These notes

This is a very preliminary draft of the notes from Cumrun Vafa's lectures at the Simons Workshop in 2004. Many references and clarifications still have to be added, and there are no doubt some errors and typos which need to be corrected. A more definitive version will appear later.

1 Calabi-Yau spaces

1.1 Definition of Calabi-Yau space

We begin with a review of the notion of "Calabi-Yau space." A Calabi-Yau space is a manifold X with a Riemannian metric g, satisfying three conditions:

- **I.** X **is a complex manifold**. This means X looks locally like $\mathbb{C}^n$ for some n, in the sense that it can be covered by patches admitting local complex coordinates

$$z_1, \dots, z_n. \tag{1.1}$$

In particular, the real dimension of X is $2n$, so it is always even. Furthermore the metric g should be Hermitian with respect to the complex structure, which means

$$g_{ij} = g_{\bar{i}\bar{j}} = 0, \tag{1.2}$$

so the only nonzero components are $g_{i\bar{j}}$.

- **II.** X **is Kähler**. This means that locally on X there is a holomorphic function K such that

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K. \tag{1.3}$$

Given a Hermitian metric g one can define its associated Kähler form, which is of type $(1,1)$,

$$k = g_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j. \tag{1.4}$$

Then the Kähler condition is $dk = 0$.

- **III.** X **is Ricci-flat.** Constructing the Ricci curvature R as usual from g, we require that

$$R_{ij} = R_{i\bar{j}} = R_{\bar{i}\bar{j}} = 0. \tag{1.5}$$

Spaces satisfying the conditions I, II, III above are a natural setting for topological string theory. Although some of these conditions can be relaxed to give "generalized Calabi-Yau spaces," with correspondingly more general notions of topological string, the examples which have played the biggest role in the development of the theory so far are honest Calabi-Yaus. Therefore, in this review we focus on the honest Calabi-Yau case.

Except in the simplest examples, it is difficult to determine the Ricci-flat Kähler metrics on Calabi-Yau spaces. Nevertheless it is important and useful to know when such a metric *exists*, even if we cannot construct it explicitly. A crucial tool in this respect is Yau's Theorem [1], which states that if X admits *some* metric satisfying conditions I and II, then it also admits a metric satisfying condition III if and only if it obeys the topological constraint

$$c_1(X) = 0. \tag{1.6}$$

Here c_1 refers to the first Chern class of the tangent bundle. The condition (1.6) is equivalent to the existence of a nonvanishing holomorphic n-form Ω on X; if Ω exists, the volume form of the Ricci-flat metric is (up to a scalar multiple)

$$\mathrm{vol} = \Omega \wedge \bar{\Omega}. \tag{1.7}$$

Strictly speaking Yau's Theorem as stated above applies to compact X, and has to be supplemented by suitable boundary conditions at infinity for non-compact X. For physical applications we do not require that X be compact; in fact, as we will see, many topological string computations simplify in the non-compact case, and this is also the case which is directly relevant for the connections to gauge theory.

1.2 Examples of Calabi-Yau spaces

1.2.1 Dimension 1

We begin with the case where the complex dimension $n = 1$. In this case one can easily list all the Calabi-Yau spaces.

The simplest example is just the complex plane $\mathbb{C}$, with a single complex coordinate z, and the usual flat metric

$$g_{z\bar{z}} = -2\mathrm{i}. \tag{1.8}$$

In this case the holomorphic 1-form is simply

$$\Omega = \mathrm{d}z. \tag{1.9}$$

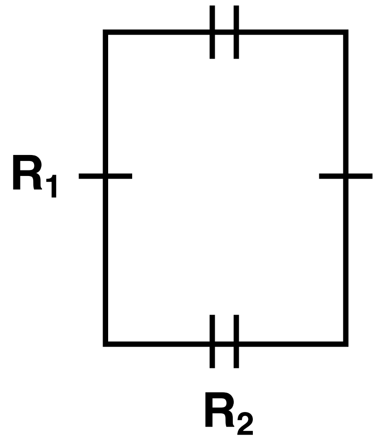

Figure 1: A rectangular torus; the top and bottom sides are identified, as are the left and right sides.

The next simplest example is $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$, with its cylinder metric

$$g_{z\bar{z}} = -2\mathrm{i}/|z|^2, \tag{1.10}$$

and holomorphic 1-form

$$\Omega = \mathrm{d}z/z. \tag{1.11}$$

Finally there is one compact example, namely the torus $T^2 = S^1 \times S^1$. We can picture it as a rectangle which we have glued together at the boundaries, as shown in Figure 1.

This torus has an obvious flat metric, namely the metric of the page; this metric depends on two parameters R_1, R_2 which are the lengths of the sides, so we say we have a two-dimensional "moduli space" of Calabi-Yau metrics on T^2, parameterized by the pair (R_1, R_2). It is convenient to repackage them into

$$A = \mathrm{i}R_1 R_2, \tag{1.12}$$

$$\tau = \mathrm{i}R_2/R_1. \tag{1.13}$$

Then A describes the overall area of the torus, while τ describes its complex structure. A remarkable fact about string theory is that the theory is in fact invariant under the exchange

$$A \leftrightarrow \tau. \tag{1.14}$$

This is the simplest example of "mirror symmetry," which we will discuss further in Section 4.1. Here we just note that the symmetry (1.14) is quite unexpected from the viewpoint

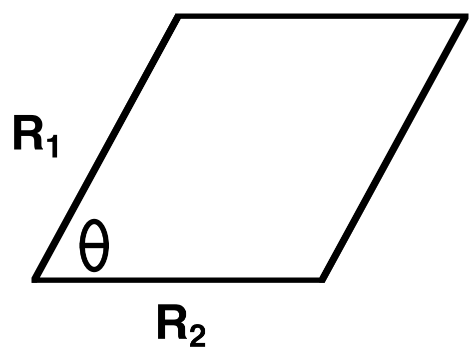

Figure 2: A torus with a more general metric; again, opposite sides of the figure are identified.

of classical geometry; for example, when combined with the obvious geometric symmetry $R_1 \leftrightarrow R_2$, it implies that string theory is invariant under $A \leftrightarrow 1/A$!

We could also consider a more general torus as in Figure 2; this is still a Calabi-Yau space. It is natural to include such tori in our moduli space by letting the parameter τ have a real part as well as an imaginary part. But then in order for the symmetry (1.14) to make sense, A should also be allowed to have a real part; in string theory this real part is naturally provided by an extra field, known as the "B field." For general X this B field is a class in $H^2(X, \mathbb{R})$, which should be considered as part of the moduli of the Calabi-Yau space along with the metric. In our case $X = T^2$, $H^2(X, \mathbb{R})$ is 1-dimensional, and it exactly provides the missing real part of A.

1.2.2 Dimension 2

Now let us move to Calabi-Yau spaces of complex dimension 2. Here the supply of examples is somewhat richer.

One can obtain simple examples by taking Cartesian products of the ones we had in dimension 1, e.g. $\mathbb{C}^2, \mathbb{C} \times \mathbb{C}^\times, \mathbb{C} \times T^2$. There are various other non-compact examples as well in $d = 2$, such as the ALE spaces; these also play an important role in string theory, but we will not discuss them here. Instead we move on to the compact examples. Up to diffeomorphism there are only two, namely the four-torus T^4 and the "K3 surface." We focus here on K3.

The fastest way to define K3 is to obtain it as a quotient $T^4/\mathbb{Z}_2$, using the $\mathbb{Z}_2$ identification

$$(x_1, x_2, x_3, x_4) \sim (-x_1, -x_2, -x_3, -x_4). \tag{1.15}$$

Strictly speaking, this quotient gives a singular K3 surface, with 16 singular points which are the fixed points of (1.15). Nevertheless, the singular points can be "blown up" (this roughly means replacing them by embedded 2-spheres, see e.g. [2]) to obtain a smooth K3 surface. In string theory both the singular K3 and the smooth K3 are allowed; the singular K3 gives a special sublocus of the moduli space of K3 surfaces.

One can also define the K3 surface directly by means of algebraic equations. To begin with we introduce an auxiliary space $\mathbb{CP}^n$, defined as follows: $\mathbb{CP}^n$ consists of all $(n+1)$-tuples $(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$, excluding the point $(0, 0, \ldots, 0)$, modulo the identification

$$(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1}), \tag{1.16}$$

for all $\lambda \in \mathbb{C}^\times$. Then $\mathbb{CP}^n$ is an n-dimensional complex manifold, roughly because we can use the identification (1.16) to eliminate one coordinate. $\mathbb{CP}^n$ is not a Calabi-Yau space by itself. To get the Calabi-Yau space K3 we consider the equation

$$P_4(z_1, \ldots, z_4) = 0, \tag{1.17}$$

where P_4 is some homogeneous polynomial of degree 4. Then we define K3 to be the set of solutions to (1.17) inside $\mathbb{CP}^3$. Since $\mathbb{CP}^3$ is 3-dimensional and (1.17) is 1 complex equation, K3 so defined will be 2-dimensional. Note that in order for this definition to make sense it is important that P_4 is a *homogeneous* polynomial — otherwise the condition (1.17) would not be well-defined after the identification (1.16).

Different choices for the polynomial P_4 give rise to different K3 surfaces, in the sense that they have different complex structures, although they are all diffeomorphic. P_4 has 20 complex coefficients, but the equation (1.17) is obviously independent of the overall scaling of P_4, so this rescaling does not affect the complex structure of the resulting K3; all the other coefficients do affect the complex structure, so one gets a 19-parameter family of K3 surfaces from this construction. [1]

So far we have only discussed K3 as a complex manifold, but it is indeed a Calabi-Yau space. It is easy to see that it is Kähler since it inherits a Kähler metric from $\mathbb{CP}^4$. To see that it has a Ricci-flat Kähler metric one can invoke Yau's Theorem, as we mentioned in Section 1.1; that reduces the task to showing that K3 has $c_1 = 0$. By using the "adjunction formula" from algebraic geometry [2] one finds that given a polynomial equation of degree d inside $\mathbb{CP}^{k-1}$, the resulting hypersurface X has

$$c_1(X) \sim (d-k) c_1(\mathbb{CP}^{k-1}). \tag{1.18}$$

[1] These are not quite all the complex moduli of K3 — there is one more complex deformation possible, for a total of 20, but after making this deformation one gets a surface which cannot be realized by algebraic equations inside $\mathbb{CP}^3$.

In this case we took $d = k = 4$, so $c_1(X) = 0$ as desired. This shows the existence of the desired Calabi-Yau metric, but its explicit form is not known except at special points in the moduli space.

1.2.3 Dimension 3

Now we move to the case which is most interesting for topological string theory. In $d = 3$ the classification problem is far more complicated, even in the compact case; while in $d = 1$ and $d = 2$ we had just T^2 and T^4, K3 respectively, in $d = 3$ it is not even known whether the number of compact Calabi-Yau spaces is finite. So we content ourselves with a few examples.

The quintic threefold. This is defined similarly to our algebraic construction of K3 above; namely we consider the equation

$$P_5(z_1, \dots, z_5) = 0, \tag{1.19}$$

where P_5 is homogeneous of degree 5. The solutions of (1.19) inside $\mathbb{CP}^4$ give a 3-dimensional space which we call the "quintic threefold." It is a Calabi-Yau space again using (1.18) just as we did for K3.

It has 101 complex moduli, and is in some sense the simplest compact Calabi-Yau threefold. As such it has been extensively studied, e.g. as the first example of full-fledged mirror symmetry [3].

Local $\mathbb{CP}^2$. One non-compact Calabi-Yau can be obtained by starting with four complex coordinates (x, z_1, z_2, z_3), subject to the condition $(z_1, z_2, z_3) \neq (0, 0, 0)$, and making the identification

$$(x, z_1, z_2, z_3) \sim (\lambda^{-3}x, \lambda z_1, \lambda z_2, \lambda z_3) \tag{1.20}$$

for all $\lambda \in \mathbb{C}^\times$. Mathematically, this space is known as the total space of the line bundle $\mathcal{O}(-3) \to \mathbb{CP}^2$; we can think of it as obtained by starting with the $\mathbb{CP}^2$ spanned by z_1, z_2, z_3 and adjoining the extra coordinate x. See Figure 3.

The rule (1.20) then characterizes the behavior of x under rescalings of the homogeneous coordinates on $\mathbb{CP}^2$, or equivalently, it determines how x transforms as one moves between different patches on $\mathbb{CP}^2$. Locally, our space has the structure of $\mathbb{CP}^2 \times \mathbb{C}$. In this sense it has "4 compact directions" and "2 non-compact directions."

Although this geometry is non-compact, it can arise naturally even if we start with a compact Calabi-Yau — namely, it describes the geometry of a Calabi-Yau space containing a $\mathbb{CP}^2$, in the limit where we focus on the immediate neighborhood of the $\mathbb{CP}^2$.

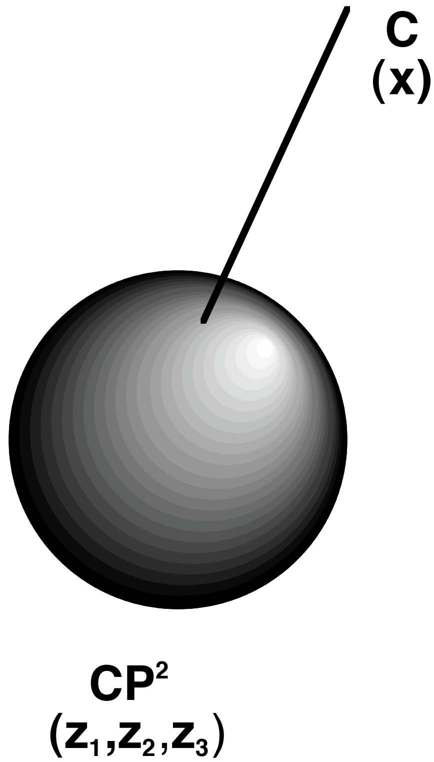

Figure 3: A crude representation of the local $\mathbb{CP}^2$ geometry, $\mathcal{O}(-3) \to \mathbb{CP}^2$.

Local $\mathbb{CP}^1$. Similarly, we can start with four complex coordinates (x_1, x_2, z_1, z_2), subject to the condition $(z_1, z_2) \neq (0,0)$, and make the identification

$$(x_1, x_2, z_1, z_2) \sim (\lambda^{-1}x_1, \lambda^{-1}x_2, \lambda z_1, \lambda z_2) \tag{1.21}$$

for all $\lambda \in \mathbb{C}^\times$. This gives the total space of the line bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. Similarly to the previous example, it is obtained by starting with $\mathbb{CP}^1$, which has "2 compact directions," and then adjoining the coordinates x_1, x_2, which contribute "4 non-compact directions." See Figure 4.

This example is also known as the "resolved conifold," a name to which we will return shortly.

Local $\mathbb{CP}^1 \times \mathbb{CP}^1$. Another standard example comes by starting with five complex coordinates (x, y_1, y_2, z_1, z_2), with $(y_1, y_2) \neq (0,0)$ and $(z_1, z_2) \neq (0,0)$, and making the identification

$$(x, y_1, y_2, z_1, z_2) \sim (\lambda^{-1}\mu^{-1}x, \lambda y_1, \lambda y_2, \mu z_1, \mu z_2) \tag{1.22}$$

for all $\lambda, \mu \in \mathbb{C}^\times$. This gives the total space of the line bundle $\mathcal{O}(-2,-2) \to \mathbb{CP}^1 \times \mathbb{CP}^1$. It

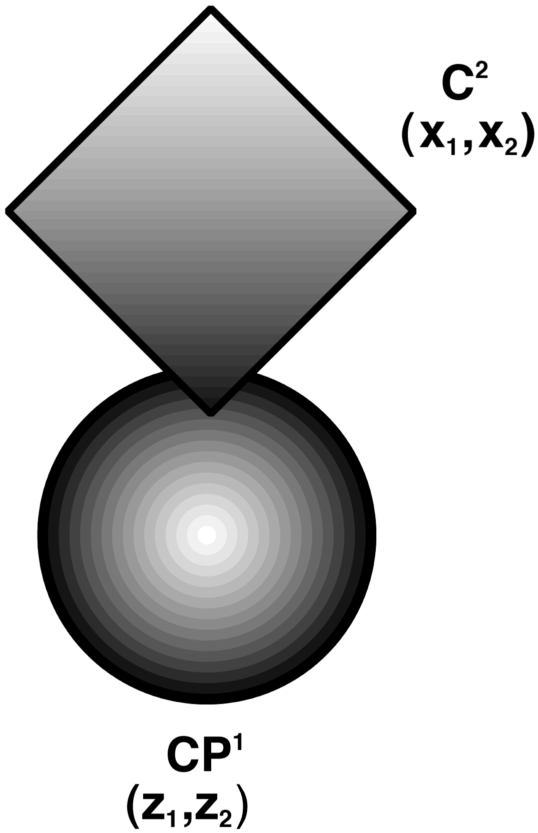

Figure 4: A crude representation of the local $\mathbb{CP}^1$ geometry, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$.

has four compact directions and two non-compact directions.

Deformed conifold. All the local examples we discussed so far were "rigid," in other words, they had no deformations of their complex structure.[2] Now let us consider an example which is not rigid. Starting with the complex coordinates $(x, y, z, t) \in \mathbb{C}^4$, this time without any projective identification, we look at the space of solutions to

$$xy - zt = \mu. \tag{1.23}$$

This gives a Calabi-Yau 3-fold for any value $\mu \in \mathbb{C}$, so μ spans the 1-dimensional moduli space of complex structures. If $\mu = 0$ then the Calabi-Yau has a singularity at $(x, y, z, t) = (0, 0, 0, 0)$, known as the "conifold singularity." For finite μ it is smooth. Since we obtain the

[2]Strictly speaking, this is a delicate statement since we should specify what kind of boundary conditions we are imposing at infinity. When we say that these local examples are rigid we essentially mean that the compact part, e.g. $\mathbb{CP}^1$ or $\mathbb{CP}^2$, has no complex deformations.

smooth Calabi-Yau from the singular one just by varying the parameter μ, which deforms the complex structure, we call the smooth version the "deformed conifold."

1.3 Conifolds

In the last section we introduced the singular conifold

$$xy - zt = 0, \tag{1.24}$$

and the deformed conifold

$$xy - zt = \mu. \tag{1.25}$$

We now want to describe another way of smoothing the conifold singularity. First rewrite (1.24) as

$$\det \begin{pmatrix} x & z \\ t & y \end{pmatrix} = 0. \tag{1.26}$$

This equation is equivalent to the existence of nontrivial solutions to

$$\begin{pmatrix} x & z \\ t & y \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \tag{1.27}$$

Indeed, away from $(x, y, z, t) = (0, 0, 0, 0)$, (1.26) just states that the matrix has rank 1, so (ξ_1, ξ_2) solving (1.27) are unique up to an overall rescaling. So away from $(x, y, z, t) = (0, 0, 0, 0)$ one could describe the singular conifold as the space of solutions to (1.27), with $(\xi_1, \xi_2) \neq (0, 0)$, and with the identification

$$(\xi_1, \xi_2) \sim (\lambda\xi_1, \lambda\xi_2) \tag{1.28}$$

where $\lambda \in \mathbb{C}^\times$. But at $(x, y, z, t) = (0, 0, 0, 0)$ something new happens: any pair (ξ_1, ξ_2) now solves (1.27). Taking into account (1.28), (ξ_1, ξ_2) parameterize a $\mathbb{CP}^1$ of solutions. In summary, (1.24) and (1.27) are equivalent, except that $(x, y, z, t) = (0, 0, 0, 0)$ describes a single point in (1.24), but a whole $\mathbb{CP}^1$ in (1.27). We refer to the latter space as the "resolved conifold." (In fact, it is isomorphic to the local $\mathbb{CP}^1$ geometry we considered above.)

