Balanced metrics and physics

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A prototypical topic at the interface of geometry and theoretical physics is the study of quantum mechanics in curved space, *i.e.* on a Riemannian manifold M [1, 2]. Many results in this area are of great interest both to physicists and to mathematicians, with some examples being the DeWitt-Seeley-Gilkey short time expansion of the heat kernel, and the relation between supersymmetric quantum mechanics and the Atiyah-Singer index theorem [3, 4, 5].

A more recent result which, although not well known by physicists, we feel also belongs in this category, is the expansion for the Bergman kernel on a Kähler manifold developed by Tian, Yau, Zelditch, Lu and Caitlin [6, 7, 8, 9]. It applies to Kähler quantization and gives an asymptotic expansion around the semiclassical limit. This has many uses in mathematics; some references are [10, 11, 12].

The goal of our work [13] is to provide a physics derivation of the asymptotic expansion of the Bergman kernel using path integrals, and explain various possible applications of this result. In physics terms, perhaps the simplest way to define the Bergman kernel is in the context of quantum mechanics of a particle in a magnetic field, in which it is the projector on the lowest Landau level. It is not hard to see that the limit of large magnetic field is semiclassical, so that one can get an expansion in the inverse magnetic field strength using standard perturbative methods. Our basic result is to rederive the Tian-Yau-Zelditch *et al* expansion as the large time limit of the perturbative expansion for the quantum mechanical path integral. We also generalize it to $N = 1$ and $N = 2$ supersymmetric quantum mechanics.

Let us state the basic result for (nonsupersymmetric) quantum mechanics. We consider a compact Kähler manifold M, and a particle in a covariantly constant magnetic field,

$$\nabla^j F_{jk} = 0.$$

One can show that, just as for a constant magnetic field in flat space, in this situation the spectrum is highly degenerate, splitting into "Landau levels." Let the lowest Landau level (LLL or ground state) be N-fold degenerate with a basis of orthonormal wave functions $\psi_I(x)$, then we define the projector on the LLL as

$$\rho(x, x') = \sum_{I=1}^{N} \psi_I^*(x')\psi_I(x).$$

We could also regard this as a density matrix describing a mixed state in which each ground state appears with equal weight, describing the the zero temperature state of maximum entropy.

We then consider scaling up the magnetic field by a parameter k, as $F \to kF$. (Note that on a compact Kähler manifold, F must satisfy a Dirac quantization condition, see below). In the large k limit, the diagonal term then satisfies

$$\rho(z, \bar{z}) \sim k^n \left(1 + \frac{\hbar}{2k}R + \frac{\hbar^2}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \mathcal{O}((\hbar/k)^3)\right) \qquad (1)$$

as an asymptotic expansion [9].

In some ways this expansion is similar to the well known short time expansion of the heat kernel, but note that it is a long time expansion, because it projects on the ground states. Unlike other analogous results for ground states, it does not require supersymmetry, either for its definition or computation. Of course, similar results can be obtained for supersymmetric theories, our point is that that they do not depend on supersymmetry.

Our original interest in this type of result came from the study of balanced metrics in [12], and a conjecture about their relevance for black holes string theory stated in [14].

A famous problem in quantum gravity is to derive the Bekenstein-Hawking entropy of a black hole by counting its microstates. In string theory, this was first done by Strominger and Vafa [15], who counted the microstates of a BPS bound state of Dirichlet branes with the same charge as the black hole, and then argued that the number of states was invariant under varying the string coupling, turning the bound state into a black hole.

This line of argument has been the basis for a great deal of work, generalizing the result to other systems and away from the semiclassical limit. One important element in such results is the claim that entropies and numbers of microstates are independent of the moduli of the background. An argument to this effect is provided by the attractor mechanism [16]. This was originally stated for BPS black holes in type II strings compactified on a Calabi-Yau manifold X, but the idea is probably more general (see [17] for a recent discussion). The attractor mechanism is based on the observation that the equations of motion for the moduli in a black hole background can be written in the form of gradient flow equations for the area of a surface of fixed radius as a function of the moduli. This flow approaches an attracting fixed point at the event horizon, with a definite value of the moduli and area. Thus, these values are insensitive to small variations of the initial conditions. By the Bekenstein-Hawking relation, this implies that the entropy is invariant under such variations.

