Much Ado about N = 2

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Supergravity at 25 – Stony Brook 2001

N = 2 supergravity : Ferrara & Van Nieuwenhuizen

reports at **Supergravity** '79 Stony Brook

emphasis on auxiliary fields & off-shell structure

Why N = 2 ? more complete unifications
ultraviolet finiteness
no off-shellness beyond 8 supercharges

matter – hypermultiplets → hyperkähler
Quaternion-Kähler

gauge theory – vector multiplets → special geometry
– tensor multiplets

supergravity

Large variety of { theories / applications }

effective actions for

type-II string compactification on CY_3
heterotic string compactification on $K3 \times T^2$
Seiberg-Witten theory
fixed point behaviour, black hole entropy
R^2 terms etc.

Participants of the Workshop on Supergravity
Stony Brook, 27-29 September, 1979

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methods

- supercontformal multiplet calculus (this talk)
  $\Longleftrightarrow$ Quotients
  insisted (originally) on off-shellness (not necessary!)
  → restricted class
- harmonic superspace
  infinite # degrees of freedom
  quantum corrections
- more straightforward approaches or geometric ones (eg group manifold)

Early work

Breitenlohner & Sohnius: Quaternionkähler quotients (in susy setting)

Bagger & Witten: imposed mathematical structure (still unresolved issues);

Alvarez-Gaumé & Freedman: target space geometry

Hitchin, Karlhede, Lindström, Roček: quotients

Early 90's

Swann, Galicki

- various quotients commute
- generalized superconformal quotient (i.e. relaxing off-shellness)
- mathematical structure of hyperkähler cone → quaternion-kähler space

SUPERSYMMETRY

Field theories with 8 supersymmetries

$N = 2 \quad D = 4$

$N = 1 \quad D = 5, 6$

"R-symmetry" group $USp(2, \mathbb{C}) \sim SU(2) \sim Sp(1) \quad D=5,6$

(relativistic internal automorphism group of susy algebra) $U(1) \times SU(2) \quad D=4$

$\uparrow$

$SO(2) \times SO(3,1) \subset SO(5,1)$

3 different massless supermultiplets 4+4 dof $(D=6)$

hypermultiplet 4 scalars + chiral spinor singlet

vector multiplet vector + chiral spinor doublet

tensor multiplet dual tensor + scalar + chiral spinor doublet

merge / exchange in 3 spacetime dimensions

$SO(5,1) \times SU(2) \longrightarrow SO(3,1) \times SU(2) \times SU(2)$

c-map

$\rightarrow N=4$ susy

Cones

start from

$$\mathcal{L} = \frac{1}{2} g_{AB} \partial_\mu \phi^A \partial^\mu \phi^B - \frac{1}{4} \frac{d-2}{d-1} \chi(\phi) R$$

"improvement" term

scale invariance:

$$D_A \partial_B \chi = g_{AB}$$ **homothety**

$$\rightarrow \quad \chi = \frac{1}{2} g^{AB} \chi_A \chi_B \quad \text{(locally up to constants)}$$

$$\delta \phi^A = w \chi^A \qquad \delta g_{\mu\nu} = -2 g_{\mu\nu}$$

CONE $\quad \chi^A \frac{\partial}{\partial \phi^A} = \frac{\partial}{\partial \phi} \qquad \phi^A \rightarrow \phi, \phi^a$

$$\chi = e^{2\phi} \hat{\chi}(\phi^a)$$

choose

$$\chi = \text{constant} \sim (M_{Planck})^{d-2}$$

gauge equivalent to Einstein gravity coupled to a nonlinear sigma model with target space $\mathcal{M}$ (Weyl-Stueckelberg)

initial space : Cone over $\mathcal{M}$.

generalize this to supersymmetry

requires CONFORMAL SUPERGRAVITY

(exists up to 16 supercharges)

$\rightarrow$ superconformal quotient

$N=0$ no further structure

$N=1$ χ + phase, Kähler (+ S susy)
Kähler quotient

$N=2$ vector χ + phase, Kähler (+ S susy)
Kähler quotient → special geometry

$N=2$ hyper $\chi + S^{t3}$, hyperkähler (+ S susy)
superconformal quotient
→ quaternion-Kähler

Note: compensating chiral, vector, hypermultiplet

χ invariant under R symmetry
(incorporated in conformal s.g)
i.e. $U(1)$ or $SU(2) \sim Sp(1)$

+ "additional" isometries.
(needs qualification)