Mathematically this discussion would be summarized by saying that the resolved conifold is obtained by making a "small resolution" of the conifold singularity. We emphasize, however, that physically it is natural to consider this as a continuous process, contrary to the usual mathematical description in which it seems to be a discrete jump. This is because physically we consider the full Calabi-Yau metric rather than just the complex structure.

Namely, the resolved conifold has a single Kähler modulus for its Calabi-Yau metric,[3] naturally parameterized by

$$t = \mathrm{vol}(\mathbb{CP}^1). \tag{1.29}$$

In the limit $t \to 0$ the $\mathbb{CP}^1$ shrinks to a point and the Calabi-Yau metric on the resolved conifold approaches the Calabi-Yau metric on the singular conifold. So the resolved conifold is obtained by a Kähler deformation of the metric without changing the complex structure,[4] while the singular conifold is obtained by deforming the complex structure.

Since the conifold is such an important example it will be useful to describe it in another way. Namely, by a change of variables we can rewrite (1.25) as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r. \tag{1.30}$$

Describing it this way it is easy to see that there is an S^3 in the geometry, namely, just look at the locus where all $x_i \in \mathbb{R}$. The full geometry where we include also the imaginary parts of x_i is in fact isomorphic to the cotangent bundle, T^*S^3.

This space is familiar to physicists as the phase space of a particle which moves on S^3; it has three "position" variables labeling a point $x \in S^3$ and three "momenta" spanning the cotangent space at x. Now we want to describe its geometry "near infinity," i.e. at large distances, similar to how we might describe the infinity of Euclidean $\mathbb{R}^3$ as looking like a large S^2. In the case of T^*S^3 the position coordinates are bounded, so looking near infinity means choosing large values for the momenta, which gives a large S^2 in the cotangent space $\mathbb{R}^3$. Therefore the infinity of T^*S^3 should look like some S^2 bundle over the position space S^3, i.e. locally on S^3 it should look like $S^2 \times S^3$. It turns out that this is enough to imply that it is even *globally* $S^2 \times S^3$.

So at infinity the deformed conifold has the geometry of $S^2 \times S^3$. As we move toward the origin both S^2 and S^3 shrink until the S^2 disappears altogether, leaving just the S^3 with radius r which is the core of the T^*S^3 geometry (the zero section of the cotangent bundle.) See Figure 5.

As $r \to 0$ the metric approaches the metric of the singular conifold; the singularity at the "tip" of the cone can be seen in Figure 5. Also, from this perspective, the S^2 which appears when we go to the resolved conifold seems very natural; in some sense it was in the game to begin with, as we see from the S^2 at infinity. The resolved conifold geometry just

[3]Once again, we are here considering only variations of the metric which preserve suitable boundary conditions at infinity.

[4]Mathematically, the resolved conifold and the singular conifold are not the same as complex manifolds, but they are birationally equivalent. Physically we want to consider birationally equivalent spaces as really having the same complex structure.

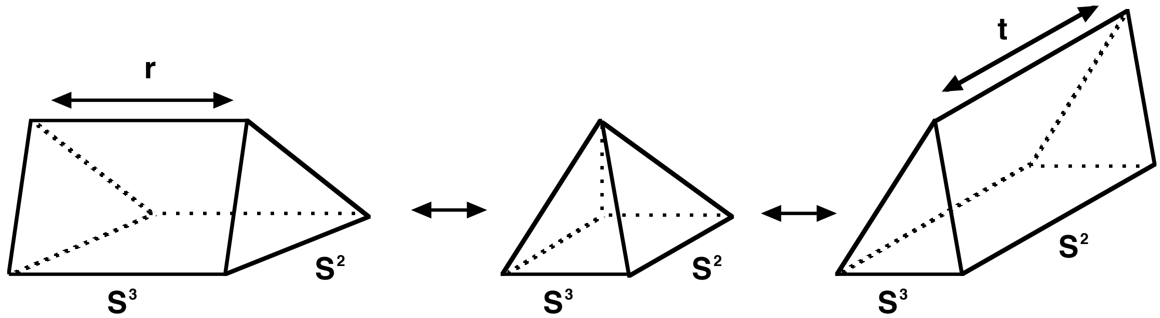

Figure 5: The three conifold geometries: deformed, singular and resolved.

corresponds to giving this S^2 a finite size even at the tip of the cone. So all three cases — deformed, singular, and resolved — look the same at infinity; they differ only near the tip of the cone. This is exactly what we expect since we were trying to study only localized deformations (normalizable modes, in physics language.)

In summary, we have two different non-compact Calabi-Yau geometries: the deformed conifold, which has one complex modulus and no Kähler moduli, and the resolved conifold, which has no complex moduli but one Kähler modulus; and we can interpolate from one moduli space to the other by passing through the singular conifold geometry.

We will return to the conifold repeatedly in later sections. For more information about its geometry, including the explicit Calabi-Yau metrics, see [4].

2 Toric geometry

Now we want to introduce a particularly convenient representation of a special class of algebraic manifolds, which includes and generalizes some of the examples we considered above. Mathematically this representation is called *toric geometry*; for a more detailed review than we present here, see e.g. [5].

$\mathbb{C}^n$. We begin with $\mathbb{C}^n$, with complex coordinates $(z_1, \ldots, z_n)$ and the standard flat metric, and parameterize it in an idiosyncratic way: writing

$$z_i = |z_i| e^{\mathrm{i}\theta_i}, \tag{2.1}$$

we choose the coordinates $((|z_1|, \theta_1), \ldots, (|z_n|, \theta_n))$. This coordinate system emphasizes the symmetry $U(1)^n$ which acts on $\mathbb{C}^n$ by shifts of the θ_i. It is also well suited to describing the

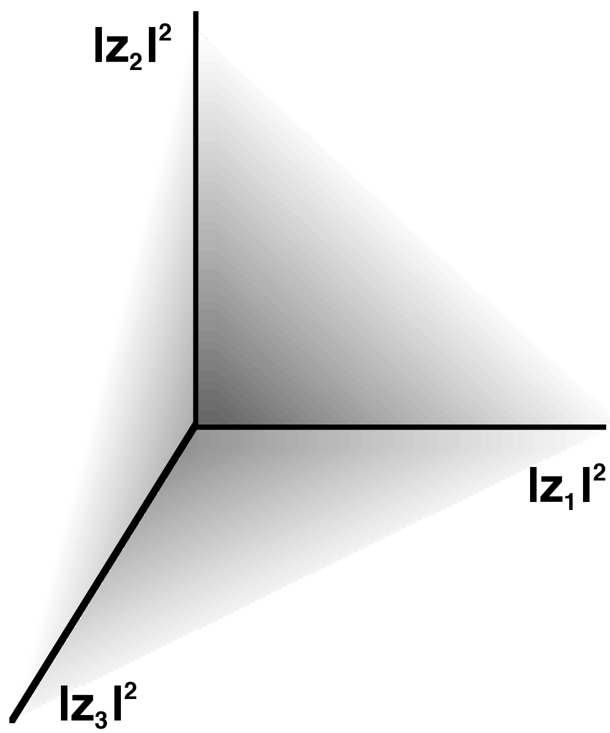

Figure 6: The positive octant $\mathcal{O}^{3+}$, which we identify as the toric base of $\mathbb{C}^3$.

symplectic structure given by the Kähler form k:

$$k = \sum_i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i = \sum_i \mathrm{d}|z_i|^2 \wedge \mathrm{d}\theta_i. \tag{2.2}$$

Roughly, splitting the coordinates into $|z_i|^2$ and θ_i gives a factorization

$$\mathbb{C}^n \approx \mathcal{O}^{n+} \times T^n, \tag{2.3}$$

where $\mathcal{O}^{n+}$ denotes the "positive orthant" $\{|z_i|^2 \geq 0\}$, represented (for $n = 3$) in Figure 6.

Namely, at each point of $\mathcal{O}^{n+}$ we have the product of n circles obtained by fixing $|z_i|$ and letting θ_i vary. However, when $|z_i|^2 = 0$ the circle $|z_i|e^{\mathrm{i}\theta_i}$ degenerates to a single point. Therefore (2.3) is not quite precise, because the "fiber" T^n degenerates at each boundary of the "base" $\mathcal{O}^{n+}$; which circle of T^n degenerates is determined by which $|z_i|^2$ vanishes, or more geometrically, by the direction of the unit normal to the boundary. When $m > 1$ of the $|z_i|^2$ vanish, the corresponding m circles of T^n degenerate, until at the origin all n cycles have degenerated and T^n shrinks to a single point. In this sense all the information about the symplectic manifold $\mathbb{C}^3$ is contained in Figure 6, which is called the "toric diagram"

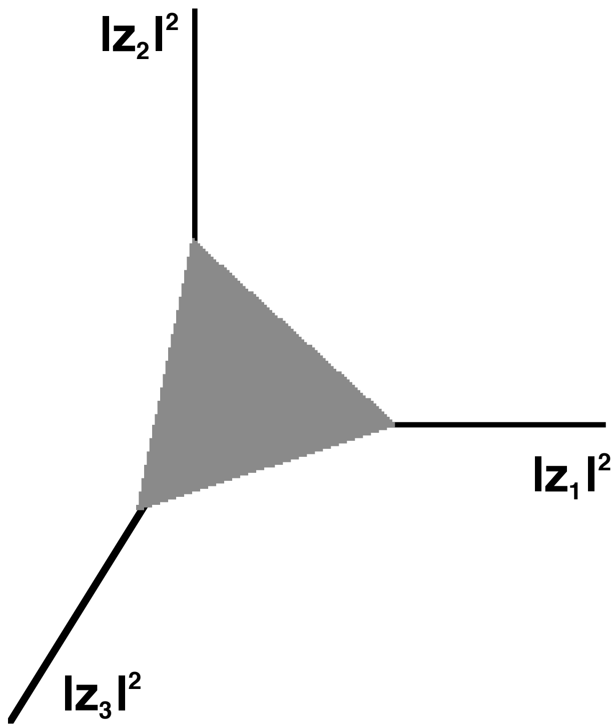

Figure 7: The toric base of $\mathbb{CP}^2$; geometrically it is just a triangle, but here we show it naturally embedded in $\mathbb{R}^3$ and cut out by the condition (2.4).

for $\mathbb{C}^3$; when looking at this diagram one always has to remember that there is a T^3 over the generic point, and that this T^3 degenerates at the boundaries in a way determined by the unit normal. Despite the fact that the T^3 becomes singular at the boundaries, the full geometry of $\mathbb{C}^3$ is of course smooth. (Of course, all this holds for general n as well as $n = 3$, but the analogue of Figure 6 would be hard to draw in the general case.)

$\mathbb{CP}^n$. Next we want to give a toric representation for $\mathbb{CP}^n$. We first give a slightly different quotient presentation of this space than the one we used in (1.16): namely, for any $r > 0$, we start with the $2n+1$-sphere

$$|z_1|^2 + \cdots + |z_{n+1}|^2 = r, \tag{2.4}$$

and then make the identification

$$(z_1, \ldots, z_{n+1}) \sim (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1}) \tag{2.5}$$

for all real θ. This is equivalent to our original "holomorphic quotient" definition, where we did not impose (2.4) but worked modulo arbitrary rescalings of the z_i instead of just phase

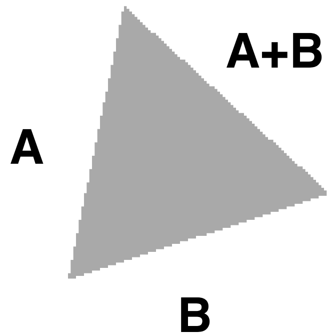

Figure 8: The toric base of $\mathbb{CP}^2$. Over each boundary a cycle of the fiber T^2 collapses; if we label the basis cycles as A and B, then the collapsing cycle over each boundary is as indicated.

rescalings; indeed, starting from that definition one can make a rescaling to impose (2.4), and afterward one still has the freedom to rescale by a phase as in (2.5). The presentation we are using now is more closely rooted in symplectic geometry. It is also natural from the point of view of a supersymmetric linear sigma model with $U(1)$ gauge symmetry. Specifically [6], the z_i appear as the scalar components of 4 chiral superfields, all with $U(1)$ charge 1. In that context $\mathbb{CP}^n$ is the moduli space of vacua; the constraint (2.4) is imposed by the D-terms, and the quotient (2.5) is the identification of gauge equivalent field configurations.

Note that in this presentation of $\mathbb{CP}^n$ we have the parameter $r > 0$, which did not appear in the holomorphic quotient. This parameter appears naturally in the gauged linear sigma model, where one sees directly that it corresponds to the size of $\mathbb{CP}^n$.

Now we want to use this presentation to draw the toric diagram. As we did for $\mathbb{C}^n$, we draw the toric base using the coordinates $|z_i|^2$; in the present case we also have to impose (2.4), so the base is an n-dimensional simplex; for example, in the case of $\mathbb{CP}^2$ the base is just a triangle, as shown in Figure 7. Over each point of the base we have a T^2 fiber generated by shifts of θ_i (naively this would give a T^3 for $\theta_1, \theta_2, \theta_3$, but the identification (2.5) reduces this to T^2.) A cycle of T^2 collapses over each boundary of T^2, as indicated in Figure 8.

Local $\mathbb{CP}^2$. To get a toric presentation of a Calabi-Yau manifold we have to take a non-compact example. The construction is closely analogous to what we did above for $\mathbb{CP}^n$; namely, for $r > 0$, we start with

$$-3|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = r, \tag{2.6}$$

and then make the additional identification

$$(z_0, z_1, z_2, z_3) \sim (e^{-3\mathrm{i}\theta} z_0, e^{\mathrm{i}\theta} z_1, e^{\mathrm{i}\theta} z_2, e^{\mathrm{i}\theta} z_3), \tag{2.7}$$

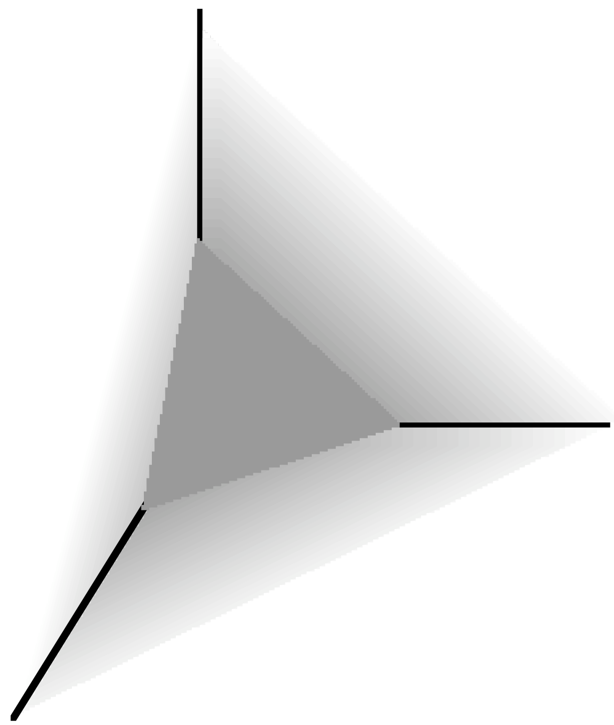

Figure 9: The toric base of the local $\mathbb{CP}^2$ geometry.

for any real θ. In the gauged linear sigma model of [6] this would be realized by taking four chiral superfields with $U(1)$ charges $(-3, 1, 1, 1)$. Actually, the fact that the local $\mathbb{CP}^2$ geometry is Calabi-Yau can also be understood naturally in the gauged linear sigma model: the condition $c_1 = 0$ turns out to be equivalent to the statement that the sum of the $U(1)$ charges vanishes, which in turn implies vanishing of the 1-loop beta function.

We can also draw the toric diagram for this case. Introducing the notation $p_i = |z_i|^2$, the base is spanned by the four real coordinates p_0, p_1, p_2, p_3, subject to the condition (2.6), which can be solved to eliminate p_0,

$$p_0 = \frac{1}{3}(p_1 + p_2 + p_3 - r). \tag{2.8}$$

The condition that all $p_i > 0$ then becomes

$$p_1 + p_2 + p_3 > r, \tag{2.9}$$

$$p_1 > 0, \tag{2.10}$$

$$p_2 > 0, \tag{2.11}$$

$$p_3 > 0. \tag{2.12}$$

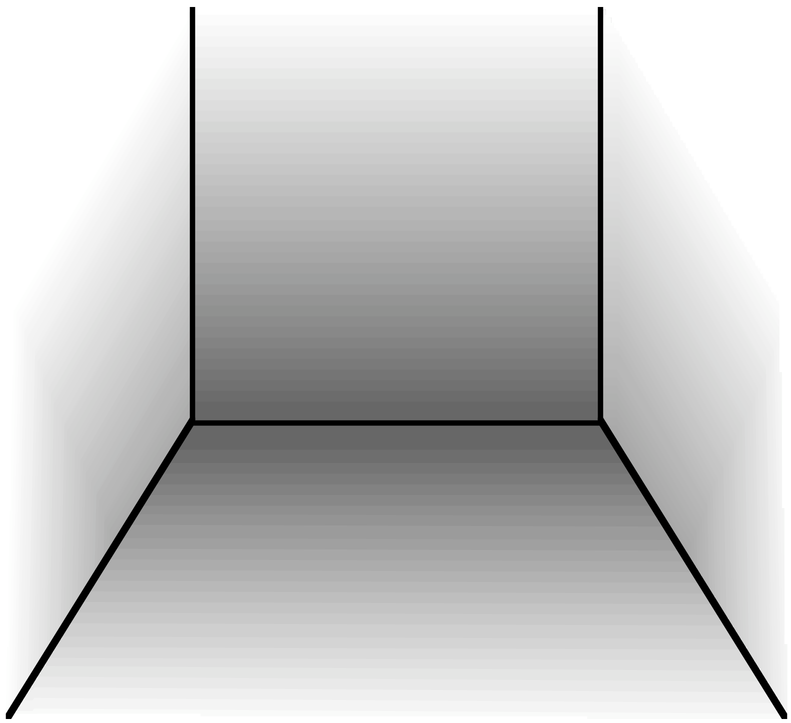

Figure 10: The toric base of the local $\mathbb{CP}^1$ geometry.