It is plausible that other properties of the black hole microsystem share this type of universal behavior. For example, we might conjecture that not only the Kähler moduli of the Calabi-Yau metric near a black hole take universal values, but that the entire metric is universal, determined only by the charge and structure of the black hole and independent of the asymptotic moduli.

What would this mean? In classical supergravity, of course the metric is determined by the Einstein equation, reducing to the Ricci flatness condition for the source-free case. Thus the stronger conjecture is quite reasonable and indeed follows directly from the validity of supergravity. On the other hand, for a finite charge black hole preserving eight or fewer supercharges, one knows that these equations will get string theoretic (α' and g_s) corrections. Thus, while such a stronger conjecture still appears reasonable, it is not *a priori* clear either what the attractor CY metric should be, or what equations determine it.

Now, one reason the general question of finding exact metrics or even precisely defining corrected supergravity equations is hard, is that the metric and equations can be changed by field redefinitions, with no obvious preferred definition. For example, the metric g_{ij} could be redefined as $g_{ij} \to g_{ij} + \alpha R_{ij} + \beta R_{ij}^2 + \ldots$. Unless we postulate an observable which singles out one definition, say measurements done by a point-like observer who moves on geodesics, there is no way to say which definition is right. This problem shows up in computing α' corrections in the sigma model as the familiar question of renormalization scheme dependence; in general there is no preferred scheme. We must first answer this question, to give meaning to the "CY attractor metric."

A nice way to answer this question is to introduce a probe brane, say a D0-brane, and study its world-volume theory. The kinetic term for its transverse coordinates is observable, and defines a unambiguous metric on the target space, including any α' corrections. While one can still make field redefinitions in the action, now these are just coordinate transformations.

To make this argument straightforward, one requires that the mass (or tension) of the probe be larger than any other quantities under discussion, so that the action can be treated classically, and the metric read off from simple measurements.[1] For example, this is true for D0-branes in weakly coupled string theory, as their mass goes as $1/g_s$. One can then (in principle) define any term in the g_s expansion this way.

Both on general grounds [19] and in examples [20], the moduli space metric seen by a D-brane probe gets α' corrections, and for a finite size Calabi-Yau background it is not Ricci flat. The existing results are consistent with the first such correction arising from the standard $\alpha'^3 R^4$ correction to supergravity [21], but pushing this to higher orders seems difficult.

Perhaps this problem becomes simpler in a black hole background. Rather than the D0, the probe brane we will use is a D2 or M2-brane wrapped on the black hole horizon. As discussed in the works [22, 23, 24, 25], such a brane, and D0-branes as well, in a near horizon BPS black hole background can preserve $SU(1,1|2)$ superconformal invariance. This is a symmetry of the $AdS_2 \times S^2$ near horizon geometry and thus this is as expected if multi-D0 quantum mechanics can be used as a dual gauge theory of the black hole. In these works, this quantum mechanics was argued to factorize into a space-time part, and an internal (Calabi-Yau) part; this second part describes motion of the probe in the Calabi-Yau and can be used to define a probe metric.

Given this system and its relation to the black hole, we can give a physical argument, based on the idea that a black hole must have "maximal entropy" no matter how this is defined, that suggests that the probe metric in such a black hole background is in fact the "balanced metric" introduced and studied in the mathematics literature [6, 10, 11, 12]. The balanced metric by definition satisfies the equation

$$\rho(z,\bar{z}) = const$$

for the expansion (1).

Our physical argument is based on the assumption that the most symmetric state of a BPS black hole is a state of "maximum entropy." We believe this is a physically reasonable claim which is implicit in all work on this subject. What does this mean? One way to define "maximal entropy" is to look at the Hilbert space of BPS states of the black hole, call this $\mathcal{H}$. By standard arguments going back to [15], these are BPS states of the quantum system describing the black hole, here a bound state of D0 and D4 branes. Let us denote an orthonormal basis of $\mathcal{H}$ as $|h_\alpha>$.

Now, the states $|h_\alpha>$ are pure states in the usual sense of quantum mechanics. The maximal

[1] This was the point of view taken in [18, 19]. Actually, one can in principle reconstruct a manifold with metric from quantum measurements (the spectrum and some position space observables), so one can work without this assumption.

entropy state of such a system is a mixed state, described by the density matrix

$$\rho = \frac{1}{\dim \mathcal{H}} \sum_{\alpha} |h_\alpha >< h_\alpha|, \tag{2}$$

in which each pure state appears with equal probability. Thus, we have a clear definition of "maximal entropy" of the black hole.