PM:
Kähler and hyperkähler quotients can be imposed supersymmetrically

Vector multiplets

gauge fields W_μ^I $\overset{\text{susy}}{\longleftrightarrow}$ scalars X^I

from $D=6$ $\quad W_{5,6}^I(x) \to X^I$ $\quad$ } real for $D=5$, complex for $D=4$

superconformal quotient

Kähler cone $\longrightarrow$ special Kähler space

$$\mathcal{K}(X,\bar{X}) = i\left(F_I(X)\,\bar{X}^I - X^I\,\bar{F}_I(\bar{X})\right)$$

(note: EM invariant)

projective geometry

X^I sections of a line bundle

section follows from $\delta_S \Omega^I{}_\alpha$

on the cone:

complex structure $J^2=-1$

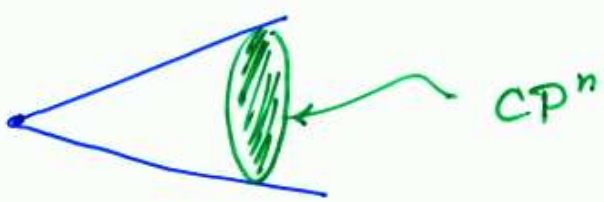
covariantly constant

$U(1)$ isometry $\quad K^A_{U(1)} = J^A{}_B\, \mathcal{X}^B$

$\mathcal{X}$: homothety.

$$D_A K_B^{U(1)} = -J_{AB}$$

example: Kähler cone ~ flat space



(in supergravity: noncompact versions)

Hypermultiplets

no preferred basis.

$$\mathcal{L} \sim g_{AB}\, \partial_\mu \phi^A \partial^\mu \phi^B + \mathcal{R}\chi(\phi)$$

g_{AB} → hyperkähler

3 complex structures
holonomy $\subset Sp(n)$
Ricci flat.

$A = 1, \dots 4n$

spinors ξ^α $\xi^{\bar\alpha}$ $\quad \alpha, \bar\alpha = 1, \dots 2n$

$$\mathcal{L} \sim G_{\alpha\beta}\, \bar\xi^\alpha \not{D} \xi^\beta$$

supersymmetry $\quad \delta\phi^A \propto \gamma^A_{i\bar\alpha}\, \bar\epsilon^i \xi^{\bar\alpha} + h.c$

$$\delta\xi^\alpha \propto V^\alpha_{Ai}\, \not\partial \phi^A\, \epsilon^i$$

quaternionic vielbeine γ, V covariantly constant

connections : target space diffs
fermion frame rotations.

$$\gamma_{Ai\bar\alpha}\, \bar V_B^{j\bar\alpha} = \varepsilon_{ik}\, J_{AB}^{kj} + \tfrac{1}{2}\, g_{AB}\, \delta_i^{\ j}$$

J_{AB}^{kj} → complex structures
g_{AB} → hyperkähler metric

ETC

Hyperkähler cone

hyperkähler potential

homothety $\mathcal{D}_A \partial_B \chi = g_{AB}$

SU(2) isometries $\vec{K}_A = \vec{J}_{AB} \chi^B$

triholomorphic isometries $\boxed{\vec{\mu} = -\frac{1}{2} \vec{K}_A K^A}$

$\vec{\mu}$ moment map: $\partial_A \vec{\mu} = \vec{J}_{AB} K^B$

(general) $\Delta \vec{\mu} = 0$

$\vec{K}^A \chi^A$ 4 orthogonal vectors of length^2 $= 2\chi$

$\chi^A K_A = 0$

horizonal: $\vec{K}^A R_{ABC}{}^D = \chi^A R_{ABC}{}^D = 0$

associated quaternionic bundle (Swann)

$A_i{}^\alpha(\phi)$ section of $Sp(1) \times Sp(n)$ bundle.

$\mathcal{D}_B A_i{}^\alpha = V_{Bi}{}^\alpha$

(pseudoreal)

$\boxed{\text{cf } X^I}: \delta_s \xi^\alpha$

$\bar{\Omega}_{\alpha\beta} A_i{}^\alpha A_j{}^\beta = \varepsilon_{ij} \chi$

$\bar{\Omega}_{\alpha\beta} A_i{}^\alpha \mathcal{D}_B A_j{}^\beta = \frac{1}{2} \varepsilon_{ij} \chi_B + K_{ij} B$ etc

supersymmetry:

$$\mathcal{L} \propto - G_{\bar{\alpha}\beta} \left\{ \tfrac{1}{2} \partial_\mu A_i{}^\beta \partial\!\!\!/ A^{i\bar{\alpha}} + \bar{\xi}^{\bar{\alpha}} \overleftrightarrow{\partial\!\!\!/} \xi^\beta \right\}$$

$$- W_{\alpha\beta\bar{\gamma}\bar{\delta}} (\bar{\xi}^{\bar{\alpha}} \gamma^\mu \xi^\beta)(\bar{\xi}^{\bar{\gamma}} \gamma_\mu \xi^{\bar{\delta}})$$

$+$ potential (gauging)

$W \propto R_{ABC}{}^D$.