So the toric base is the positive octant in $\mathbb{R}^3$ with a corner chopped off, as shown in Figure 9. The triangle at the corner represents the $\mathbb{CP}^2$ at the core of the geometry, just as in the previous example.

Local $\mathbb{CP}^1$. A similar construction gives the toric diagram for the local $\mathbb{CP}^1$ geometry. One obtains in this case Figure 10. One feature of interest is the $\mathbb{CP}^1$ at the core of the geometry, which can be easily seen as the line segment in the middle. (To see that the line segment indeed represents the topology of $\mathbb{CP}^1$, recall that along this segment two of the three circles of the fiber T^3 are degenerate, so that one just has an S^1 in the fiber; moving along the segment, this S^1 then sweeps out a $\mathbb{CP}^1$; indeed, the S^1 degenerates at the two ends of the segment, which are identified with the north and south poles of $\mathbb{CP}^1$.) Furthermore it is easy to read off the volume of this $\mathbb{CP}^1$ from the toric diagram: the Kähler form in this geometry is $k = \mathrm{d}p_i \wedge \mathrm{d}\theta_i$, and integrating it just gives $2\pi\Delta p$, i.e. the length of the line segment![5]

Local $\mathbb{CP}^1 \times \mathbb{CP}^1$. We can give a toric construction for this case as well, again parallel

[5] We are using a fact about Kähler geometry, namely, the volume of a holomorphic cycle is just obtained by integrating k over the cycle.

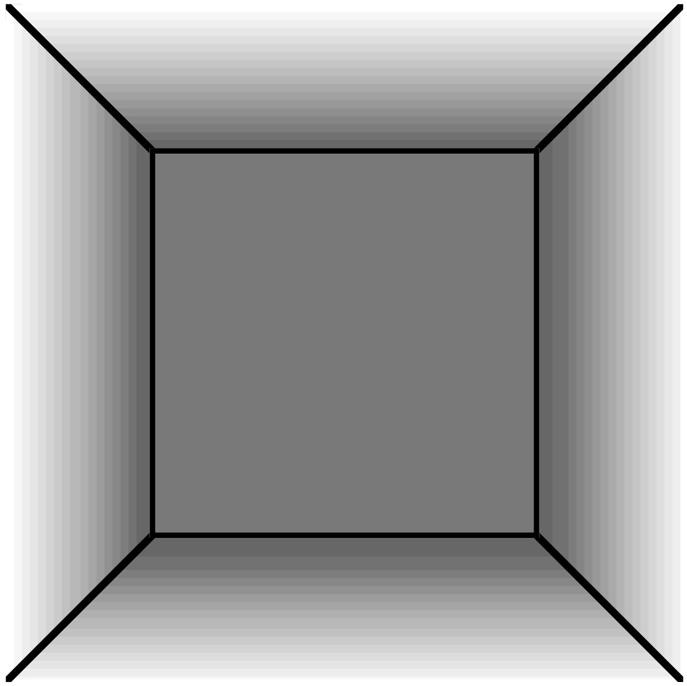

Figure 11: The toric base of the local $\mathbb{CP}^1 \times \mathbb{CP}^1$ geometry.

to the holomorphic construction we gave above; in nonlinear sigma model terms it would correspond to having 5 chiral superfields and two $U(1)$ gauge groups, with the charges $(-2, 1, 1, 0, 0)$ and $(-2, 0, 0, 1, 1)$. (Note that the charges under both $U(1)$ groups sum to zero as required for one-loop conformality.) The corresponding toric diagram is the "oubliette" shown in Figure 11.

Our list of examples has focused on the non-compact case, but we should note that it is also possible to construct compact Calabi-Yaus using the techniques of toric geometry. Indeed, we have already done so in the last section, where we started with the toric manifold $\mathbb{CP}^n$ and then imposed some algebraic equations to obtain a Calabi-Yau. A similar construction can be performed starting with a more general toric manifold, and this gives a large class of interesting examples. From the point of view of the nonlinear sigma model this construction corresponds to introducing a superpotential.

2.1 Why Calabi-Yau?

Now we want to briefly explain the role that these Calabi-Yau spaces play in superstring theory.

Generally, the reason that Riemannian manifolds are important for string theory is that they provide a class of candidate backgrounds on which the strings could propagate. The requirement that X be complex and Kähler turns out to have a rather direct consequence for the physics of observers living in the target space: namely, it implies that these observers will see supersymmetric physics. Since supersymmetry is interesting both phenomenologically and mathematically, this is a natural condition to impose. The requirement that X be Ricci-flat is even more fundamental: string theory would not even make sense without it, as we will discuss in the next section.

So we are in a remarkable situation: the class of Calabi-Yau spaces, which were studied by mathematicians well before their relevance to string theory was appreciated, turns out quite independently to be crucial for physical considerations!

3 Sigma models and topological twisting

3.1 Sigma models and $\mathcal{N} = (2, 2)$ supersymmetry

Now let us sketch what the topological string actually is.

The string theories in which we will be interested (both the ordinary physical version and the topological version) have to do with maps from a surface Σ to a target space X. Roughly, in string theory one integrates over all such maps as well as over metrics on Σ, weighing each map by its "energy" which is given by the Polyakov action:[6]

$$\int_{\mathrm{Map}(\Sigma,X)} \mathrm{D}X \ \mathrm{D}g \ e^{-\int_\Sigma |\partial X|^2}. \tag{3.1}$$

This path integral defines a two-dimensional quantum field theory which is called a "sigma model into X;" its saddle points are locally area-minimizing surfaces in X. Because we are integrating both over maps $\Sigma \to X$ and over two-dimensional metrics, one often describes the string theory as obtained by coupling the sigma model to two-dimensional quantum gravity.

Classically, the sigma model action depends only on the conformal class of the metric g, so that the integral over metrics can be reduced to an integral over conformal structures — or equivalently, to an integral over complex structures on Σ. For the theory to be well defined we need this property to persist at the quantum level, but this turns out to be a nontrivial restriction on the allowed X; namely, requiring that the theory should be conformally invariant even after including one-loop quantum effects on Σ, one finds the condition

[6] Actually, this is the Polyakov action for the "bosonic string"; we are really interested in the superstring, for which there are extra fermionic degrees of freedom, but we are suppressing those for simplicity.

that X should be Ricci flat as well as the condition that the total dimension should be 10, both of which we discussed above.

For generic X one might expect even more conditions to appear when one considers higher-loop quantum effects; this does happen in the bosonic string, but mercifully not in the superstring provided that X is Kähler. The reason why the Kähler condition is so effective in suppressing quantum corrections is that it is related to $(2, 2)$ supersymmetry of the 2-dimensional sigma model.[7] This $(2, 2)$ supersymmetry is crucial for the definition of the topological string, so we now discuss it in more detail.

The statement of $\mathcal{N} = 2$ supersymmetry simply means that there are 4 currents

$$J, G^+, G^-, T, \tag{3.2}$$

with spins $1, 3/2, 3/2, 2$ respectively, and with prescribed operator product relations. These operators get interpreted as follows: T is the usual energy-momentum tensor; $G^\pm$ are conserved supercurrents for two worldsheet supersymmetries; J is the conserved current for the $U(1)$ R-symmetry which rotates $G^\pm$ into one another. The modes of these currents act on the Hilbert space of the theory.

In the case of the sigma model on X, these currents can be identified with the operators

$$\deg, \bar\partial, \bar\partial^\dagger, \Delta \tag{3.3}$$

acting on $\Omega^*(LX)$, the space of differential forms on the loop space of X. This identification suggests that among the operator product relations of the $\mathcal{N} = (2, 2)$ algebra should be

$$(G^+)^2 \sim 0, \tag{3.4}$$

$$(G^-)^2 \sim 0, \tag{3.5}$$

$$G^+G^- \sim T + J; \tag{3.6}$$

these relations indeed hold and they will play a particularly important role for us below.

In the case where X is Calabi-Yau, so that the sigma model is conformal, we can make a further refinement, splitting the algebra (3.2) into two copies, which we write $(J, G^\pm, T)$ and $(\bar J, \bar G^\pm, \bar T)$, both obeying the same operator products; this split structure is referred to as $\mathcal{N} = (2, 2)$ supersymmetry. The structure of $\mathcal{N} = (2, 2)$ superconformal field theory — the operators listed above as well as the Hilbert space on which they act — should be regarded as an invariant associated to the manifold X.

[7] Note that this "worldsheet" supersymmetry is different from the spacetime supersymmetry we discussed in the previous section, although the Kähler condition on X is ultimately responsible for both, and there are indirect arguments which relate one to the other.

3.2 Twisting the $\mathcal{N} = (2,2)$ supersymmetry

Given an $\mathcal{N} = (2,2)$ superconformal field theory as described in the previous section, there is an important construction which produces a "topological" version of the theory. One can think of this procedure as analogous to the passage from the de Rham complex $\Omega^*(M)$ to its cohomology $H^*(M)$: while the cohomology contains less information than the full de Rham complex, the information it does contain is far more easily organized and understood. So how do we construct this topological version of the SCFT? Guided by the relation $(G^+)^2 = 0$ and the above analogy, we might try to form the cohomology of one of the modes of G^+. In fact this is not quite possible, because G^+ has the wrong spin, namely 3/2; in order to define a scalar supercharge which makes sense on arbitrary curved Σ, we need an operator with spin 1. This problem can be overcome, as explained in [7] (see also [8]) by "twisting" the sigma model. The twist can be understood in various ways, but one way to describe it is as a shift in the operator T:

$$T_{\mathrm{new}} = T_{\mathrm{old}} - \frac{1}{2}\partial J. \tag{3.7}$$

This shift has the effect of changing the spins of all operators by an amount proportional to their $U(1)$ charge,

$$S_{\mathrm{new}} = S_{\mathrm{old}} - \frac{1}{2}q. \tag{3.8}$$

After this shift the operators (G^+, J) have spin 1 while (T, G^-) have spin 2.[8] Now we can define $Q = G_0^+$, which makes sense on arbitrary Σ and obeys $Q^2 = 0$, and pass to the cohomology of Q. In this context one often calls Q a "BRST operator." Here we have not obtained it from the usual BRST procedure, but in fact the structure of the twisted $\mathcal{N} = (2,2)$ algebra is isomorphic to one which *is* obtained from the usual BRST procedure, namely that of the *bosonic* string. In that case one has operators (Q, J_{ghost}) of spin 1 and (T, b) of spin 2, where (Q, b) are the BRST charge and antighost corresponding to diffeomorphisms on the bosonic string worldsheet.

It turns out that the b antighost is the crucial element which is needed for the computation of correlation functions in the bosonic string — more specifically, it provides the link between CFT correlators, computed on a fixed worldsheet Σ, and string correlators, which involve integrating over all metrics on Σ. Via the Faddeev-Popov procedure this integral over metrics on Σ gets reduced to an integral over the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces,

[8] Note that although $G^\pm$ now have integer spin, they still obey fermionic statistics!

with the b ghosts providing the measure: the genus g free energy is[9]

$$\int_{\mathcal{M}_g} \langle | \prod_{i=1}^{3g-3} b(\mu_i) |^2 \rangle. \tag{3.9}$$

Here the symbol $\langle \cdots \rangle$ denotes a CFT correlation function. The $3g-3$ μ_i are "Beltrami differentials," 1-forms on Σ with values in the holomorphic tangent bundle; they span the space of infinitesimal deformations of the $\bar{\partial}$ operator on Σ, which is the tangent space to $\mathcal{M}_g$. Then $b(\mu_i)$ is an operator obtained by integrating the b-ghost against μ_i:

$$b(\mu) = \int_\Sigma b_{zz} \mu^z_{\bar{z}}. \tag{3.10}$$

More abstractly, b is an operator-valued 1-form on $\mathcal{M}_g$, so the expectation value of the product of $3g-3$ copies of b gives a holomorphic $3g-3$-form; taking both the holomorphic and antiholomorphic pieces we then get a $6g-6$-form, which can be integrated over $\mathcal{M}_g$.

Now comes the important point: since the twisted $\mathcal{N} = 2$ superconformal algebra is isomorphic to the algebra appearing in the bosonic string, we can now promote the correlation functions of the $\mathcal{N} = (2,2)$ SCFT on fixed Σ to correlation functions of the topological string, by repeating the same formula with b replaced by G^-:

$$F_g = \int_{\mathcal{M}_g} \langle | \prod_{i=1}^{3g-3} G^-(\mu_i) |^2 \rangle. \tag{3.11}$$

The formula (3.11) can also be understood as coming from coupling the twisted $\mathcal{N} = (2,2)$ theory to topological gravity — see [7].

One then defines the full topological string free energy to be

$$\mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2-2g} F_g, \tag{3.12}$$

where λ is the "string coupling constant" weighing the contributions at different genera.[10] Finally, the partition function is defined as

$$Z = \exp \mathcal{F}. \tag{3.13}$$

From our present point of view, the construction of the topological string would have made sense starting from any $\mathcal{N} = (2,2)$ SCFT, and in particular, the sigma model on any Calabi-Yau space X would suffice.

[9] Strictly speaking this is the answer for $g > 1$; the expression has to be slightly modified for $g = 0, 1$ because the sphere and torus admit nonzero holomorphic vector fields.

[10] This expression is only perturbative; it should be understood in the sense of an asymptotic series in λ.

On the other hand, for the physical string, there is a good reason to focus on Calabi-Yau threefolds. Namely, if we focus our attention on backgrounds which could resemble the real world, we find an obvious constraint: we seem to live in (to a good approximation) 4-dimensional Minkowski space M. On the other hand one-loop conformal invariance requires the total dimension of spacetime to be 10. Therefore a natural class of backgrounds would be $M \times X$, where X is some compact 6-dimensional space, small enough that it cannot be seen directly, either by the naked eye or by any experiment we have so far been able to do. Studying string theory on $M \times X$, what one finds is that the *internal* properties of X lead to physical consequences for the observers living in M. Conversely, the four-dimensional perspective on the string theory computations sheds a great deal of light on the geometry of X.

Remarkably, it turns out that the case of Calabi-Yau threefolds is special for the topological string as well. Namely, although one can define F_g for any Calabi-Yau d-fold, this F_g actually vanishes for all $g \neq 1$ unless $d = 3$! This follows from considerations of charge conservation: namely, the topological twisting turns out to introduce a background $U(1)$ charge $d(g-1)$. In order for the correlator appearing in (3.11) to be nonvanishing, the insertions which appear must exactly compensate this background charge; but the insertions consist of $3g-3$ G^- operators, so they have total charge $-3(g-1)$, hence the correlator vanishes unless $d = 3$.[11]

3.3 A and B twists

In the last subsection we glossed over an important point; we chose the operator G^- for our BRST supercharge Q, but we could equally well have chosen G^+. The latter possibility corresponds to an opposite twist where we replace (3.7) by

$$T_{\text{new}} = T_{\text{old}} + \frac{1}{2}\partial J. \tag{3.14}$$

With this twist it is G^+ rather than G^- which will have spin 1. We have a similar freedom in the antiholomorphic sector, so altogether there are four possible choices of twist,

[11]For $g = 0$ one can get an interesting correlator even for $d \neq 3$, by inserting some other operators to absorb the background charge, but for $g > 1$ there is really nothing to be done.

corresponding to choosing for the BRST operators

$$(G^+, \bar{G}^+) : \text{A model} \tag{3.15}$$

$$(G^-, \bar{G}^-) : \bar{\text{A}} \text{ model} \tag{3.16}$$

$$(G^+, \bar{G}^-) : \text{B model} \tag{3.17}$$

$$(G^-, \bar{G}^+) : \bar{\text{B}} \text{ model} \tag{3.18}$$

We have listed each choice together with the name usually given to the corresponding topological string. The $\bar{\text{A}}$ model is related to the A model in a trivial way, namely, all correlators are just related by an overall complex conjugation; so essentially we have two distinct choices here for a given Calabi-Yau X, namely the A and B models.

Now, what is the geometric content of the topological string? In the A model case, the BRST operator $Q + \bar{Q}$ turns out to be the d operator on X, and the BRST cohomology is the de Rham cohomology $H^*_{\mathrm{d}R}(X)$. There is an additional "physical state" constraint which leads to considering only the degree $(1,1)$ part of this cohomology. A $(1,1)$ form corresponds to a deformation of the Kähler form, so finally the observables of the A model correspond to deformations of the Kähler moduli of X. In the B model case one again gets objects of bidegree $(1,1)$, but this time the complex in question is the $\bar{\partial}$ cohomology with values in $\wedge^* TX$, so the observables are $(0,1)$-forms with values in TX, i.e. Beltrami differentials on X. So the observables of the B model correspond to deformations of the complex structure of X.

In fact, one can show directly that the A model is *independent* of the complex structure deformations, and the B model is *independent* of the Kähler deformations; namely, one shows that these deformations are Q-exact, so that they decouple from the computation of the string amplitudes. In this sense the A and B models are decoupled. In sum,

$$\text{A model on } X \leftrightarrow \text{Kähler moduli of } X, \tag{3.19}$$

$$\text{B model on } X \leftrightarrow \text{complex moduli of } X. \tag{3.20}$$

How do we actually compute the correlation functions in the A and B models? In each case we are computing a path integral over maps to X, but this path integral is significantly simplified by the fermionic Q symmetry. Indeed, integrating the Q-invariant functional e^{-S} over the space of maps gives a sum of local contributions from the fixed points of Q; the rest of the field space contributes zero, because one can introduce field space coordinates in which Q acts by infinitesimal shifts of a Grassman coordinate θ, and then note that the integral over that one coordinate gives

$$\int d\theta e^{-S} = 0. \tag{3.21}$$

This follows from the rule for Grassman integration, and the fact that Q is a symmetry of the path integral, so that S is independent of θ.

So the path integral is localized on Q-invariant configurations. In the B model these turn out to be simply the constant maps $\Sigma \to X$, obeying $dX = 0$. In this sense the string worldsheet reduces to a point on X, so the B model is "local," and its correlation functions are those of a field theory on X. In the A model, on the other hand, one finds the condition $\bar{\partial} X = 0$, which requires only that the map $\Sigma \to X$ be holomorphic; such a map is called a *worldsheet instanton*. In nontrivial instanton sectors the string worldsheet does not reduce to a point. The sum over instanton sectors is a complicated structure, non-local from the point of view of X, and therefore the A model does not reduce straightforwardly to a field theory on X.

From this point of view the Kähler structure dependence in the A model is easy to understand; it arises simply because each worldsheet instanton is weighted by the factor

$$e^{-\int_C k} \tag{3.22}$$

i.e. the area of the curve $C \subset X$ which is the image of the string worldsheet in X. The fact that the B model depends on the complex structure is more subtle, but it turns out that the B model computes quantities determined by the *periods* of the holomorphic 3-form Ω, which are sensitive to changes in the complex structure. Note that the complex structure moduli (the periods) are naturally complex numbers themselves, while the A model moduli (volumes of 2-cycles) are real numbers, so we seem to have a serious asymmetry between the two moduli spaces and hence between the A and B models; as we mentioned earlier, the symmetry between the two moduli spaces is restored by including an extra class $B \in H^2(X, \mathbb{R})$. When B is included, the weighting factor for a worldsheet instanton becomes

$$e^{-\int_C k + iB}. \tag{3.23}$$

We will combine k and B into a single modulus $t = k + iB \in H^2(X, \mathbb{C})$.

