Holomorphic Anomaly in Open Topological String

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August 17th, 2007

Talk at the Fifth Simons Workshop in Mathematics and Physics
Stony Brook University, July 30 - August 31, 2007

In my talk I reported on progress in mirror symmetry for open strings and topological strings on compact Calabi-Yau threefolds with D-branes in the background. The emphasis is on *compact* Calabi-Yau threefolds for which most of the methods developed for the exact solution of local geometries (matrix models, topological vertex, etc.) are not applicable. On the other hand, one of the main motivations is indeed to compare the results with the many lessons that have been learned from the study of those non-compact models, including in particular dualities between open and closed strings.

The results up to now appear in the papers [2, 3, 4, 5], and the story can be summarized as follows.

The idea that mirror symmetry could yield exact results for open strings also on compact Calabi-Yau manifolds was originally formulated in a talk I gave at Strings '04 in Paris [1]. It had been suggested in previous studies of D-branes in Gepner models and associated geometries that the Lagrangian D-branes defined as the real submanifolds of certain Calabi-Yau hypersurfaces can be identified with certain boundary states in the rational conformal field theories. Via mirror symmetry and the correspondence with Landau-Ginzburg models, one obtains natural extensions to entire families of Calabi-Yau manifolds with D-branes on top, thus providing a natural setting for mirror symmetry with open strings. In particular, it was proposed in [1] that a certain puzzle in the spectrum of open strings ending on the real quintic could have an explanation in Floer homology, once worldsheet disk instantons are properly counted and accounted for. In fact, it was suggested that the knowledge of the mirror B-model should be sufficient for the exact solution of this problem.

This promise was fulfilled in [2], however with a significant departure from the original strategy. The generating function of open Gromov-Witten invariants ending on the real quintic was identified with the tension of a domainwall between certain $\mathcal{N} = 1$ vacua on the worldvolume of a D-brane wrapped on the real quintic. This domainwall tension was computed as the solution of a proposed differential equation which is an extension of the standard Picard-Fuchs equation governing closed string mirror symmetry on the quintic. While the qualitative expectations expressed in [1] could be reproduced using the results of [2], there are to this date significant gaps in explaining the exact quantitative connection with Floer homology.

A rigorous mathematical proof of the enumeration of disks predicted in [2] was given in [3]. The definition of open Gromov-Witten invariants in this situation is due to Jake Solomon. The method of proof is an extension of that used to establish the classical mirror theorems of Kontsevich, Givental, Lian-Liu-Yau, and others. It uses localization on the moduli space of maps to $\mathbb{P}^4$ with respect to the two-dimensional torus that is left invariant by the anti-holomorphic involution (complex conjugation).

The B-model origin of the differential equation of [2] was understood in Hodge theoretic terms in the work [4]. One can argue that in generic D-brane setups on Calabi-Yau manifolds, domainwall tensions (i.e., superpotential differences) are always given by integrating the holomorphic three-form over certain three-chains. While most easily justified for branes wrapping curves, this statement is expected to be valid for any B-brane (i.e., any object in the derived category of coherent sheaves of the Calabi-Yau manifold). The pleasant aspect is that such chain integrals naturally connect the open B-model with classical Hodge theory. This connection involves the Abel-Jacobi map to the intermediate Jacobian of the Calabi-Yau manifold and identifies the invariant holomorphic data of a B-brane as a Poincaré normal function in the sense of Griffiths. In [4], the three-chain relevant for the mirror of the real quintic is identified by following arguments of mirror symmetry and homological Calabi-Yau/Landau-Ginzburg correspondence. The differential equation then follows by an explicit analytic computation on the mirror quintic.

The combination of the results of [2], [3], and [4] provides a thorough understanding of the tree-level data associated with open strings ending on the real quintic, and strongly suggests that a similar structure can be assumed for a generic D-brane setup on a compact Calabi-Yau manifold. Based on this foundation, the computations were extended to loop amplitudes in the open+closed topological string in [5]. The main result of that work is an extension of the standard BCOV holomorphic anomaly equation [6]. The extended equation can be solved in the example of the real quintic, giving several valuable insights into properties of open topological string amplitudes on compact Calabi-Yau manifolds.

Four basic ingredients, called “facts” in [5], enter the extended holomorphic anomaly equation.

(F1) Slogan: “Ignore open string moduli”. More detailed version: For generic values of the closed string moduli, the topological amplitudes $\mathcal{F}^{(g,h)}$ do not depend on any continuous open string moduli.

(F2) The total topological charge of the D-brane configuration should vanish. This is a kind of tadpole cancellation condition for the topological string, and can probably also be achieved using orientifolds.

(F3) Recall that the basic closed string tree-level data entering the holomorphic anomaly equation is the three-point function on the sphere, often called the Yukawa coupling. The open counterpart of the Yukawa coupling is the two-point function on the disk. It can be identified mathematically as the Griffiths infinitesimal invariant of a normal function known from Hodge theory. This identification involves fixing a certain ambiguity in a particular (non-holomorphic!) way, which to my knowledge is new.

(F4) Degenerations of the string worldsheet with open strings in intermediate channels do not contribute to the RHS of the extended holomorphic anomaly equation. Although this statement might eventually follow from the previous ones, it is at this point probably the most controversial one.

I expect the extended holomorphic anomaly equation to be valid in fairly wide generality and can envisage several further applications to certain questions about the open topological string, open/closed string duality, background independence, and BPS state counting.

Acknowledgments. This work was supported in part by the Roger Dashen Membership at the Institute for Advanced Study, and by the NSF under grant number PHY-0503584.

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