The original description of the black hole Hilbert space $\mathcal{H}$ [28] was in terms of a postulated bound state of D0-branes at each triple intersection of D4-brane on the Calabi-Yau. A later argument to the same effect [29] proceeds by lifting the black hole to M theory on $X \times S^1$, in which it becomes a wrapped M5-brane. First compactifying on X, a wrapped five-brane on a 4-cycle (or divisor) D becomes a black string. The string is then compactified on S^1 to obtain the black hole. More recently, a related but (at least to us) not obviously identical description of the black hole Hilbert space has been developed, motivated by the idea that the black hole should be described by a superconformal matrix quantum mechanics of q_0 D0-branes in the D4 background. [24, 25] In this picture, the basic object is a bound state of n D0-branes which can be thought of as a "fuzzy D2-brane," which arises from the matrix D0 theory by a Myers-type effect [30]. The main difference between this argument and the previous one which is relevant for us, is that in this argument, the supersymmetric quantum mechanics is postulated to have as target space the Calabi-Yau manifold X, with a non-trivial $U(1)$ magnetic field, of topological type exactly that of the bundle L. One simple argument which leads to such a QM is to consider a D2-brane wrapped on the black hole horizon, an S^2. Such a D2-brane will couple to the three-form magnetic field produced by the D4-branes of the black hole. Integrating over the horizon, one obtains precisely this magnetic field.

Now, granting this identification of the black hole with a system including a D2-brane, we can reinterpret part of $\mathcal{H}$, namely the part with a single bound state of n D0-branes, as also describing states of the D2 probe quantum mechanics. More generally, if the black hole is in another state with several D0 bound states, we can trace over the degrees of freedom of all but one of these, to again get a single D2 state. The motion of this single D2 on the CY is governed by supersymmetric quantum mechanics with target X and magnetic field $F = \omega$, for which the BPS Hilbert space is $\mathcal{H}_1$. Thus the black hole density matrix can be projected onto $\mathcal{H}_1$, resulting in another maximal entropy density matrix over the Hilbert space $\mathcal{H}_1$.

To summarize, the distribution of black hole microstates implies a distribution of states in the D2 probe quantum mechanics, and a corresponding density matrix. Let us ask, what is the probability to find the D2 probe at a given point $z \in X$. Given a density matrix P for the D2, this probability will be

$$\rho(z, \bar{z}) =< z|\rho|z > . \tag{3}$$

In general, the D2 will have "spin" degrees of freedom as well, corresponding to the degrees (p, q) of cohomology; let us fix these in the $p = q = 0$ sector. By inserting explicit wave functions $\psi_\alpha(z, \bar{z})$, the density matrix can be written in position space as a kernel,

$$\rho(z_1,\bar{z}_1,z_2,\bar{z}_2) = \frac{1}{\dim H^0}\sum_\alpha \psi^*_\alpha(z_1,\bar{z}_1)\ \psi_\alpha(z_2,\bar{z}_2), \tag{4}$$

with its values on the diagonal $z_1 = z_2 = z$.

Now, since the black hole has maximal entropy, and the probe is in some sense dual to (or a possible constituent of) the black hole, one would expect that this probability does not favor any particular point in moduli space, in other words

$$\rho(z,\bar{z}) = const. \tag{5}$$

But this is not at all obvious from what we have said so far; we might regard it as a second, independent interpretation of the claim that the black hole has maximal entropy.

While from the point of view of an asymptotic observer, the first definition (2) of maximal entropy seems more natural, if we can only make measurements with the probe, the second definition seems more natural. Going further, to the extent that (following the arguments above) the probe can also be thought of as a constituent of the black hole, we might be able to reformulate black hole thermodynamics in terms of the second definition. Thus, while not self-evident, it is an attractive hypothesis that the entropy should be maximal in both senses. Actually, the two definitions of maximal entropy are not directly in conflict. Indeed, we could compute (3) from the definition (4), and check whether they agree. But since the actual wave functions and thus (4) depend on the details of the probe world-volume theory, in particular the metric, we need to know the probe metric to make this check. Turning around this logic, we can regard the conjunction of (2) and (5) as a non-trivial condition on the probe metric. In fact, this is a known condition: it implies that the probe metric is the balanced metric.

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