So the A and B models each depend on only "half" the moduli of X. In fact even more is true: in each case the partition function factorizes into a chiral and anti-chiral part, and if we focus on the chiral part, it *formally* depends only holomorphically on its moduli. One sees this by trying to compute the antiholomorphic derivative of the free energy, which amounts to inserting the operator corresponding to the anti-holomorphic deformation into the path integral. It turns out that this operator is Q-exact and so it is formally decoupled. Actually, this statement has to be modified slightly; because of the G^- insertions in the definition of the correlation function, what the Q-exactness really shows is that the integrand is a total

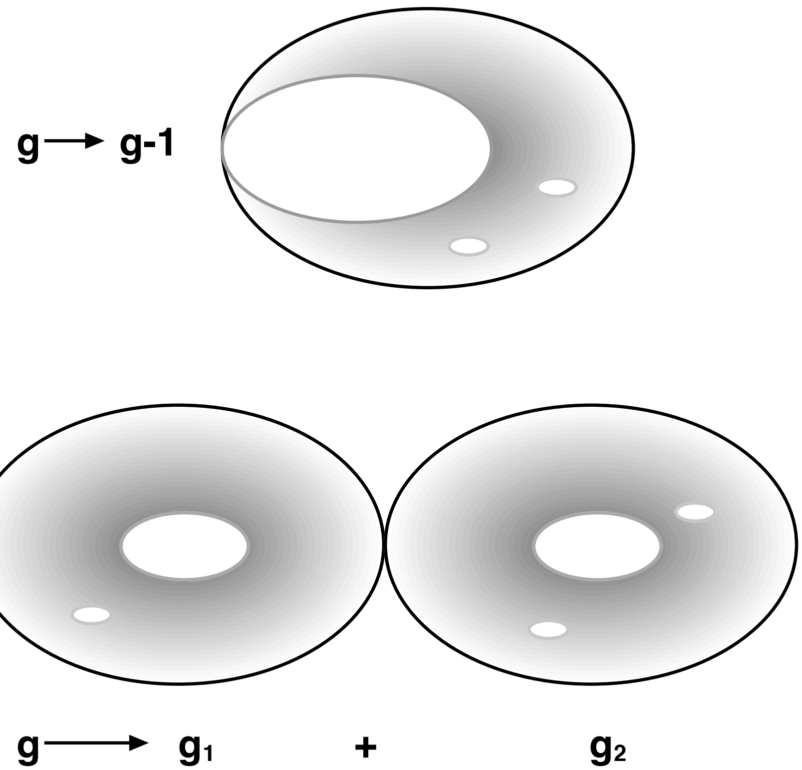

Figure 12: Degenerations of a Riemann surface of genus g, corresponding to boundary components of the moduli space $\mathcal{M}_g$.

derivative over $\mathcal{M}_g$; there can be contributions from the boundary of moduli space. Indeed there are such contributions, so the partition function is not quite holomorphic. Nevertheless the antiholomorphic dependence can be determined precisely; it is expressed in terms of a "holomorphic anomaly equation" derived in [9, 10]. Through the anomaly equation $\bar{\partial} F_g$ gets related to the $F_{g'}$ with $g' < g$, corresponding to boundaries of moduli space where some cycle of the genus g surface shrinks — see Figure 12.

The holomorphic anomaly is familiar to mathematicians, particularly in the case of the B model in genus 1, where it is related to the curvature of the determinant line bundle which obstructs the construction of a holomorphic $\det \bar{\partial}$ [13]. The full holomorphic anomaly including all genera can be interpreted as saying that the partition function transforms (in an appropriate sense) as a *wavefunction* [11, 12].

3.4 Genus zero

In our study of the topological string it is natural to begin with the simplest case, namely genus zero; it turns out that this case already contains a lot of interesting geometrical information. In the A model case one finds

$$F_0 = \int_X k \wedge k \wedge k + \sum_{n \in H_2(X,\mathbb{Z})} \sum_{m=1}^{\infty} d_n \frac{e^{-\langle n,t\rangle m}}{m^3}. \tag{3.24}$$

The first term is the classical contribution in the sense of worldsheet perturbation theory; it corresponds to the zero-instanton sector, where the string reduces to a point, and just gives the volume of X. The second term is more interesting since it contains information about worldsheet instantons. Its form is intuitive, at least if we focus on the $m = 1$ term: we sum over all $n \in H_2(X,\mathbb{Z})$, the homology classes of the image of the worldsheet, and weigh each instanton by the factor $e^{-\langle n,t\rangle}$ giving the complexified volume. The interesting information is then contained in the number d_n which counts the number of holomorphic maps in the homology class n.[12] The sum over m reflects the subtlety that one has to consider "multi-wrappings," in other words maps $\Sigma \to X$ which are m-to-one; these lead to a universal correction, which is independent of the particular X and just determined by the geometry of maps $S^2 \to S^2$. It is captured by the factor $1/m^3$.

To write the B model partition function we introduce a convenient coordinate system for the complex moduli space. To describe it we first discuss the space $H^3(X,\mathbb{C})$, which is decomposed into

$$\begin{array}{ccccccccc} H^3 & = & H^{3,0} & \oplus & H^{2,1} & \oplus & H^{1,2} & \oplus & H^{0,3}, \\ h^3 & = & 1 & + & h^{2,1} & + & h^{2,1} & + & 1. \end{array} \tag{3.25}$$

Therefore $H_3(X,\mathbb{R})$ has real dimension $2h^{2,1} + 2$. Now we choose a symplectic basis of $H_3(X,\mathbb{Z})$; this amounts to choosing 3-cycles A^i, B_j, for $i = 1,\dots,h^{2,1}+1$ and $j = 1,\dots,h^{2,1}+1$, with intersection numbers

$$A^i \cap A^j = 0, \quad B_i \cap B_j = 0, \quad A^i \cap B_j = \delta^i_j. \tag{3.26}$$

Note that $h^{2,1}(X)$ is the complex dimension of the moduli space of complex structures (this identification is obtained by using the holomorphic 3-form to convert Beltrami differentials

[12] Sometimes this number needs some extra interpreting from the mathematical point of view: it could be that the holomorphic maps are not isolated, so that there is a whole moduli space of such maps. Nevertheless, the virtual or "expected" dimension of this moduli space is always zero (for a Calabi-Yau threefold); roughly this means that one can define a sensible "number of maps" even when the actual dimension happens to be nonzero. The index computation showing that the virtual dimension vanishes when $d = 3$ is in fact isomorphic to the charge-conservation computation which singled out $d = 3$.

to $(2,1)$-forms.) Indeed, we can get coordinates on the moduli space by defining

$$X^i = \int_{A^i} \Omega. \tag{3.27}$$

Actually this gives $h^{2,1}+1$ complex coordinates corresponding to the $h^{2,1}+1$ A cycles, one more than needed to cover the moduli space. The reason for this overcounting is that Ω is not quite unique for a given the complex structure — it is unique only up to an overall complex rescaling, so from (3.27), the X^i are also ambiguous up to an overall rescaling. Thus we have the right number of coordinates after accounting for this rescaling; and indeed the periods over the A cycles do determine the complex structure. Hence the X^i give homogeneous coordinates on the moduli space.

One could ask, what about the periods over the B cycles? Writing[13]

$$F_i = \int_{B_i} \Omega \tag{3.28}$$

it follows from the above that they must be expressible in terms of the A periods,

$$F_i = F_i(X^j). \tag{3.29}$$

(Of course, since our choice of symplectic basis was arbitrary, and in particular we could have interchanged the A and B cycles, one could equally well write $X^i = X^i(F_j)$.)

To describe the free energy we need one more fact, namely the statement of "Griffiths transversality." Recall that $\Omega \in H^{3,0}$. Now work in a local complex coordinate system in which $\Omega = f(z)\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3$, and consider a variation μ of complex structure, which changes the local complex coordinates by $\mathrm{d}z_i \mapsto \mathrm{d}z_i + \mu_i^{\bar{j}} \mathrm{d}\bar{z}_j$. Then expanding in $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ one sees that to first order in μ, the variation of Ω satisfies $\delta\Omega \in H^{3,0} \oplus H^{2,1}$, and the second-order variations similarly have $\delta\delta\Omega \in H^{3,0} \oplus H^{2,1} \oplus H^{1,2}$. This implies

$$\int_X \delta\Omega \wedge \Omega = 0, \tag{3.30}$$

$$\int_X \delta\delta\Omega \wedge \Omega = 0, \tag{3.31}$$

which in turn implies that

$$\frac{\partial}{\partial X^i} F_j = \frac{\partial}{\partial X^j} F_i, \tag{3.32}$$

so that the F_i can be integrated:

$$F_i = \frac{\partial}{\partial X^i} F. \tag{3.33}$$

[13] There is an unfortunate clash of notation here; the F_i we define here are *not* the genus i free energy, although below we will consider the genus 0 free energy, which we will write simply as F!

The F so defined is the genus zero free energy of the B model. Strictly speaking, F is not quite defined on the complex moduli space, because it depends on the choice of the overall scaling of Ω; under $\Omega \mapsto \xi\Omega$ one has $F \mapsto \xi^2 F$. So F is better described as a section of a line bundle over the moduli space.[14]

Note that in contrast to the A model, which involved an infinite sum over worldsheet instantons and involved the integral coefficients d_n, the B model free energy is determined purely by "classical" geometry and seems to have no underlying integral structure. In this sense one could say that the B model is easy to compute, while the A model is hard. (On the other hand, it is the A model partition function which is easier to *define*, at least formally — it just counts holomorphic maps!)

4 Computing the topological amplitudes

4.1 Mirror symmetry

In the last section we concluded that while the A model computes some interesting geometric information, it is the B model which is easier to compute. Remarkably, it is possible to exploit the simplicity of the B model to make computations in the A model! Namely, the A model on a Calabi-Yau space M is often equivalent to a B model on a "mirror" Calabi-Yau space W. Therefore computations of the periods of W can be exploited to count holomorphic curves in M.

To understand how such a surprising duality could be true, we consider an example which is in some sense underlying the whole phenomenon: string theory on a circle S^1 of radius R. The spectrum of states of this theory has one obvious quantum number, namely the number w of times the string is wound around S^1. It also has a second quantum number n corresponding to the momentum of the center of mass of the string going around the circle; this momentum is quantized in units of $1/R$, as is familiar from point particle quantum mechanics in compact spaces. The contribution to the energy of a state from these two quantum numbers is (in units with $\alpha' = 1$)

$$E_{n,w} = wR + \frac{n}{R}. \tag{4.1}$$

Note that the set of possible $E_{n,w}$ is invariant under the interchange $R \leftrightarrow 1/R$ — namely $E_{n,w}$ at radius R is the same as $E_{w,n}$ at radius $1/R$! This is the first clue that this interchange

[14]Even this more refined description is still a little misleading, because F also depends on the choice of A and B cycles, i.e. the choice of a coordinate system. If one makes a symplectic transformation of the basis, F transforms by an appropriate Legendre transform.

might be a symmetry of the full string theory; indeed, there is such a symmetry, called "T-duality." It can be rigorously understood from the worldsheet point of view, but as we will see, it has deep consequences.

Indeed, T-duality implies mirror symmetry. The simplest example is one we already mentioned in Section 1.2.1. Namely, given a rectangular torus T^2 with radii R_1, R_2 and defining

$$A = \mathrm{i} R_1 R_2, \tag{4.2}$$

$$\tau = \mathrm{i} R_2 / R_1, \tag{4.3}$$

taking $R_1 \mapsto 1/R_1$ is equivalent to swapping $A \leftrightarrow \tau$. So in this case M and its mirror W are both T^2, but with different metrics, i.e. different values of the moduli. Anyway, given that the physical string has this T-duality symmetry, one could then ask how it gets implemented in the topological theory. Since it exchanges complex and Kähler moduli it would be natural to conjecture that it exchanges the A and B models, and this is indeed the case; the A model with Kähler modulus A computes exactly the same quantity as the B model with complex modulus $\tau = A$.

Since we are looking at T^2 here, which has complex dimension $1 \neq 3$, most of the topological string is trivial as we explained before. However, one can still look at the one-loop free energy F_1, and mirror symmetry turns out to be an interesting statement already here: namely, it turns out that the B model at one loop computes the determinant of the $\bar{\partial}$ operator acting on T^2, in keeping with the general principle that the B model has to do with *local* expressions on the target space. This determinant gives the Dedekind η function. On the other hand, the A model at one loop counts maps $T^2 \to T^2$, but it should also give the η function; this gives a natural explanation of the integrality of the coefficients in the expansion of η as a function of $e^{2\pi i \tau}$. Namely, the term $e^{2\pi i n \tau}$ gets related to e^{-nA} by the mirror map, and from the A model point of view the coefficient of e^{-nA} counts maps which wrap T^2 over itself n times.

Now what about the case of maximal interest, namely Calabi-Yau threefolds? Here also one might expect a "mirror" duality. Indeed, this duality was conjectured before a single example was known, on the basis of lower-dimensional examples like the one discussed above, and also because from the point of view of the $\mathcal{N} = (2, 2)$ algebra the difference between A and B models is purely a matter of convention — considered abstractly, the SCFT has no way of knowing whether it is the A model or the B model. By now many examples of mirror pairs are known, both compact and non-compact, and the symmetry has been proven in various ways for various classes of examples.

Here we sketch a derivation given in [14] which captures the essential physical picture. We begin with a toric Calabi-Yau threefold M and realize it concretely via the gauged linear sigma model of [6]. Recall that this model is constructed from a set of chiral superfields Z_i representing the homogeneous coordinates of M, and that its space of vacua is M itself. To get the mirror of M one splits each Z_i into its modulus and phase as we did before when discussing the toric diagram,

$$Z_i \to (|Z_i|^2, \theta_i), \tag{4.4}$$

and then performs T-duality on the circle coordinatized by θ_i. The T-duality gives a new dual periodic coordinate ϕ_i, and we organize this coordinate together with $|Z_i|^2$ into a new "twisted chiral" superfield

$$Y_i = |Z_i|^2 + i\phi_i. \tag{4.5}$$

Crucially, the dual description in terms of the Y_i also turns out to have a superpotential, generated by instantons of the linear sigma model:

$$W(Y) = \sum_i e^{-Y_i}. \tag{4.6}$$

Finally, the D-term "moment map" constraints of the gauged linear sigma model,[15]

$$\sum_i Q_i |z_i|^2 = t, \tag{4.7}$$

are converted to holomorphic constraints in the dual model,

$$\sum_i Q_i Y_i = t. \tag{4.8}$$

The three equations (4.6), (4.7), (4.8) contain all the information about the dual theory, as we now see in an example: consider the local $\mathbb{CP}^2$ geometry $\mathcal{O}(-3) \to \mathbb{CP}^2$ which we discussed before. This geometry involves four chiral superfields with charges

$$\begin{array}{rcccc} Z = & (Z_0, & Z_1, & Z_2, & Z_3), \\ Q = & (-3, & 1, & 1, & 1). \end{array} \tag{4.9}$$

The corresponding D-term constraint

$$-3|z_0|^2 + \sum_i |z_i|^2 = t \tag{4.10}$$

[15] For simplicity we are writing (4.7), (4.8) in the case where there is just a single $U(1)$ gauge group, hence a single D-term constraint and a single Kähler modulus. The general case just has more indices.

becomes in the dual model

$$-3Y_0 + \sum_{i=1}^{3} Y_i = t, \tag{4.11}$$

and the superpotential is

$$W = \sum_{i=0}^{3} e^{-Y_i}. \tag{4.12}$$

It is convenient to make the change of variables

$$y_i = e^{Y_i/3}. \tag{4.13}$$

Then, after eliminating Y_0 using (4.11), we are left with the superpotential

$$W = y_1^3 + y_2^3 + y_3^3 + e^{t/3} y_1 y_2 y_3. \tag{4.14}$$

So we might expect that the space of vacua in the dual theory will be given the locus $W = 0$. This is correct, but we have to remember that the change of variables (4.13) is not quite one-to-one; the y_i are ambiguous by cube roots of unity, and therefore we have to divide out by the group $\mathbb{Z}_3^2$ which multiplies the y_i by cube roots of unity while leaving W invariant. After so doing we obtain the mirror to the local $\mathbb{CP}^2$ geometry.

Looking at $W = 0$ one notices that, if the y_i are considered as homogeneous coordinates in projective space, it is in fact the equation describing an elliptic curve. Passing to inhomogeneous coordinates we could rewrite it as an equation in two variables,

$$F(x, z) = x^3 + z^3 + 1 + e^{t/3} xz = 0. \tag{4.15}$$

Indeed, the mirror geometry in this case is effectively an elliptic curve rather than a Calabi-Yau threefold, in the sense that the B model partition function can be computed from the geometry of the elliptic curve. This is a common phenomenon when computing mirrors of noncompact Calabi-Yaus. Nevertheless, the usual statement of mirror symmetry as we phrased it above requires a threefold mirror to a threefold; to make contact with that formulation we should add two extra variables u, v which enter in a rather trivial way:

$$F(x, z) = uv. \tag{4.16}$$

These two variables just contribute a quadratic term to the superpotential W and do not couple to the interesting part of the geometry.

We can derive mirror symmetry for compact Calabi-Yaus with linear sigma model realizations in a similar way. Recall the example of the quintic threefold; this space is obtained by starting with the linear sigma model for $\mathcal{O}(-5) \to \mathbb{CP}^4$ and then introducing a superpotential which reduces the space of vacua to the quintic hypersurface in $\mathbb{CP}^4$. Temporarily ignoring this superpotential and repeating the steps above we get the manifold defined by

$$W = y_1^5 + y_2^5 + y_3^5 + y_4^5 + y_5^5 + e^{t/5} y_1 y_2 y_3 y_4 y_5 \tag{4.17}$$

modulo a $\mathbb{Z}_5^4$ symmetry multiplying the y_i by fifth roots of unity. Now what happens if we include the superpotential? Remarkably it turns out that the only effect is to change the fundamental variables of the theory to the y_i instead of Y_i. (One might think that what is the "fundamental variable" is a matter of terminology, but concretely, it affects the measures of integration one uses when computing the B model periods.)

Finally, we briefly mention another point of view on the mirror symmetry in the compact case, which leads more directly to the same conclusion: namely, it was observed in [15] that the A model on the quintic threefold is in fact equivalent to the A model on a weighted *super* projective space $\mathbb{CP}^{1,1,1,1,1|5}$, which is compact but nevertheless torically realized without the need for a superpotential. T-dualizing on phases in this super context then leads to a new derivation of the mirror to the quintic threefold [16].