ALL FOLLOWS FROM SUSY

More on Cones and Quotients

Cone over χ = constant hypersurface

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\chi_A\chi_B\right) \qquad \text{(horizonal)}$$

(3) - Sasakian metric

quotient isometry (left invariant by χ)

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\chi_A\chi_B - \frac{1}{k^2}k_A k_B\right)$$

k^3 isometry $\rightarrow$ k^{3A} holomorphic ; χ moment map

$\rightarrow$ superconformal / Kähler quotient

metric is horizontal <u>and</u> independent of the orthogonal coordinates (special geometry / vector multiplets)

For HKC $\rightarrow$ twistor space $(2n-1)$-dimensional.

Sp(1) superconformal quotient

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\left(\chi_A\chi_B + \vec{k}_A\cdot\vec{k}_B\right)\right)$$

quaternion-Kähler metric ! $4n-4$ coordinates

requires **explicit** gauge fixing.

Summarize:

HKC : cone over 3-Sasakian space

Kähler quotient $\rightarrow$ twistor space

twistor space S^2 fibration of Qk^{4n-4}

3-Sasakian space S^3 fibration of Qk^{4n-4}

<u>ASSERTION</u> HKC and/or twistor space are the best way to deal with hypermultiplets

c.q. quaternion-Kähler spaces

Results (with Vandoren, Kleyn, Roček, ...)

- general Lagrangian (with gauging) in terms of S'wann sections, both for HKC and QK space, prior to the last (quaternionic/twistor) gauge condition

  question formulate this explicitly in twistor space ?

- explicit construction

  HKC ↔ twistor space ↔ QK space

  geometry including complex/quaternionic structure

  triholomorphic ↔ quaternionic isometries

- Legendre transform

  superconformal tensor multiplets ↔ HKC

  determine the maximal number of abelian isometries

  (does not exclude nonabelian ones)

- gauging: potential both on the HKC and the QK space formulation.

  crucial $\mu_{CONE} = \mu^{SK}_{QK}\, \chi$

  up to a phase / Sp(1) rotation

- quaternion-K gauge choice

  some of $\vec{\mu}_{QK}$ vanish

  QK quotients seem to become singular

  → change order hyperkähler quotient prior to s.c. quotient

  [relevant for Higgs branch dynamics below vector boson mass scale]

Application (work in progress)

Abelian gauging of quaternion-Kähler spaces (dim $4n$) whose target space metric has $n+1$ abelian isometries (possibly complemented to nonabelian motions)

This covers all symplectic and unitary Wolf spaces

Two functions:

$F(X)$ governs the vector multiplets

$H(x, v, \bar{v})$ governs the hyper multiplets $(v, \bar{v}, w, \bar{w})^P$

related to $v, \bar{v}, x$ by Legendre transform

Abelian isometries $w_P \to w_P + i\Lambda_P$

Each isometry has coupling constant g_P

$$\mathcal{L}_{scalar} = - g_P g_Q \left[-4 H^{PQ} \chi^P \bar{\chi}^Q + N^{PQ} (x^P x^Q + 4 v^P \bar{v}^Q) \right]$$

+ nonabelian gauging for vectors only

$$H_{PQ} = \frac{\partial^2 H}{\partial x^P \partial x^Q} \qquad N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ})$$

↑ part of HKC metric ↑ Kähler cone metric

Observe H^{PQ} N^{PQ} not necessarily positive !

$\mathcal{L}_{scalar}$ defined on the HK cone × K cone.

$$\chi = 2 \chi^I N_{IJ} \bar{\chi}^J$$

Minimal case

$F(x)$ quadratic polynomial signature $(-+++\cdots)$

→ vectors parametrize CP_m

$H(x, v, \bar{v})$

$$\chi = \sum_P \sigma_P \sqrt{(x^P)^2 + 4 v^P \bar{v}^P}$$

σ_P: $-+++\cdots$

→ hypers parametrize HP_n

(class discussed by dW, Lauwers, Philippe, Su, VanProeyen PLB 134 (1984))

$$\mathcal{L} = -4 g_P^2 \sigma_P \sqrt{(x^P)^2 + 4 v^P \bar{v}^P}\, |X^P|^2$$

$$- g_P g_Q N^{PQ} (x^P x^Q + 4 v^P \bar{v}^Q)$$

N^{PQ}: constant.

When taking account of the cone structure the potential is invariant under the $SU(2) \times U(1)$ R-symmetry. This degeneracy is not physical and can be disregarded.