the type IIB tensor potential which is illustrated in table 1. The resulting corrections due to D2-brane instantons was then given in [10]

$$\chi^{\rm IIA}_{\rm A-D2} = -\frac{r^0\tau_2}{2\pi^2} \sum_{k_\Lambda} n_{k_\Lambda} \sum_{m \neq 0} \frac{|k_\Lambda z^\Lambda|}{|m|}\, K_1\big(2\pi\tau_2\,|m\,k_\Lambda z^\Lambda|\big)\, \mathrm{e}^{-2\pi i m k_\Lambda A^\Lambda}\,. \tag{9}$$

Here, $k_\Lambda = \big(n, k_a\big)$, $z^\Lambda = \big(1, z^a\big)$, $A^\Lambda = \big(A^1, A^a\big)$, and the sum over k_a now includes the zero-vector $k_a = 0$, but $k_\Lambda = 0$ is excluded. The type IIA "instanton numbers" are

$$n_{(n,k_a=0)} = \frac{1}{2}\chi_E(X)\,, \qquad n_{(n,k_a)} = n_{k_a} \text{ as in type IIB}\,, \tag{10}$$

with $\chi_E(X)$ the Euler number of the CY_3 and n_{k_a} the Gopakumar-Vafa invariants of the *mirror* CY_3. This indicates a deep connection between the properties of holomorphic two-cycles (counted by the Gopakumar-Vafa invariants) and special Lagrangian three-cycles of the mirror manifold.

IIB HM	SL(2, $\mathbb{Z}$)-invariant	IIA HM	composed from IIB terms
$\chi_{\rm cl}$	$= \chi_{\rm cl}$	$\chi_{\rm tree}$	$= \chi_{\rm cl} + \chi_{\rm ws-pert} + \chi_{\rm ws-inst}$
$\chi_{(-1)}$	$= \chi_{\rm ws-pert} + \chi_{\rm loop} + \chi_{D(-1)}$	$\chi_{\rm loop}$	$= \chi_{\rm loop}$
$\chi_{(1)}$	$= \chi_{\rm ws-inst} + \chi_{\rm D1}$	$\chi_{\rm A-D2}$	$= \chi_{\rm D(-1)} + \chi_{\rm D1}$

Table 1: Rearrangement of contributions of the type IIB tensor potential under mirror symmetry.

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