4.2 Holomorphy and higher genera

So far we just discussed computation of topological amplitudes at genus zero. More generally we can compute all the F_g using the fact that they depend only holomorphically on moduli. Actually, as we mentioned earlier, this statement is not quite true; but the antiholomorphic dependence is completely determined by the anomaly equation of [10] and does not qualitatively affect the discussion to follow. So we can think of F_g as a holomorphic section of a line bundle over the moduli space. Such objects are highly constrained once their boundary behavior is determined — recall that a bundle over a compact space has only finitely many sections. The Calabi-Yau moduli spaces under consideration are also compact, or can be compactified by adding some points at infinity, where the order of the singular behavior can be constrained by geometrical considerations; hence the F_g are basically determined by holomorphy up to a finite-dimensional ambiguity at each g. With some work this ambiguity can also be fixed, leading to a practical method for computing the F_g, which has been applied to degrees and genera of order 10 [**?**].

4.3 Target space approach

There is also a target space approach to computing the topological string partition function [17]. Namely, suppose we study the A model on a non-compact threefold which has a toric realization as we discussed above. Then by mirror symmetry we obtain the B model on a Calabi-Yau of the form

$$F(x, z) = uv \tag{4.18}$$

with the corresponding holomorphic 3-form

$$\Omega = \frac{\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}z}{u}. \tag{4.19}$$

The u and v variables are basically playing a trivial role here; the important part of the geometry is captured by the equation $F(x, z) = 0$, which characterizes the degeneration locus of the fiber spanned by u and v. Contour-integrating in u reduces Ω to $\omega = \mathrm{d}x \wedge \mathrm{d}z$. So we have an algebraic curve $F(x, z) = 0$ embedded in the (x, z) space equipped with the two-form ω. What are the symmetries of this structure? If F were identically zero, then we would just have the group of ω-preserving diffeomorphisms, which form the so-called "W_∞" symmetry. This infinite-dimensional symmetry is extremely powerful. Indeed, even when $F \neq 0$ and the W_∞ symmetry is spontaneously broken, it nevertheless generates Ward identities which act on the possible deformations of F. But these deformations exactly correspond to complex structure deformations of the Calabi-Yau geometry, which are the objects of study in the B model! It turns out that the Ward identities are sufficient to completely determine the B model partition function at all genera (and hence the A model partition function on the original toric threefold) — see [17].

4.4 Large N dualities

Yet another approach to computing the F_g depends on the notion of "large N duality." Such dualities have played a starring role in the physical string theory over the last few years [18, 19]; as it turns out, they are equally important in the topological string [20, 21].

4.4.1 D-branes in the topological string

These dualities relate open string theory in the presence of D-branes to closed string theory in the gravitational background those D-branes produce; so in order to discuss their topological realization, we have to begin by explaining the notion of D-brane in the topological string.

From the worldsheet perspective, a D-brane simply corresponds to a boundary condition which can be consistently imposed on worldsheets with boundaries. In the topological case we require that the boundary condition preserves the BRST symmetry. In the A model this condition implies that the boundary should be mapped into a Lagrangian submanifold L

of the target Calabi-Yau X [22] (recall that "Lagrangian" means that the dimension of L is half that of X and the Kähler form ω vanishes when restricted to L). This L should be thought of as a real section of X — a typical 1-dimensional model is the upper half-plane, which ends on the real axis L.

(As an aside, it is interesting that the branes which appear in the A model are wrapping Lagrangian cycles, which are 3-cycles for which the volume is naturally measured by the holomorphic 3-form Ω — the natural object in the B model! Similarly, in the B model the branes turn out to wrap holomorphic cycles, whose volume is measured by the A model field k. This crossover between the A and B models may be a hint of a deeper relation, which is currently under investigation.)

Now how do the topological D-branes affect the closed string background? The answer to this question will be crucial since it defines the dual geometry. In the physical superstring D-branes are sources of flux; in the A or B model topological string the flux in question should be the Kähler 2-form or holomorphic 3-form respectively. More precisely, consider a Lagrangian subspace L. Since the total dimension is 6, we can consider a 2-cycle C which "links" L, similar to the way two curves can "link" one another in dimension 3. This means that $C = \partial S$ for some 3-cycle S which intersects L once; so C is homologically trivial in the full geometry of X, although it becomes nontrivial if we consider instead $X \setminus L$. Because C is homologically trivial we must have $\int_C k = 0$ in the original closed string geometry of X. Then the effect of wrapping N branes on L is to create a flux of the Kähler form through C, namely

$$\int_C k = N g_s. \tag{4.20}$$

This can be understood by saying that the branes act as a δ-function source for k, i.e., the usual closed string relation $\mathrm{d}k = 0$ is replaced by

$$\mathrm{d}k = N g_s \delta(L). \tag{4.21}$$

Similarly, a B model brane on a 2-cycle induces a flux of Ω over the linking 3-cycle. Note that this phenomenon actually suggests a privileged role for 2-cycles; we could also have B model branes on 0, 4, or 6-cycles, but these branes do not induce gravitational backreaction since there is no candidate field for them to source.

What about the target space description of the topological D-brane? In the case of physical D-branes we know that the open strings produce a gauge theory on the brane in the low energy limit; namely, in the case of a stack of N coincident branes, the fact that strings can end on any of the N branes leads to a $U(N)$ gauge theory. See Figure 13.

In the topological A model one can work out the exact open string field theory; it is again a gauge theory, but this time a topological gauge theory, namely Chern-Simons. To see this

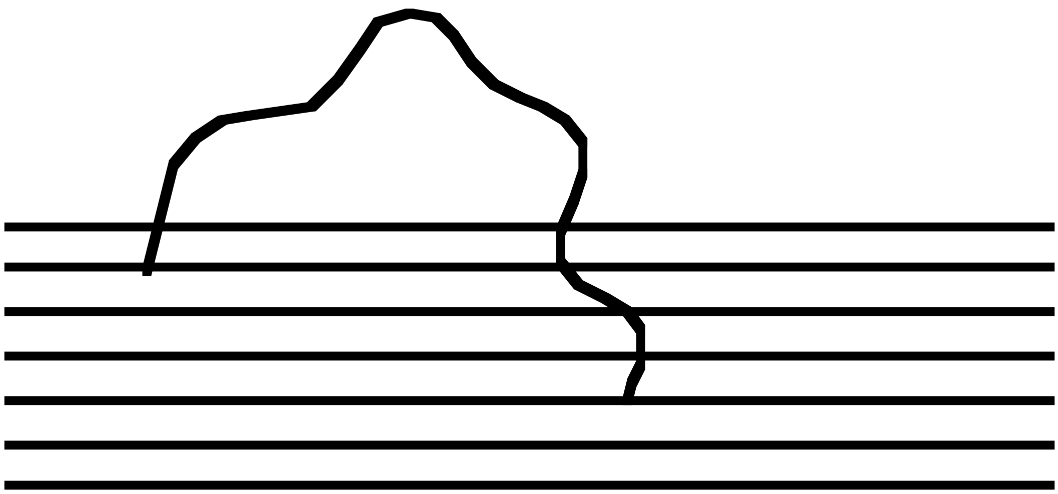

Figure 13: A stack of N branes carries a $U(N)$ gauge symmetry; the fundamental and antifundamental gauge indices arise from strings which can end on any of the N branes.

we first note that our construction of the topological string (and specifically its coupling to gravity) was modeled on the bosonic string, and therefore it is reasonable that the open string field theory should also be the one that appeared for the open bosonic string. In the bosonic string it was shown in [23] that the OSFT is an abstract version of Chern-Simons, written

$$S = \int A * QA + \frac{2}{3} A * A * A. \tag{4.22}$$

Re-running those arguments in the topological context, using the dictionary $Q \leftrightarrow \mathrm{d}$, shows that this abstract Chern-Simons is in this case simply the standard Chern-Simons action [22] — possibly corrected by terms involving holomorphic instantons ending on L. (In fact, one might ask how the appearance of the Chern-Simons action is consistent with the localization of the open string path integral on holomorphic configurations; the answer is that the localization has to be interpreted carefully because of the non-compactness of the field space. One has to include contributions from "degenerate instantons" in which the Riemann surface has collapsed to a Feynman diagram, and these diagrams precisely account for the Chern-Simons action.) In some interesting cases there are no holomorphic instantons and we just get pure Chern-Simons; this happens in particular in the case of branes wrapping the S^3 in the deformed conifold T^*S^3.

One can similarly consider the open string field theory on a B model brane. In the case of a brane which wraps the full Calabi-Yau threefold X, one gets a holomorphic version of

Chern-Simons, with action [23]

$$\int \Omega \wedge \mathrm{Tr}(A\bar{\partial}A + \frac{2}{3}A^3), \tag{4.23}$$

where A is a $u(N)$-valued $(0,1)$ form on X, which we are combining with the $(3,0)$ form Ω so that the full action is a $(3,3)$ form as required. Starting from (4.23), one can also obtain the action for B model branes which wrap holomorphic 0,2,4-cycles inside X, by realizing such lower-dimensional branes as defects in the gauge field on a brane that fills X.

4.4.2 The geometric transition

After these preliminaries on branes in the topological string, we are ready to use them to compute closed string amplitudes. The simplest example is the A model on T^*S^3. This geometry is uninteresting from the point of view of the closed A model, since it has no 2-cycles and hence no Kähler moduli; but it contains the Lagrangian 3-cycle S^3 on which we can wrap branes, obtaining the open string partition function $Z(g_s, N)$ for N branes. As we discussed in the last section, this $Z(g_s, N)$ is nothing but the partition function of $U(N)$ Chern-Simons theory on S^3, with level $k = 2\pi i/(k+N)$; this partition function is readily computable in practice. Furthermore, the effect of the branes on the closed string geometry is to create a flux Ng_s on the S^2 which links S^3. Now, a la AdS/CFT, let us try to describe this geometry in terms of a background without branes. There is an obvious guess for the answer: as we discussed earlier, in addition to the deformed conifold which has a nontrivial S^3 at its core, there is also the resolved conifold which has a nontrivial S^2, and both geometries look the same at long distances. So it is natural to conjecture that the dual geometry is the resolved conifold, where the nontrivial S^2 has volume $t = Ng_s$ [20]. In the resolved conifold there are no branes anymore — we have just closed strings — and indeed there is not even a nontrivial cycle where the branes could have been wrapped! The passage from one geometry to the other is referred to as a "geometric transition."

The conjecture that the open A model on the deformed conifold with N branes should be equivalent to the closed A model on the resolved conifold with $t = Ng_s$ leads immediately to a prediction for the partition function of the latter: namely, from the Chern-Simons side one expects

$$Z(g_s, t) = \prod_{n=1}^{\infty} (1 - q^n Q)^n, \tag{4.24}$$

where $q = e^{-g_s}$ and $Q = e^{-t}$. Note that this expansion has integral coefficients, which seems remarkable from the point of view of the closed string, and might make us wonder whether the closed string partition function has an interpretation as the answer to some counting

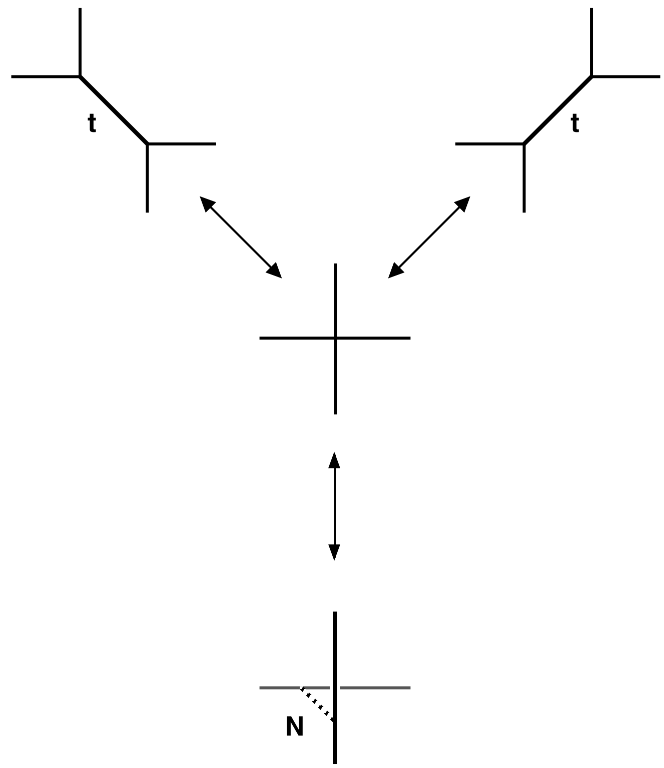

Figure 14: The geometric transition between the resolved conifold with Kähler parameter t (above) and deformed conifold with N branes (below).

problem. We will understand this structure later when we discuss the physical applications of the topological string.

The geometric transition we just discussed can be summarized in a toric picture as shown in Figure 14.

One can also use open/closed duality to compute the partition function in more complicated geometries [24]. For example, consider the local $\mathbb{CP}^2$ geometry. As shown in Figure 15, we can obtain this geometry as the $t_i \to \infty$ limit of a geometry with three compact $\mathbb{CP}^1$'s, and each of these $\mathbb{CP}^1$'s in turn can be obtained by a geometric transition from an S^3. In this way the closed string on local $\mathbb{CP}^2$ is related to the open string on a geometry with three S^3's, each supporting a stack of branes. Naively, then, we would expect that the closed string partition function would be the product of three copies of the Chern-Simons partition function. However, we have to remember that the Chern-Simons theory is corrected by worldsheet instantons. One can show that the only instantons which contribute are ones in which the worldsheets form tubes connecting two S^3's, as shown in Figure 16. Each such

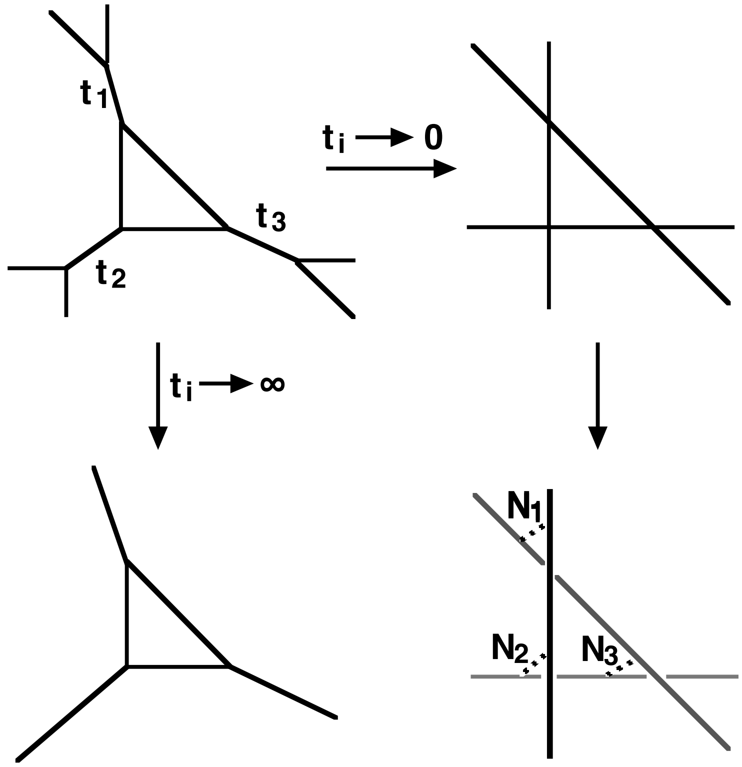

Figure 15: The geometric transition and $t_i \to \infty$ limit relating local $\mathbb{CP}^2$ to a rigid geometry with $N_i \to \infty$ branes.

tube ends on an unknotted circle in S^3; so in a generic instanton configuration each S^3 has two such circles on it, and a careful analysis shows that they are in fact linked, forming the "Hopf link." One therefore has to compute the Chern-Simons partition function including an operator associated to the link, given by a sum of link invariants [25]. The full partition function at all genera is a sum over representations of $U(N)$:

$$Z = \sum_{R_1,R_2,R_3} e^{-t|R_1|} S_{R_1R_2} e^{-t|R_2|} S_{R_2R_3} e^{-t|R_3|} S_{R_3R_1}, \tag{4.25}$$

where $S_{RR'}$ is the Chern-Simons knot invariant of the Hopf link with representations R and R' on the two circles, as defined in [26], and $|R|$ is the number of boxes in a Young diagram representing R.

4.5 The topological vertex

Although the geometric transitions we described above lead to an all-genus formula for the A model partition function in the local $\mathbb{CP}^2$ geometry, the method of computation is

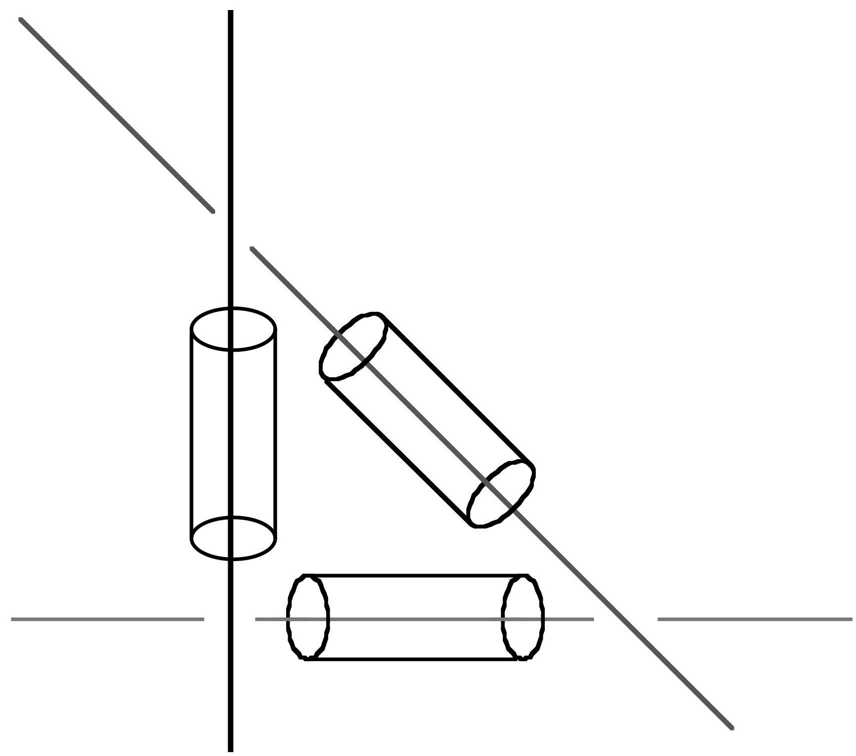

Figure 16: Worldsheet instantons, with each boundary on an S^3, which contribute to the A model amplitudes after the transition, or dually, to the A model amplitudes on local $\mathbb{CP}^2$.

somewhat unsatisfactory: we obtained local $\mathbb{CP}^2$ only after taking the $t_i \to \infty$ limit of a more complicated geometry. One might have hoped for a more intrinsic method of computation. Indeed there is such a method, and it generalizes to arbitrary toric diagrams, whether or not they come from geometric transitions! This method exploits the similarity between the toric diagram (with fixed Kähler parameters) and a Feynman diagram with trivalent vertices and fixed Schwinger parameters. Namely, one can define a "topological vertex," $C_{R_1 R_2 R_3}(g_s)$, depending on three Young diagrams R_1, R_2, R_3 and on the string coupling g_s [27]. See Figure 17. Then the full partition function is obtained by assigning a Young diagram R to each internal edge of the toric diagram, with a propagator $e^{-t|R|+mC_2(R)}$, and a factor $C_{R_1 R_2 R_3}(g_s)$ to each vertex. (The integer m appearing in the propagator is related to the relative orientation of the 2-surfaces on which the propagator ends.)

Of course, the actual vertex $C_{R_1 R_2 R_3}(g_s)$ is extremely complicated! It was originally determined in [27] using Chern-Simons theory along the lines discussed above. Since then two other methods of computing the vertex have appeared. One way uses the W_∞ symmetry

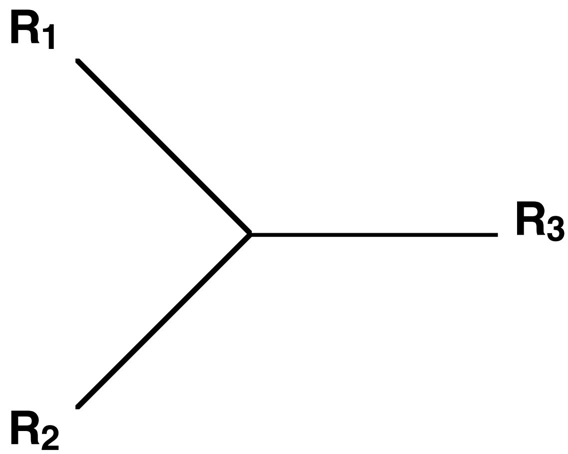

Figure 17: The topological vertex, which assigns a function of g_s to any three Young diagrams R_1, R_2, R_3.

of the target space in the mirror B model [17], as we discussed above.

One can also obtain the vertex by a direct A model closed string target space computation [28, 29, 30]. Namely, the target space description of the A model is a theory of "Kähler gravity" [31], which roughly sums over Kähler geometries with the weight $e^{-\int k^3/g_s^2}$. In fact, we can describe exactly which Kähler geometries contribute to the A model sum that gives the A model partition function, at least if we restrict to studying quantities which are torically invariant in a suitable sense. For example, consider the case of the A model on the non-compact Calabi-Yau $\mathbb{C}^3$. In this case one can obtain a new Kähler geometry which is still toric by blowing up the origin; this replaces that point by a $\mathbb{CP}^2$ as shown in Figure 18. Of course, this new geometry is not Calabi-Yau; the only Calabi-Yau geometry which is asymptotically $\mathbb{C}^3$ is $\mathbb{C}^3$ itself. Nevertheless, it should be included in the target space A model sum; this is not unexpected, since a theory of quantum gravity should sum over off-shell configurations as well as on-shell ones. One can also do more complicated blow-ups. Such blow-ups are more difficult to describe in words, but their algebraic description is straightforward: the possible blow-ups correspond to toric ideals in the space of algebraic functions on $\mathbb{C}^3$, i.e. invariant ideals in the ring $\mathbb{C}[X, Y, Z]$ of polynomials in three variables. Such ideals correspond to 3-dimensional Young diagrams D, or equivalently to configurations of a melting crystal; this was the point of view taken in [28]. (The simplest blow-up which we discussed above corresponds to the Young diagram with a single box, or the ideal $\langle X, Y, Z \rangle$.)

The weight $e^{-\int k^3/g_s^2}$ for such a geometry obtained by blowing up an ideal is simply $q^{|D|}$, where $q = e^{-g_s}$ and $|D|$ is the number of boxes of the 3-dimensional Young diagram D, or

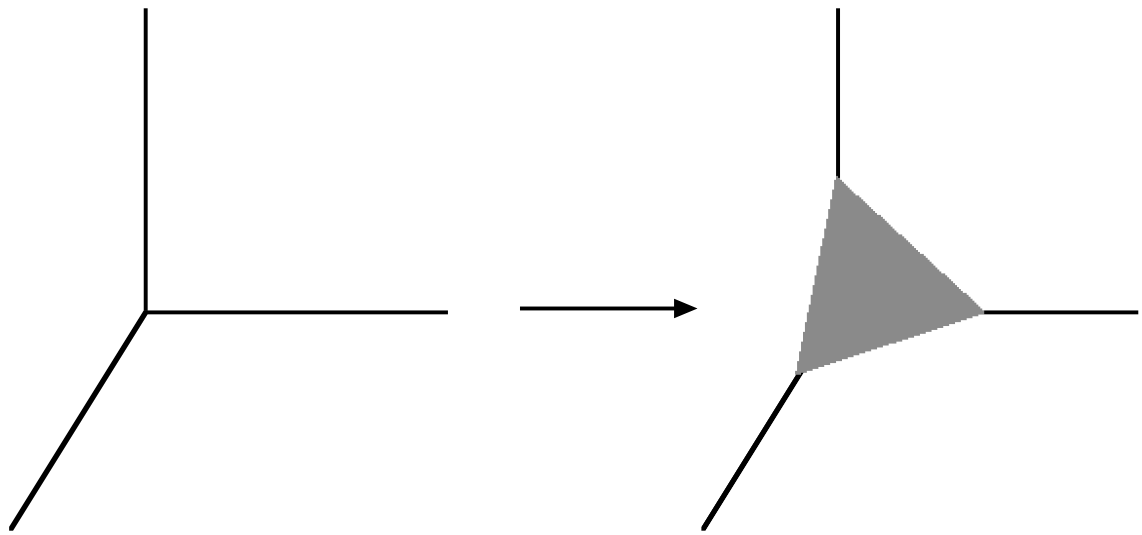

Figure 18: Blowing up the origin in $\mathbb{C}^3$ gives a new geometry which is not Calabi-Yau but still contributes to the target space sum in the A model.

equivalently the codimension of the corresponding ideal. The sum over all such diagrams with this weight gives the A model partition function on $\mathbb{C}^3$,

$$Z_{\mathbb{C}^3} = \sum_D q^{|D|} = \prod_{i=1}^{n} (1 - q^n)^{-n}. \tag{4.26}$$

This is the special case of the topological vertex where all three representations R_1, R_2, R_3 are trivial. More generally one can consider infinite 3-d Young diagrams, which asymptote to fixed 2-d diagrams R_1, R_2, R_3 along the x, y, z directions; in this case the sum over diagrams gives the full topological vertex!

5 Physical applications

So far we have mostly discussed the topological string in its own right. Now we turn to its physical applications. At first it might be a surprise that there are any physical applications at all, but they do exist; speaking broadly, the reason for this is that the topological string is a *localized* version of the physical string, i.e. it receives contributions only from special path-integral configurations as we discussed above, and there are some "BPS" observables of the physical string which are also localized on the same special configurations.

The main examples which have been explored so far are summarized in the table below:

physical theory	physical observable	topological theory
$\mathcal{N}=2$, $d=4$ gauge theory	prepotential	A model
$\mathcal{N}=1$, $d=4$ gauge theory	superpotential	B model with branes
spinning black holes in $d=5$	BPS states	A model
charged black holes in $d=4$	BPS states	nonperturbative?

Now we will discuss these applications in turn.

5.1 $\mathcal{N}=2$ gauge theories

One area in which the topological string connects to ordinary physical theories is in the context of $\mathcal{N}=2$ gauge theories in $d=4$. To understand this connection we begin by discussing the physical theory obtained by compactifying the Type II string on a Calabi-Yau X. The curvature of X breaks 3/4 of the supersymmetry, leaving 8 supercharges which make the $\mathcal{N}=2$ algebra in $d=4$; the massless field content in $d=4$ can be organized into multiplets of $\mathcal{N}=2$ supergravity as follows:

	vector	hyper	gravity
IIA on X	$h^{1,1}(X)$	$h^{2,1}(X)+1$	1
IIB on X	$h^{2,1}(X)$	$h^{1,1}(X)+1$	1

The topological string computes particular F-terms in the effective $d=4$ action. They can be written conveniently in terms of the $\mathcal{N}=2$ graviphoton multiplet, which is a chiral superfield $\mathcal{W}_{\alpha\beta}$ with lowest component $F_{\alpha\beta}$. [16] Namely, forming the combination

$$\mathcal{W}^2 = \mathcal{W}_{alpha\beta}\mathcal{W}_{alpha'\beta'}\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}, \tag{5.1}$$

the F-terms in question can be written as

$$\int \mathrm{d}^4x \int \mathrm{d}^4\theta F_g(X^I)(\mathcal{W}^2)^g. \tag{5.2}$$

The crucial link between physical and topological strings here is as follows: the $F_g(X^I)$ which appears in (5.2) is precisely the genus g topological string free energy, written as a function of the vector multiplets X^I (so if we study Type IIB then the F_g appearing is the B model free energy, since the vector multiplets in that case parameterize the complex deformations, while for Type IIA it is the A model free energy that appears.) Arguments for this statement can be found in [32, 10]; one can also see it more directly using the Berkovits hybrid formalism.

[16] Here the "graviphoton" F is the field strength for the $U(1)$ vector in the supergravity multiplet, and α, β are spinor indices labeling the self-dual part of the full field strength $F_{\mu\nu}$, i.e. $F_{\mu\nu} = F_{\alpha\beta}(\gamma_\mu)^{\alpha\dot\sigma}(\gamma_\nu)^{\beta}_{\dot\sigma} + F_{\dot\alpha\dot\beta}(\gamma_\mu)^{\dot\alpha}_{\sigma}(\gamma_\nu)^{\dot\beta\sigma}$.

It is noteworthy that each F_g contributes to a different term in the effective action and hence to a different physical process. To see this more clearly we can expand (5.2) in components; one term which appears is (for $g > 1$)

$$\int \mathrm{d}^4 x F_g(X^I)(R^2 F^{2g-2}), \tag{5.3}$$

so $F_g(X^I)$ contributes a gravitational correction to the amplitude for scattering of $2g-2$ graviphotons. In the application to $\mathcal{N}=2$ gauge theory we will mostly be interested in F_0, which gets identified with the prepotential of the gauge theory.

Now let us focus on the specific geometries which will lead to interesting $\mathcal{N}=2$ gauge theories. In order to decouple gravity we should consider a non-compact Calabi-Yau space. The simplest example is an ALE space; these are four-dimensional Calabi-Yaus obtained as $\mathbb{C}^2/G$, where G is a finite subgroup of $SU(2)$ acting linearly on $\mathbb{C}^2$. More precisely, the ALE space is not quite $\mathbb{C}^2/G$; that quotient has a singularity at the origin, and one obtains the ALE space by resolving that singularity, introducing a number of $\mathbb{CP}^1$'s localized near the origin. For each such $\mathbb{CP}^1$ obtained by resolving the singularity there is a Kähler parameter t_i for its size; in the limit $t_i \to 0$ the metric reduces to that of the singular space $\mathbb{C}^2/G$. In this sense one can think of the singularity of $\mathbb{C}^2/G$ as containing a number of zero size $\mathbb{CP}^1$'s. Then considering Type II string theory on $\mathbb{C}^2/G$ one obtains a gauge theory in six dimensions; the massless gauge bosons arise from D2-branes which wrap around these zero size $\mathbb{CP}^1$'s, and the particular gauge group we get depends on the group G. The simplest example is $G = \mathbb{Z}_n$, which gives $SU(n)$ gauge symmetry in the six-dimensional gauge theory, but one can also get $SO(2n)$ or E_6, E_7, E_8; this is called the "ADE classification" of finite subgroups of $SU(2)$.

But $\mathbb{C}^2/G$ is not quite the example we want; we want to get down to $d=4$ rather than $d=6$, and we also want to get down to 8 supercharges rather than 16. These goals can be simultaneously accomplished by fibering $\mathbb{C}^2/G$ over a genus g Riemann surface Σ_g; this can be done in a way so that the resulting six-dimensional space is a Calabi-Yau threefold X, and the Type II string on X gives an $\mathcal{N}=2$ theory with gauge group determined by G and with g adjoint hypermultiplets [33]. (The origin of these hypermultiplets can be understood by starting with the gauge theory in $d=6$ and compactifying it on Σ_g; then the electric and magnetic Wilson lines of the gauge theory give rise to the $4g$ scalar components of the g hypermultiplets.)

An interesting special case is $g=1$. In this case the fibration of $\mathbb{C}^2/G$ over the Riemann surface T^2 is trivial, so the $\mathcal{N}=2$ supersymmetry should be enhanced to $\mathcal{N}=4$; this agrees with the fact that we get a single adjoint hypermultiplet, which is the required matter content for the $\mathcal{N} = 4$ theory. Furthermore, there is a relation

$$\mathrm{V}ol(T^2) = 1/g_{YM}^2. \tag{5.4}$$

T-dualizing on the two circles of T^2 then implies that the theory with coupling g_{YM} is equivalent to the theory with coupling $1/g_{YM}$ — so the existence of a string theory realization already implies the highly nontrivial Montonen-Olive duality of $\mathcal{N} = 4$ super Yang-Mills!

One could also consider the case $g > 1$, but in this case the gauge theory is not asymptotically free. We therefore focus on $g = 0$, which gives pure $\mathcal{N} = 2$ gauge theory; and for simplicity we consider the case $G = \mathbb{Z}_N$, which gives $SU(N)$ gauge theory. To "solve" this gauge theory a la Seiberg and Witten [34], one wants to compute its prepotential F_0, as a function of the Coulomb branch moduli. As we remarked above, this prepotential coincides with the F_0 computed by the A model topological string; namely, the Coulomb branch moduli get identified with the Kähler moduli t_i which give the sizes of the $\mathbb{CP}^1$'s resolving the singularity, while the volume of the base $\mathbb{CP}^1$ controls the bare gauge coupling at the string scale,

$$\mathrm{V}ol(\mathbb{CP}^1) = 1/g_{YM}^2. \tag{5.5}$$

So we have two different kinds of $\mathbb{CP}^1$ in the geometry, playing quite distinct roles. Indeed, from (5.5) one sees that the worldsheet instantons which wrap n times around the base $\mathbb{CP}^1$ contribute with a factor e^{-n/g_{YM}^2} to F_0, and hence they should correspond to target space instanton number n.

So far we have elided one subtlety: at generic values of g_{YM}^2 and t_i, the string theory actually contains more information than just the 4-dimensional gauge theory. This is to be expected since the F_0 of the gauge theory depends just on the moduli t_i, while our F_0 also depends on the size of the base which we identified with g_{YM}^2 at the string scale. To isolate the 4-dimensional content we have to take a decoupling limit in which g_{YM}^2 and t_i approach zero, which sends the string scale to infinity while keeping the masses of the W bosons on the Coulomb branch fixed [35]. If we do not take this decoupling limit, we get a theory which includes information about compactification from 5 to 4 dimensions; from that point of view the instantons can be interpreted as particles of the 5-dimensional theory which are running in loops.

We have just reformulated the problem of solving the IR dynamics of the $\mathcal{N} = 2$ gauge theory as the problem of computing the A model F_0 in a particular Calabi-Yau geometry. Let us see how this procedure works out in the simplest example, namely the pure $SU(2)$ theory. In this case we have to consider the A_1 singularity fibered over $\mathbb{CP}^1$, and it turns out that this geometry is nothing but the local $\mathbb{CP}^1 \times \mathbb{CP}^1$ geometry we discussed before.

The two $\mathbb{CP}^1$ factors appear symmetrically in the geometry, although we are interpreting them quite differently (one of them is the "base" which controls g_{YM}^2 while the other is the "fiber" which gives the Coulomb branch modulus) and in particular the decoupling limit breaks the symmetry between them. Now we want to obtain F_0 for this geometry, and this can be done using mirror symmetry; recalling that we have a toric realization as shown in Figure 11, the techniques we illustrated in Section 4.1 can be straightforwardly applied. The mirror geometry is again of the form $F(x, z) = uv$, where the Riemann surface $F(x, z) = 0$ is precisely the Seiberg-Witten curve encoding the solution of the model [36].

A similar procedure can be applied to any ADE gauge group by choosing the appropriate geometry, and conversely, anytime we have a toric geometry where the Kähler parameters arise by resolving some singularity, we expect that that toric geometry can be interpreted in terms of gauge theory. The zoo of $\mathcal{N} = 2$ theories one can "geometrically engineer" in this way includes cases with arbitrarily complicated product gauge groups and bifundamental matter content, as well as some exotic conformal fixed points in higher dimensions; see e.g. [35, 36, 37, 33, 38, 39]. To obtain the prepotentials for the geometrically-engineered theories is in principle straightforward via mirror symmetry, and it has been worked out in many cases, but it is not always easy — e.g. for the E_k singularities one would have a more difficult job, because to realize these geometries torically one has to include a superpotential, which makes the mirror procedure and computation of the mirror periods less straightforward.

5.2 $\mathcal{N} = 1$ gauge theories

So far we have discussed geometric engineering of $\mathcal{N} = 2$ theories, but it turns out that string theory also has something to say about the $\mathcal{N} = 1$ case. How can we geometrically engineer an $\mathcal{N} = 1$ theory? Starting with compactification on a Calabi-Yau space, we need to introduce an extra ingredient which reduces the supersymmetry by half. There are two natural possibilities: we can add either D-branes or fluxes. In both cases we want to preserve the four-dimensional Poincare invariance; so if we use D-branes we have to choose them to fill the four uncompactified dimensions, and if we use fluxes we have to choose them entirely in the Calabi-Yau directions (i.e. the 0, 1, 2, 3 components of the flux should vanish.) In fact, the two possibilities are sometimes equivalent because of the possibility of a geometric transition in which branes are replaced by flux, as we discussed above.

Let us first focus on the case where we introduce a stack of N branes, which are wrapped on some cycle in the Calabi-Yau and also fill the four dimensions of spacetime. Then we obtain an $\mathcal{N} = 1$ theory in four dimensions, with $U(N)$ gauge symmetry. (Note the difference from the geometric engineering we did in the $\mathcal{N} = 2$ case; there we obtained the gauge symmetry from a geometric singularity, but in the $\mathcal{N}=1$ case it just comes from the usual Chan-Paton mechanism, while the geometry is responsible for details of the gauge theory, specifically the form of the bare superpotential.)

Now what does this have to do with topological strings? We have seen above that the free energy F_g (and particularly F_0) compute F-terms relevant for the $\mathcal{N}=2$ theory. After introducing D-branes in the topological string, we need not consider only closed worldsheets anymore; an open string worldsheet naturally corresponds to a Riemann surface with boundaries. Therefore we can define a free energy $F_{g,h}$, obtained by integrating over worldsheets with genus g and h holes, with each hole mapped to one of the D-branes; and we can ask whether this $F_{g,h}$ computes something relevant for the $\mathcal{N}=1$ theory. The answer is of course "yes." As we did in the $\mathcal{N}=2$ case, we will focus on $g=0$ at first; higher genera are related to gravitational corrections.

To write the terms which the topological string computes in the $\mathcal{N}=1$ theory with branes, we need the "glueball" superfield S, with lowest component $\mathrm{Tr}\psi_\alpha\psi^\alpha$. Then we organize the $F_{0,h}$ into a generating function:

$$F(S)=\sum_{h=0}^{\infty}F_{0,h}S^h. \tag{5.6}$$

The F-term the topological string computes in the $\mathcal{N}=1$ theory can then be written [10]

$$\int \mathrm{d}^4x\int \mathrm{d}^2\theta N\frac{\partial F}{\partial S}. \tag{5.7}$$

This is a superpotential for the glueball S, and it turns out that the this superpotential captures a lot of the relevant infrared dynamics. More precisely, in addition to (5.7), one also has to include the term

$$\int \mathrm{d}^4x\int \mathrm{d}^2\theta\tau S, \tag{5.8}$$

which is simply the super Yang-Mills action in superfield notation, with

$$\tau=\frac{4\pi i}{g_Y^2M}+\frac{\theta}{2\pi}. \tag{5.9}$$

After including this extra term, one can determine the vacuum structure of the theory just by extremizing the glueball superpotential — different vacua are distinguished by different values of the glueball condensate.

Now what about the case where we introduce fluxes instead of branes? Consider Type IIB on a Calabi-Yau X. Recall from the last section that this theory has a prepotential term

$$\int \mathrm{d}^4x\int \mathrm{d}^4\theta F_0(X^I), \tag{5.10}$$

where F_0 is the B model topological string free energy at genus zero, and the X^I are the vector superfields, whose lowest components parameterize the complex structure moduli of X. How does this term change if we introduce N^I units of three-form flux on the I-th A cycle?[17] In the $\mathcal{N} = 2$ supergravity language, it turns out that this flux corresponds to the θ^2 component of the superfield X^I; turning on a vacuum expectation value for this component absorbs two θ integrals from (5.10), leaving behind an F-term [40],

$$\int \mathrm{d}^4x \int \mathrm{d}^2\theta N^I \frac{\partial F_0}{\partial X^I}. \tag{5.11}$$

There is a natural extension of this formula to include a flux τ_I on the I-th B cycle (which need not be quantized since the B cycle is non-compact):

$$\int \mathrm{d}^4x \int \mathrm{d}^2\theta N^I \frac{\partial F_0}{\partial X^I} + \tau_I F^I. \tag{5.12}$$

This form of the superpotential was derived in [41].

There is an obvious analogy between (5.7) and (5.11). Note though that the lowest component of the X^I which appears in (5.11) is an honest scalar field parameterizing a complex structure modulus, while the S which appears in (5.7) is a fermion bilinear, which naively cannot have a classical vacuum expectation value. Nevertheless, the analogy between the two sides seems to be suggesting that we should treat S also as an honest scalar, and we will do so in what follows.

So what do (5.7) and (5.11) have to do with one another? The crucial link is provided by the notion of "geometric transition" which we discussed before, but now in the context of the Type IIB superstring rather than the topological string:[18] start with a Calabi-Yau X which has a 2-cycle. Then wrap D5-branes on these 2-cycles, obtaining a $U(N)$ gauge theory. There is a dual geometry where the D5-branes disappear and are replaced by a 3-cycle A; in this dual geometry there are N units of flux on the dual cycle B, the remnant of the vanished D5-branes. The claim is that the string theories on these two geometries are equivalent, after we identify the glueball superfield S with the period of Ω over the A cycle in the dual geometry.[19] With this identification (5.7) and (5.11) are identical. One can therefore use either one to compute the glueball superpotential.

[17] Recall that in writing the $\mathcal{N} = 2$ supergravity Lagrangian we have chosen a splitting of $H_3(X)$ into A and B cycles, with the X^I representing the A cycle periods.

[18] See [40] for a detailed discussion of the superstring version of the large N duality in the Type IIA case.

[19] On the face of it this claim might sound bizarre since the theory with branes should have $U(N)$ gauge symmetry in four dimensions; but since we are now talking about the effective theory in $d = 4$, what we should really compare is the IR dynamics, and we know that $\mathcal{N} = 1$ gauge theories confine, which reduces the $U(N)$ to $U(1)$ in the IR.

The simplest example of this phenomenon is provided by the geometry we discussed before, namely the resolved conifold, which just has a single 2-cycle $\mathbb{CP}^1$. So suppose we wrap M D5-branes on the $\mathbb{CP}^1$ of the resolved conifold. As one might expect, this simplest possible geometry leads to the simplest possible gauge theory in $d = 4$, namely $\mathcal{N} = 1$ super Yang-Mills. Now using the glueball superpotential we can recover the standard lore about this theory's infrared behavior. Let us derive this superpotential from the geometric transition and (5.11). The dual geometry after the transition is the deformed conifold, which has a compact S^3 and its dual B cycle, with corresponding periods

$$X = \int_A \Omega = \mu, \tag{5.13}$$

$$F = \int_B \Omega = \mu \log \mu. \tag{5.14}$$

Here to compare with the gauge theory we have to identify $\mu = S$ as we stated above. (A simple way to check the formula for F in terms of μ is to note that it has the correct monodromy; as $\mu \to e^{2\pi i}\mu$ the B cycle gets transformed into a linear combination of the B cycle and the A cycle, corresponding to the fact that F gets shifted by the A period μ.) From the periods we immediately obtain the closed string F_0,

$$F_0 = \frac{1}{2} X F = \mu^2 \log \mu, \tag{5.15}$$

which leads to the superpotential

$$N \frac{\partial F_0}{\partial S} - \tau S = N S \log S - \tau S. \tag{5.16}$$

This is the standard Veneziano-Yankielowicz glueball superpotential for $\mathcal{N} = 1$ super Yang-Mills [42]. By extremizing it as a function of S one finds the expected N vacua of $\mathcal{N} = 1$ super Yang-Mills,[20]

$$S = \Lambda^3 \exp\left(2\pi \mathrm{i} j / N\right), \tag{5.17}$$

where $j = 1, \ldots, N$.

So far we have not used much of our topological-string machinery. But now we can consider more elaborate examples: instead of the singular conifold geometry

$$u^2 + v^2 + y^2 + x^2 = 0, \tag{5.18}$$

which just has a single shrunk $\mathbb{CP}^1$, we could look instead at

$$u^2 + v^2 + y^2 + W'(x)^2 = 0, \tag{5.19}$$

[20]We have not been careful to keep track of the cutoff Λ_0; if one does keep track of it, one finds that it combines with the bare coupling τ to give the QCD scale Λ which appears in (5.17).

for some polynomial $W(x)$ of degree $n+1$. This geometry has n conifold singularities at the n critical poitns of W. The singularities can be resolved by blowing up to obtain n $\mathbb{CP}^1$'s at these n points (all these $\mathbb{CP}^1$'s are homologous, however, so in particular there is only one Kähler modulus describing the resolution.)

We want to use this geometry to engineer an interesting $\mathcal{N}=1$ gauge theory. Namely, if we have M D5-branes we can wrap M_1 of them on the first $\mathbb{CP}^1$, M_2 on the second and so on, obtaining gauge symmetry $U(M_1)\times\cdots\times U(M_n)$. Actually, all such configurations could be viewed as different vacua of a single gauge theory describing the M branes; this is natural because the gauge theory on the branes should include an adjoint chiral multiplet Φ, whose lowest component represents the x-coordinate of the brane. [21] Now what can we say about this gauge theory? The supersymmetric vacua should arise from configurations in which the eigenvalues of Φ are distributed among the critical points of W. It would therefore be natural to guess that the gauge theory in question has a bare superpotential $\mathrm{Tr}W(\Phi)$. This is indeed the case; one can derive this result from the holomorphic Chern-Simons action which, as we discussed earlier, is the topological open string field theory of the brane [43]. Namely, one shows from the holomorphic Chern-Simons action that, as one moves the 2-brane along a path, sweeping out a 3-cycle C, the classical action is shifted by $\int_C \Omega$; combined with the explicit form of Ω in the geometry (5.19) this gives the classical action for the brane at x as $W(x)$. This classical action in the topological string turns out to be the superpotential of the physical superstring. This superpotential computation can also be interpreted directly in the worldsheet language as coming from disc diagrams with boundary on the brane; to see this from the topological string one notes that $F_{0,1}$ contributes an S-independent term to (5.7), which gets interpreted as the desired bare superpotential.

So we have geometrically engineered an $\mathcal{N}=1$ gauge theory, with $U(M_1)\times\cdots\times U(M_n)$ gauge group, one adjoint chiral multiplet Φ, and a superpotential $\mathrm{Tr}W(\Phi)$. To answer questions about the vacuum structure of this theory we now want to find the appropriate glueball superpotential. As in the case of the simple conifold geometry, one way to do this is to consider the dual geometry in which the branes have disappeared and each of the n $\mathbb{CP}^1$'s has been replaced by an S^3: this geometry is written

$$u^2+v^2+y^2+W'(x)^2=f(x), \tag{5.20}$$

where $f(x)$ is a polynomial of degree $n-1$ giving the deformation. This $f(x)$ depends on the M_i and is completely fixed by the requirement that the period of Ω on the i-th S^3 is

[21] The adjoint scalar Φ is present even in the conifold case which we considered above, but there (as we will see below) it is accompanied by a quadratic superpotential $W(\Phi)=\Phi^2$, so Φ can be harmlessly integrated out to leave pure $\mathcal{N}=1$ super Yang-Mills.

$M_i g_s$. This approach was followed in [44].

Alternatively, one can avoid the geometric transitions altogether and compute directly in the gauge theory on the D5-branes. Since the glueball superpotential is computed by the topological string, one can avoid all the complexities of Yang-Mills theory, and use instead the topological open string field theory; as we explained above, in the case of the B model this is (the dimensional reduction of) holomorphic Chern-Simons. Working this out one finds that the whole computation of the topological string free energy is reduced to a computation in a holomorphic matrix model [45, 46, 47]. Specifically, to compute the planar free energy $F_{0,h}$ one is interested in the *planar limit* of the matrix model, while the higher $F_{g,h}$ correspond directly to higher genera in the 't Hooft expansion of the matrix model. These models have turned out to be a quite powerful tool, which is applicable to geometries more general than the case we described here. They are also related in a beautiful way to the geometric transitions we described above: namely, the planar limit of the matrix model can be described as a saddle-point expansion around a particular distribution of the eigenvalues, and this distribution turns out to capture the dual geometry in a precise way. In this sense the smooth geometry seems to be an emergent property, which only makes sense at large N!

5.3 BPS black holes in $d = 5$

So far we have discussed applications of the topological string to gauge theory, which (at least if we do not ask about gravitational corrections) involved only the genus zero free energy F_0. Now we want to discuss an application to black hole entropy, which is somewhat more sophisticated in the sense that it naturally involves all of the F_g. We ask the following question: given a compactification of M theory to five dimensions on a Calabi-Yau threefold X, how many BPS black hole states are there with a particular spin and charge?

First, what do we mean by "charge"? M-theory compactified on X has a $U(1)$ gauge field for each 2-cycle of X, obtained by dimensional reduction of the M-theory 3-form C on the 2-cycle, i.e. via the ansatz $C_{\mu\alpha\beta} = A_\mu \omega_{\alpha\beta}$, where $\omega_{\alpha\beta}$ is the harmonic 2-form dual to the cycle in question. So we get $U(1)^n$ gauge symmetry, where $n = b_2(X)$ is the number of independent 2-cycles. We also naturally get states which are charged under this $U(1)^n$; namely, an M2-brane wrapped on a 2-cycle gives a particle state charged under the corresponding gauge field. Hence the charges in the theory are classified by elements of the second homology, $Q \in H_2(X, \mathbb{Z})$.

So we could ask for the number of BPS states with given Q. But actually there is a finer question we can ask: namely, it turns out that in five dimensions it is possible for a black hole to have spin and still be BPS. The little group for a massive particle in this dimension is $SO(4) = SU(2)_L \times SU(2)_R$, giving rise to spins (j_L, j_R), and one can get BPS states so long as one requires either $j_L = 0$ or $j_R = 0$. So fixing, say, $j_R = 0$, we can ask for the number of BPS states with charge Q and spin j_L.

A convenient way of packaging this information is suggested by the notion of *elliptic genus*, which we now quickly recall in a related context, namely $\mathcal{N} = (1,1)$ theories in two dimensions [48]. The partition function on a torus with modular parameter τ with the natural boundary conditions is

$$\mathrm{Tr}(-1)^F q^{L_0} \bar{q}^{\bar{L_0}}. \tag{5.21}$$

This partition function is relatively "boring" in the sense that it just computes the Witten index [49], which is independent of q and $\bar{q}$. But in an $\mathcal{N} = (1,1)$ one can define separate left and right-moving fermion number operators F_L, F_R, and we can use these to define a more interesting object, the *elliptic genus*,

$$\mathrm{Tr}(-1)^{F_R} q^{L_0} \bar{q}^{\bar{L_0}}. \tag{5.22}$$

The usual argument shows that (5.22) gets contributions only from states which have $\bar{L_0} = 0$, so it is independent of $\bar{q}$, but it is a nontrivial function of q, which has modular properties. Like the usual Witten index it has some rigidity properties, namely, it does not depend on small deformations of the theory (moduli of the target space); this follows from the fact that the coefficients in the q expansion are integral.

Returning to the $d = 5$ black hole, note that we have a splitting into left and right similar to the one for $\mathcal{N} = (1,1)$ theories, so instead of computing the ordinary index

$$\mathrm{Tr}(-1)^J e^{-\beta H} \tag{5.23}$$

we can consider an elliptic genus analogous to (5.22),

$$\mathrm{Tr}(-1)^{J_R} q^{J_L} e^{-\beta H}. \tag{5.24}$$

Like (5.22), this elliptic genus has a rigidity property: it is independent of the complex structure moduli of X, although it can and does depend continuously on the Kähler moduli. This property is reminiscent of the A model topological string, and indeed it turns out that the A model partition function $Z_A(g_s, t_i)$ is precisely the elliptic genus (5.24), with the identification

$$q = e^{-g_s}, \tag{5.25}$$

as we will see below. In this sense the spin-dependence of the BPS state counting gets related to the genus-dependence of the topological string.

Now, *why* is the A model partition function counting BPS states? Such a connection seems reasonable; after all, the A model counts holomorphic maps, and the image of a holomorphic map is a supersymmetric cycle on which a brane could be wrapped to give a BPS state. There is a more precise argument which explains the relation; it was worked out in [50, 51] and goes roughly as follows. If we are interested in computing (5.24) we should study the five-dimensional theory on a circle S^1 of radius β. However, since (5.24) is an index it is independent of β, and can be evaluated in the limit $\beta \to 0$ in which the theory becomes four-dimensional. Interpreting S^1 as the M-theory circle, in this limit we are studying the weakly coupled Type IIA string on X. As we mentioned earlier, there are certain F-terms in the effective four-dimensional action of this theory which are computed by the A model topological string, namely

$$\int \mathrm{d}^4x \int \mathrm{d}^4\theta F_g(t)(\mathcal{W}^2)^g + \text{c.c.}, \tag{5.26}$$

which give rise to

$$\int \mathrm{d}^4x F_g(t)(R^2F^{2g-2}). \tag{5.27}$$

If we consider the Euclidean version of the theory, then in four dimensions we can turn on a self-dual graviphoton background $F_{\alpha\beta} \neq 0$, $F_{\bar{\alpha}\bar{\beta}} = 0$, i.e. $\mathcal{W} \neq 0$, $\bar{\mathcal{W}} = 0$. Substituting this background into (5.27) we get a sum correcting the R^2 term,

$$\left(\sum_{g=0}^{\infty} F_g(t)F^{2g-2}\right) R^2. \tag{5.28}$$

Note that in (5.28) we get a sum over all genus topological string amplitudes, with the role of the topological string coupling played by the graviphoton field strength F.

To establish the relation between the topological string and the elliptic genus, we now want to show that one can compute the R^2 correction in a graviphoton background in a different way which manifestly gives the elliptic genus. This second computation is based on Schwinger's computation of the correction to the vacuum energy from pair-production of charged particles in a background electric field. In the present context the relevant charged particles are the BPS states we have been considering; for a D2 brane wrapped on the cycle Q, bound to k D0 branes, the central charge is

$$Z = \langle Q, t\rangle + ik, \tag{5.29}$$

and the mass of the corresponding BPS state is $m = |Z|$. We compute the corrections to the effective action due to pair production of such states in the self-dual graviphoton background F. Since these states come in hypermultiplets of the $\mathcal{N} = 2$ supersymmetry, their contribution to the vacuum energy cancels, but it turns out that they make a simple contribution to the R^2 term: for example, a multiplet whose lowest component is scalar contributes precisely as a scalar would have contributed to the vacuum energy.

Let us focus on the contribution to the R^2 correction from a particular homology class Q, supporting a BPS hypermultiplet with lowest component scalar. Actually, we will get one such hypermultiplet for each value of the D0 brane charge k. The Schwinger computation expresses the contribution from each of these hypermultiplets as a one-loop determinant; summing over all k gives

$$\sum_k \log\det(\Delta + m_k^2) = \sum_{k=-\infty}^{\infty} \int_\epsilon^\infty \frac{\mathrm{d}s}{s} \frac{e^{-2\pi s(\langle Q,t\rangle + ik)}}{\left(2\sinh\frac{sF}{2}\right)}^2 . \tag{5.30}$$

(Here F enters the determinant through the non-commutation of the covariant derivatives which appear in Δ.) The integral appearing in (5.30) looks formidable, but luckily we do not have to do it: the sum over D0 brane charge k gives a δ-function which cancels the integral and also removes the awkward dependence on the cutoff ϵ. We get a simple result,

$$\sum_{n=0}^{\infty} \frac{1}{n} \frac{e^{-2\pi n\langle Q,t\rangle}}{\left(2\sinh\frac{nF}{2}\right)}^2 . \tag{5.31}$$

This is the contribution to the R^2 correction coming from a single BPS multiplet with scalar lowest component. Now identifying $F = g_s$ to compare with the topological string, and taking the $g_s \to 0$ limit, we recover

$$\frac{1}{g_s^2} \sum_n \frac{e^{-2\pi n\langle Q,t\rangle}}{n^3}, \tag{5.32}$$

which is precisely the formula we wrote in (3.24) for the contribution of an isolated genus zero curve to the A model F_0! In particular, the counting of BPS states reproduces the tricky $\sum_n 1/n^3$, which arose from multi-covering maps $S^2 \to S^2$ in the A model. Indeed, from counting BPS states one obtains formulas for the multi-covering contributions at all genera, as well as "bubbling" terms which occur when part of the worldsheet degenerates to a surface of lower genus.

Looking at (5.31) one does not see any obvious integer structure, but after exponentiating to get Z one finds

$$\prod_{n=1}^{\infty} (1 - q^n e^{-\langle Q,t\rangle})^n, \tag{5.33}$$

which indeed has integer coefficients. This is an expected property, because from what we have said, (5.33) is supposed to be the contribution to the elliptic genus from an M2-brane wrapped on the homology class Q. At first glance this formula might be surprising — one might naively have expected to get simply $e^{-\langle Q,t\rangle}$ — but one has to remember that the M2-brane gives rise to a whole field ϕ in five dimensions, and this ϕ can have excitations which are not Poincare invariant but are still BPS. Namely, choosing complex coordinates z_1, z_2 for the Euclidean time-slice $\mathbb{R}^4$, we can write

$$\phi = \phi(z_1, z_2, \bar{z}_1, \bar{z}_2), \tag{5.34}$$

and the BPS excitations are the ones where we only excite the z_i, so expanding

$$\phi = \sum_{l,m\geq 0} \phi_{lm} z_1^l z_2^m \tag{5.35}$$

we get a collection of fermionic creation operators ϕ_{lm}. The operator ϕ_{lm} creates $SU(2)_L$ spin $l+m$, so there are $n+1$ of them that create spin n. The second quantization of these operators then accounts for the degeneracy (5.33). From this discussion one easily sees how to modify the answer if the lowest component were not scalar but rather had $SU(2)_L$ spin j: one just has to replace q^n by q^{n+j} in (5.33).

Now we are ready to write the general form of the topological A model free energy in terms of the five-dimensional BPS content. It is convenient to choose a slightly exotic basis for the representation content: namely, we introduce the symbol $[j]$ for the $SU(2)_L$ representation $[2(0) \oplus (\frac{1}{2})]^{\otimes j}$. Any representation of $SU(2)_L$ can be written as a sum of the representations $[j]$ with integer coefficients (not necessarily positive). Then write $\mathcal{N}_{j,Q}$ for the number of times $[j]$ appears in the $SU(2)_L$ content of the BPS spectrum obtained by wrapping M2 branes on Q. Combining our results above we can now write

$$F(t, g_s) = \sum_{j\geq 0} \sum_{Q\in H_3(X,\mathbb{Z})} \mathcal{N}_{j,Q} \left(\sum_{n\geq 0} \left(2\sinh \frac{n g_s}{2} \right)^{2j-2} e^{-n\langle Q,t\rangle} \right). \tag{5.36}$$

The formula (5.36) expresses all the complexity of the A model topological string at all genera in terms of the integer invariants $\mathcal{N}_{j,Q}$. Conversely, it gives an algorithm for computing the numbers $\mathcal{N}_{j,Q}$, which capture the degeneracy of BPS states, using the topological string.

Despite the formidable computational techniques which are known for the topological string, it has not yet been possible to use it to verify one of the simplest predictions from black hole physics: namely, the asymptotic growth of the $\mathcal{N}_{j,Q}$ with Q should agree with the scaling of the black hole entropy.

5.4 BPS black holes in $d = 4$

Remarkably, it turns out that the topological string is also relevant to black hole entropy in $d = 4$! This application is somewhat subtler than the $d = 5$ case, however. In the $d = 5$

case, one could recover the number of BPS states with fixed charge and spin j just by looking at the A model amplitudes up to genus j. In $d = 4$ the perturbative topological string will only give us coefficients of the asymptotic *growth* of the number of states as a function of the charge; to get the actual number of states with a given fixed charge would require some sort of nonperturbative completion of the topological string.

What kind of black holes will we study in $d = 4$? Unlike in $d = 5$, there are no spinning BPS black holes, so we just want to determine the number of BPS states as a function of the charge. The charges in $d = 4$ are also a little more subtle than in $d = 5$; each $U(1)$ in the gauge group leads to both an electric and a magnetic charge. In Type IIA on X, there would be a natural splitting into electric and magnetic charges; namely, D0 and D2 branes on X give electrically charged states, while D4 and D6 branes give magnetically charged states. In Type IIB, on the other hand, all of the charges are realized by D3 branes wrapping 3-cycles, and a splitting into electric and magnetic is obtained only after making a choice of symplectic basis (A and B cycles), as we've discussed before:

$$A_I \cap B^J = \delta_I^J. \tag{5.37}$$

So a general combination of electric and magnetic charges can be realized by a D3 brane wrapping a general 3-cycle, i.e. a choice of $C \in H_3(X, \mathbb{Z})$. Now, how can the Calabi-Yau space X give us the number of BPS states, as a function of this C?

The crucial ingredient here is the *attractor mechanism* of $\mathcal{N} = 2$ supergravity [52]. Namely: suppose we consider the supergravity theory obtained by compactifying Type IIB on X and look for classical solutions describing a BPS black hole with charge C. There are various such solutions, depending on a choice of boundary condition: namely, the supergravity theory includes scalar fields corresponding to the moduli of X, and we can choose the expectation values of those scalar fields at infinity arbitrarily. Then studying the evolution of the scalar fields as we move in from infinity toward the black hole horizon, one finds a remarkable phenomenon: the vector multiplet moduli, describing the complex structure of X, approach fixed values as we approach the horizon. These fixed values depend only on the charge C of the black hole; they are independent of the boundary condition.[22] It is not easy to describe the map from the charge C to the holomorphic 3-form Ω (which determines the complex structure), but the inverse map is straightforward: choosing a basis of 3-cycles and

[22]This statement needs to be slightly qualified: the moduli at the horizon are *locally* independent of the moduli at infinity, but there can be multiple basins of attraction [53]. This phenomenon is related to the existence of lines of marginal stability for BPS states in $\mathcal{N} = 2$ theories.

corresponding electric-magnetic splitting $C = (P^I, Q_J)$, the relation is

$$P^I = \int_{A^I} \mathrm{Re}\Omega, \qquad Q_J = \int_{B_J} \mathrm{Re}\Omega, \tag{5.38}$$

or more invariantly, $\mathrm{Re}\Omega \in H^3(X, \mathbb{R})$ is the Poincare dual of $C \in H_3(X, \mathbb{Z})$. (Note that the counting of parameters works out correctly: the complex structure moduli, when augmented to include the choice of overall scaling of Ω, make up $2b_3(X)$ real parameters, and this is also the number of possible black hole charges.)

Given the black hole charges C we now want to compute the number of BPS states. In fact it will be convenient to express the answer in terms of S, the entropy, To leading order in C, the answer is remarkably simple: the entropy is given by the "holomorphic volume" of the Calabi-Yau at the attractor value of Ω,

$$S(\Omega) = \frac{\mathrm{i}\pi}{4} \int_X \Omega \wedge \bar{\Omega}. \tag{5.39}$$

Note that this entropy has the expected scaling with the size of the black hole: namely, from (5.38) we see that a rescaling $C \mapsto \lambda C$ (which also rescales the size of the black hole by λ thanks to the BPS relation between mass and charge) rescales the attractor moduli by $\Omega \mapsto \lambda\Omega$, and hence $S \mapsto \lambda^2 S$. This is the expected behavior for the entropy of a black hole in four dimensions.

Now we want to highlight a connection between (5.39) and the topological string. To do so, we begin by noting that if we choose an electric-magnetic splitting we can rewrite (5.39) as

$$S(P, Q) = \frac{\mathrm{i}\pi}{4}(X^I \bar{F}_I - \bar{X}^I F_I). \tag{5.40}$$

This expression is quadratic in the periods of Ω, which is reminiscent of the tree level B model free energy F_0. Indeed, form the combination

$$\frac{\mathrm{i}\pi}{4}(F_0(X) - \bar{F}_0(X)) = \frac{\mathrm{i}\pi}{4}(X^I F_I - \bar{X}^I \bar{F}_I). \tag{5.41}$$

Now (5.40) and (5.41) are not quite equal, but they are related, as explained in [54]: namely, beginning with (5.41), we introduce the notation $\Phi^I = X^I - \bar{X}^I$, and then make a Legendre transform from Φ^I to a dual variable Q_I. We call the dual variable Q_I for a reason: namely, according to (5.38) the black hole electric charge $Q_I = F_I + \bar{F}_I$, and substituting this for the Q_I we just introduced in the Legendre transform of (5.41) one recovers (5.40)!

One thus obtains, to leading order in P and Q, the relation

$$\sum_Q \rho(P, Q) e^{-Q_I \Phi^I} = |Z_B(P + \mathrm{i}\Phi)|^2. \tag{5.42}$$

On the left side of (5.42), $\rho(P, Q)$ is the number of BPS black holes with electric and magnetic charges (P, Q), while on the right side $Z_B(P+\mathrm{i}\Phi)$ is the B model partition function, evaluated at the complex structure determined by the A cycle periods $X^I = P^I + \mathrm{i}\Phi^I$.

In other words, the partition function of the grand canonical ensemble of black holes where we fix the magnetic charges P and the electric potential Φ, then sum over all electric charges, is given by $|Z_B|^2$! This is a beautiful relation and it is very natural to conjecture that it holds not just at tree level but in fact to all genera [54].

Now what is the evidence for this conjecture beyond tree level? The major source of evidence comes from a reconsideration of the corrections to $\mathcal{N} = 2$ supergravity computed by the topological string, which we wrote in (5.2):

$$\int \mathrm{d}^4x \int \mathrm{d}^4\theta F_g(X^I)(\mathcal{W}^2)^g + \text{c.c.} \tag{5.43}$$

In the background of the charged black hole, the graviphoton field $\mathcal{W}$ and $\bar{\mathcal{W}}$ both have nonzero expectation values (and create a nontrivial gravitational backreaction.) The terms (5.43) therefore lead to terms proportional to $F_g(X^I)$. A careful study of these terms shows that they correct the black hole entropy, as explained in [**?**], and these corrections can be shown to be consistent with the conjecture. There are also formulas for the one-loop correction to the black hole entropy which agree with the conjecture.

Finally, there is at least one example (so far) where one can see directly that the square of the B model partition function is related to counting of black holes. This example, studied in [55], is obtained by putting the B model on the Calabi-Yau threefold geometry $\mathcal{L} \oplus \mathcal{L}^{-1} \to T^2$, where $\mathcal{L}$ is a particular complex line bundle over T^2. First consider the counting of black hole BPS states in this geometry. The relevant black holes are obtained by wrapping D4 branes on $\mathcal{L}^{-1} \to T^2$ as well as wrapping D2-D0 bound states on T^2. One can then argue that the theory on the N D4 branes is a topological $U(N)$ gauge theory, and furthermore that it localizes to a bosonic $U(N)$ Yang-Mills theory on T^2. The latter theory was studied in detail in [**?**], where it was shown that the exact partition function can be obtained as a sum over representations of $U(N)$:

$$Z_{YM} = \sum_R e^{-\lambda C_2(R) + \mathrm{i}\theta|R|} \tag{5.44}$$

where $|R|$ is the number of boxes in the Young diagram representing R. Furthermore, expanding around the large N limit, one finds that this Z_{YM} is the square of a holomorphic function to all orders in $1/N$, $Z_{YM} = |Z|^2$. (This splitting into "chiral" and "anti-chiral" parts is obtained by splitting up the Young diagrams R into short diagrams, with a finite number of boxes, and large diagrams, for which the size of the first column differs from N

by a finite number; from this description it is manifest that the splitting only makes sense in the large N limit.) So the partition function on the branes, which counts BPS states, is indeed of the form $|Z|^2$. Furthermore, using the topological vertex techniques we mentioned earlier, one can see that this Z actually agrees with the B model partition function, $Z = Z_B$!

The example of Yang-Mills on T^2 thus provides a striking confirmation of the conjecture that $|Z_B|^2$ counts BPS black hole states. It also gives us some insight into the nonperturbative topological string. Namely, as we noted above, the factorization of Z_{YM} into $|Z_B|^2$ is only valid in perturbation theory; but whatever the nonperturbative topological string is, we want it to count BPS states and hence to agree with Z_{YM}. Therefore we might expect that Z_B itself probably only makes sense perturbatively in general — the object that has a chance to have a nonperturbative completion is $|Z_B|^2$, but the nonperturbative completion probably is *not* generally factorized into chiral and anti-chiral parts.

References

[1] S.-T. Yau, "On the Ricci curvature of a compact Kaehler manifold and the complex Monge-Ampere equation. I.," *Comm. Pure Appl. Math.* **31** (1978) 339–411.

[2] P. A. Griffiths and J. Harris, *Principles of algebraic geometry.* John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[3] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," *Nucl. Phys.* **B359** (1991) 21–74.

[4] P. Candelas and X. C. de la Ossa, "Comments on conifolds," *Nucl. Phys.* **B342** (1990) 246–268.

[5] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. P. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, vol. 1 of *Clay Mathematics Monographs.* American Mathematical Society, Providence, RI, 2003.

[6] E. Witten, "Phases of $\mathcal{N} = 2$ theories in two dimensions," *Nucl. Phys.* **B403** (1993) 159–222, `hep-th/9301042`.

[7] E. Witten, "Topological sigma models," *Commun. Math. Phys.* **118** (1988) 411.

[8] E. Witten, "Mirror manifolds and topological field theory," `hep-th/9112056`. In Yau, S.T. (ed.): *Mirror symmetry I*, 121-160.

[9] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Holomorphic anomalies in topological field theories," *Nucl. Phys.* **B405** (1993) 279–304, `hep-th/9302103`.

[10] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, hep-th/9309140.

[11] E. Witten, "Quantum background independence in string theory," hep-th/9306122.

[12] R. Dijkgraaf, E. Verlinde, and M. Vonk, "On the partition sum of the NS five-brane," hep-th/0205281.

[13] D. Quillen, "Determinants of Cauchy-Riemann operators on Riemann surfaces," *Functional Anal. Appl.* **19** (1985), no. 1, 31–34.

[14] K. Hori and C. Vafa, "Mirror symmetry," hep-th/0002222.

[15] A. Schwarz, "Sigma models having supermanifolds as target spaces," *Lett. Math. Phys.* **38** (1996) 91–96, hep-th/9506070.

[16] M. Aganagic and C. Vafa, "Mirror symmetry and supermanifolds," hep-th/0403192.

[17] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, "Topological strings and integrable hierarchies," hep-th/0312085.

[18] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large n field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, hep-th/9905111.

[19] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.

[20] R. Gopakumar and C. Vafa, "On the gauge theory/geometry correspondence," *Adv. Theor. Math. Phys.* **3** (1999) 1415–1443, hep-th/9811131.

[21] H. Ooguri and C. Vafa, "Worldsheet derivation of a large N duality," *Nucl. Phys.* **B641** (2002) 3–34, hep-th/0205297.

[22] E. Witten, "Chern-Simons gauge theory as a string theory," *Prog. Math.* **133** (1995) 637–678, hep-th/9207094.

[23] E. Witten, "Noncommutative geometry and string field theory," *Nucl. Phys.* **B268** (1986) 253.

[24] M. Aganagic, M. Marino, and C. Vafa, "All loop topological string amplitudes from Chern-Simons theory," hep-th/0206164.

[25] H. Ooguri and C. Vafa, "Knot invariants and topological strings," *Nucl. Phys.* **B577** (2000) 419–438, hep-th/9912123.

[26] E. Witten, "Quantum field theory and the Jones polynomial," *Commun. Math. Phys.* **121** (1989) 351.

[27] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, "The topological vertex," `hep-th/0305132`.

[28] A. Okounkov, N. Reshetikhin, and C. Vafa, "Quantum Calabi-Yau and classical crystals," `hep-th/0309208`.

[29] A. Iqbal, N. A. Nekrasov, A. Okounkov, and C. Vafa, "Quantum foam and topological strings," `hep-th/0312022`.

[30] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov-Witten theory and Donaldson-Thomas theory," `math.AG/0312059`.

[31] M. Bershadsky and V. Sadov, "Theory of Kahler gravity," *Int. J. Mod. Phys.* **A11** (1996) 4689–4730, `hep-th/9410011`.

[32] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "Topological amplitudes in string theory," *Nucl. Phys.* **B413** (1994) 162–184, `hep-th/9307158`.

[33] S. Katz and C. Vafa, "Matter from geometry," *Nucl. Phys.* **B497** (1997) 146–154, `hep-th/9606086`.

[34] N. Seiberg and E. Witten, "Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, `hep-th/9407087`.

[35] S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, "Nonperturbative results on the point particle limit of $\mathcal{N} = 2$ heterotic string compactifications," *Nucl. Phys.* **B459** (1996) 537–558, `hep-th/9508155`.

[36] S. Katz, A. Klemm, and C. Vafa, "Geometric engineering of quantum field theories," *Nucl. Phys.* **B497** (1997) 173–195, `hep-th/9609239`.

[37] S. Katz, P. Mayr, and C. Vafa, "Mirror symmetry and exact solution of 4d $\mathcal{N} = 2$ gauge theories. I," *Adv. Theor. Math. Phys.* **1** (1998) 53–114, `hep-th/9706110`.

[38] D. R. Morrison and N. Seiberg, "Extremal transitions and five-dimensional supersymmetric field theories," *Nucl. Phys.* **B483** (1997) 229–247, `hep-th/9609070`.

[39] A. D. Shapere and C. Vafa, "Bps structure of argyres-douglas superconformal theories," `hep-th/9910182`.

[40] C. Vafa, "Superstrings and topological strings at large N," *J. Math. Phys.* **42** (2001) 2798–2817, `hep-th/0008142`.

[41] S. Gukov, C. Vafa, and E. Witten, "CFT's from Calabi-Yau four-folds," *Nucl. Phys.* **B584** (2000) 69–108, `hep-th/9906070`.

[42] G. Veneziano and S. Yankielowicz, "An effective Lagrangian for the pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory," *Phys. Lett.* **B113** (1982) 231.

[43] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs," `hep-th/0012041`.

[44] F. A. Cachazo, K. A. Intriligator, and C. Vafa, "A large N duality via a geometric transition," *Nucl. Phys.* **B603** (2001) 3–41, `hep-th/0103067`.

[45] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," `hep-th/0206255`.

[46] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," `hep-th/0207106`.

[47] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," `hep-th/0208048`.

[48] E. Witten, "On the Landau-Ginzburg description of $\mathcal{N} = 2$ minimal models," *Int. J. Mod. Phys.* **A9** (1994) 4783–4800, `hep-th/9304026`.

[49] E. Witten, "Constraints on supersymmetry breaking," *Nucl. Phys.* **B202** (1982) 253.

[50] R. Gopakumar and C. Vafa, "M-theory and topological strings. I," `hep-th/9809187`.

[51] R. Gopakumar and C. Vafa, "M-theory and topological strings. II," `hep-th/9812127`.

[52] S. Ferrara, R. Kallosh, and A. Strominger, "$\mathcal{N} = 2$ extremal black holes," *Phys. Rev.* **D52** (1995) 5412–5416, `hep-th/9508072`.

[53] G. W. Moore, "Arithmetic and attractors," `hep-th/9807087`.

[54] H. Ooguri, A. Strominger, and C. Vafa, "Black hole attractors and the topological string," `hep-th/0405146`.

[55] C. Vafa, "Two dimensional yang-mills, black holes and topological strings," `hep-th/0406058`.